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QUANTUM ($\mathfrak{sl}_n$,$\wedge V_n$) LINK INVARIANT AND MATRIX FACTORIZATIONS

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ABSTRACT. I told about my Ph.D thesis [25] in Conference on Quantum Groups and Quantum Topology at RIMS Kyoto University. In this paper, I give a summary of my Ph.D. thesis.

1. INTRODUCTION

The purpose of this study is to construct a homology whose Euler characteristic is the quantum link invariant associated to the quantum group $U_q(\mathfrak{sl}_n)$ and its fundamental representations $\wedge V_n$ using matrix factorizations. That is, we try to construct a generalization of Khovanov-Rozansky homology[9].
We have a state model of the quantum ($\mathfrak{sl}_n$, $\wedge V_n$) link invariant using planer diagrams (intertwiners between tensor product representations of fundamental representations) given by Murakami, Ohtsuki and Yamada[15].

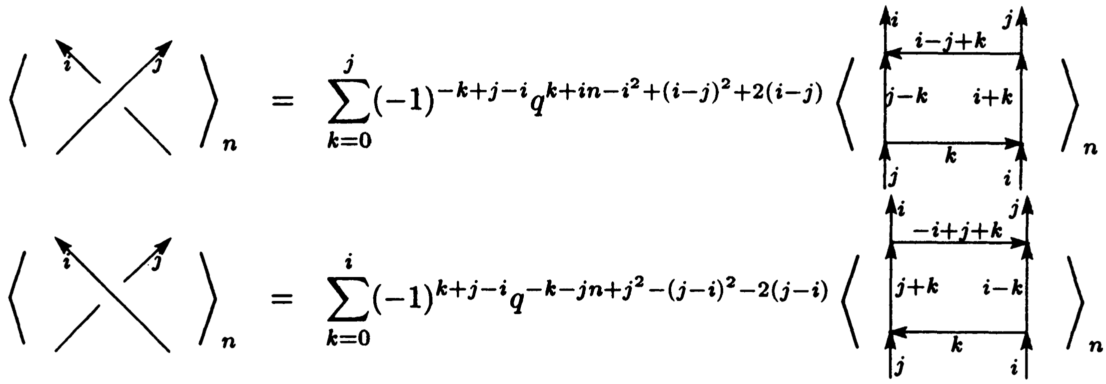


FIGURE 1. Reduction for $[i,j]$-crossing of quantum ($\mathfrak{sl}_n$,$\wedge V_n$) link invariant

We imitate Khovanov and Rozansky's construction to give the homology whose Euler characteristic is the quantum ($\mathfrak{sl}_n$, $\wedge V_n$) link invariant.

Step 1

Define matrix factorizations for planer diagrams appearing in the state model of the quantum ($\mathfrak{sl}_n$, $\wedge V_n$) link invariant.

Step 2

Define a complex of matrix factorizations for colored crossing in the state model.

Step 2

Check that complexes for colored link diagrams which are transformed into each other by the colored Reidemeister moves are isomorphic (in homotopy category).

It was my dream to make a generalization of Khovanov-Rozansky homology by this construction when I was a doctor course student...

2. Step 1: matrix factorization for a colored planar diagram

2.1. Notation. Notation of polynomials in this paper.
Let $t_{1,a}, ..., t_{m,a}$ be variables with the grading 2, where a is an index.
Let $e_{1,a}, ..., e_{m,a}$ be elementary symmetric polynomial of $t_{1,a}, ..., t_{m,a}$.
Put

$$s(m) = \begin{cases} 1 & m \geq 0 \\ -1 & m < 0 \end{cases}$$

We consider indexes $(a_1, ..., a_k)$ and integers $(m_1, ..., m_k)$. Let

$$X^{(m_1,...,m_k)}_{(a_1,...,a_k)} := \prod_{\alpha=1}^{k} \left(\sum_{\beta=1}^{m_\alpha} e_{\beta,a_\alpha} \right)^{s(m_\alpha)} .$$

Let $X^{(m_1,...,m_k)}_{j,(a_1,...,a_k)}$ be homogeneous terms with grading j of $X^{(m_1,...,m_k)}_{(a_1,...,a_k)}$.
We denote the sequence of homogeneous polynomials $X^{(m_1,...,m_k)}_{j,(a_1,...,a_k)}$ $(j = 1, 2, 3, ...)$ by $\mathbb{X}^{(m_1,...,m_k)}_{(a_1,...,a_k)}$. For instance,
$X^{(m)}_{1,(a)}, ..., X^{(m)}_{m,(a)}$ are elementary symmetric polynomials of $t_{1,a}, ..., t_{m,a}$.
$X^{(-m)}_{1,(a)}, ..., X^{(-m)}_{m,(a)}, ...$ are complete symmetric polynomials of $t_{1,a}, ..., t_{m,a}$.
Let $R^{(m_1,...,m_k)}_{(a_1,...,a_k)}$ be polynomial ring over $\mathbb{Q}$ generated by variables $\mathbb{X}^{(m_1)}_{(a_1)}, \mathbb{X}^{(m_2)}_{(a_2)}, ..., \mathbb{X}^{(m_k)}_{(a_k)}$.
A power sum $t^{n+1}_{1,a} + t^{n+1}_{2,a} + ... + t^{n+1}_{m,a}$, where n is associated to $\mathfrak{sl}_n$, is a symmetric polynomial, The power sum has the description by elementary symmetric polynomials, denoted by $F_m(\mathbb{X}^{(m)}_{(a)})$.

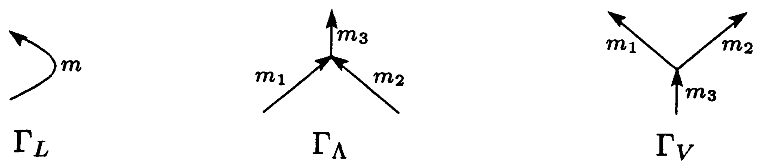
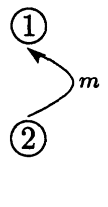


FIGURE 2. Specific intertwiners

2.2. Definition of matrix factorization for a colored planar diagram. We consider the identity of the fundamental representation $\wedge^i V_n$ of $U_q(\mathfrak{sl}_n)$ $(i = 1, ..., n)$, represented by a digram Γ_L in Figure 2, a intertwiner from $\wedge^{m_1} V_n \otimes \wedge^{m_2} V_n$ to $\wedge^{m_3} V_n$ $(1 \leq m_1 + m_2 = m_3 \leq n)$, represented by a diagram $\Gamma_\wedge$, and a intertwiner from $\wedge^{m_3} V_n$ to $\wedge^{m_1} V_n \otimes \wedge^{m_2} V_n$, represented by a diagram Γ_V.

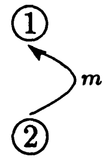
For these planar diagrams with an additional data which is an index on a boundary of a diagram, we define matrix factorizations as follows.

Definition 2.1. *We consider the following diagram with indexes.*



$(1 \leq \mathrm{m} \leq \mathrm{n})$.

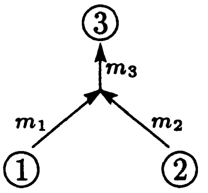
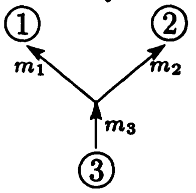
A matrix factorization for this diagram is defined by

(1) $\mathcal{C}\Big($  $\Big)_n := \boxtimes_{j=1}^{m} K\left(L^{[m]}_{j,(1;2)}; X^{(m)}_{j,(1)} - X^{(m)}_{j,(2)}\right)_{R^{(m,m)}_{(1,2)}},$

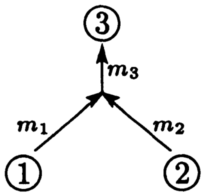
where

$$L^{[m]}_{j,(1;2)} = \frac{F_m(X^{(m)}_{1,(2)},\ldots,X^{(m)}_{j-1,(2)},X^{(m)}_{j,(1)},\ldots,X^{(m)}_{m,(1)}) - F_m(X^{(m)}_{1,(2)},\ldots,X^{(m)}_{j,(2)},X^{(m)}_{j+1,(1)},\ldots,X^{(m)}_{m,(1)})}{X^{(m)}_{j,(1)} - X^{(m)}_{j,(2)}}.$$

Definition 2.2. *We consider the following diagrams with indexes.*

  $(2 \leq m_3 = m_1 + m_2 \leq n).$

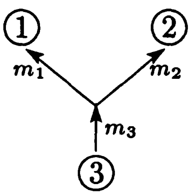
We define a matrix factorization for the left-hand side diagram to be

(2) $\mathcal{C}\Big($  $\Big)_n := \boxtimes_{j=1}^{m_3} K\left(\Lambda^{[m_1,m_2]}_{j,(3;1,2)}; X^{(m_3)}_{j,(3)} - X^{(m_1,m_2)}_{j,(1,2)}\right)_{R^{(m_1,m_2,m_3)}_{(1,2,3)}},$

where

$$\Lambda^{[m_1,m_2]}_{j,(3;1,2)} = \frac{F_{m_3}(\ldots,X^{(m_1,m_2)}_{j-1,(1,2)},X^{(m_3)}_{j,(3)},X^{(m_3)}_{j+1,(3)},\ldots) - F_{m_3}(\ldots,X^{(m_1,m_2)}_{j-1,(1,2)},X^{(m_1,m_2)}_{j,(1,2)},X^{(m_3)}_{j+1,(3)},\ldots)}{X^{(m_3)}_{j,(3)} - X^{(m_1,m_2)}_{j,(1,2)}}.$$

We define a matrix factorization for the right-hand side diagram to be

(3) $\mathcal{C}\Big($  $\Big)_n := \boxtimes_{j=1}^{m_3} K\left(V^{[m_1,m_2]}_{j,(1,2;3)}; X^{(m_1,m_2)}_{j,(1,2)} - X^{(m_3)}_{j,(3)}\right)_{R^{(m_1,m_2,m_3)}_{(1,2,3)}}\{-m_1m_2\},$

where

$$V^{[m_1,m_2]}_{j,(1,2;3)} = \frac{F_{m_3}(\ldots,X^{(m_3)}_{j-1,(3)},X^{(m_1,m_2)}_{j,(1,2)},X^{(m_1,m_2)}_{j+1,(1,2)},\ldots) - F_{m_3}(\ldots,X^{(m_3)}_{j-1,(3)},X^{(m_3)}_{j,(3)},X^{(m_1,m_2)}_{j+1,(1,2)},\ldots)}{X^{(m_1,m_2)}_{j,(1,2)} - X^{(m_3)}_{j,(3)}}.$$

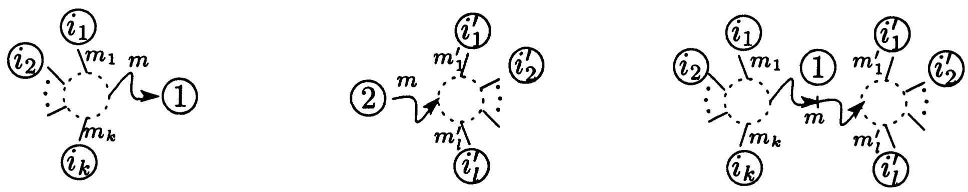
We have matrix factorizations for colored planar diagrams which appear in the state model in Figure 1,if we define gluing of matrix factorizations.

2.3. Glued diagram and matrix factorization.

Definition 2.3. *For a colored planar diagram Γ composed of the disjoint union of diagrams Γ_1 and Γ_2, we define a matrix factorization $\mathcal{C}(\Gamma)$ for Γ to be the tensor product of the factorizations for Γ_1 and Γ_2;*

$$\mathcal{C}(\ \Gamma_1\ )_n \boxtimes \mathcal{C}(\ \Gamma_2\ )_n.$$

We consider two colored planar diagrams which have an m-colored edge and is match with keeping the orientation on the edge, see the left and the middle diagrams in Figure 3. These diagrams Γ_L and Γ_R are glued at the markings ① and ② and, then, we obtain the diagram Γ_G in Figure 3.



Γ_L Γ_R Γ_G

FIGURE 3. Gluing planar diagrams

Definition 2.4. *We assume that we have a matrix factorization for Γ_L, denoted by $\mathcal{C}(\Gamma_L)_n$, and a matrix factorization for Γ_R, denoted by $\mathcal{C}(\Gamma_R)_n$. A matrix factorization $\mathcal{C}(\Gamma_G)$ for the glued diagram Γ_G is defined by*

$$\mathcal{C}(\Gamma_L)_n \boxtimes \mathcal{C}(\Gamma_R)_n\Big|_{\mathbb{X}^{(m)}_{(2)}=\mathbb{X}^{(m)}_{(1)}}.$$

Γ_T Γ_C

FIGURE 4. Diagram Γ_T and glued diagram Γ_C

We consider a colored diagram Γ_T and a diagram Γ_C obtained by joining ends of edges with the same coloring, see Figure 4.

Definition 2.5. *We assume that we have a matrix factorization $\mathcal{C}(\Gamma_T)_n$. A matrix factorization $\mathcal{C}(\Gamma_C)$ for the diagram Γ_C is defined by*

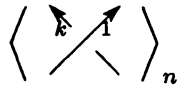
$$\mathcal{C}(\Gamma_T)_n|_{\mathbb{X}^{(m)}_{(2)}=\mathbb{X}^{(m)}_{(1)}}.$$

By this gluing of matrix factorizations, we have matrix factorizations for colored planar diagrams which appear in the state model in Figure 1.

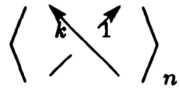
3. Part of Step 2: complex for $[k,1]$-crossing and $[1,k]$-crossing

3.1. Complex for $[k,1]$-crossing and $[1,k]$-crossing. We sketch a construction of complexes of factorizations for $[k,1]$-crossing and $[1,k]$-crossing. See Section 5 in [25] for the construction in detail.

In the case of a $[k,1]$-crossing, the state model of the quantum $(\mathfrak{sl}_n, \wedge V_n)$ link invariant takes the following forms



$$\left\langle \quad \right\rangle_n = (-1)^{1-k} q^{kn-1} \left\langle \Gamma_1^{[k,1]} \right\rangle_n + (-1)^{-k} q^{kn} \left\langle \Gamma_2^{[k,1]} \right\rangle_n,$$



$$\left\langle \quad \right\rangle_n = (-1)^{k-1} q^{-kn+1} \left\langle \Gamma_1^{[k,1]} \right\rangle_n + (-1)^{k} q^{-kn} \left\langle \Gamma_2^{[k,1]} \right\rangle_n,$$

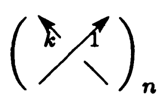
where $\Gamma_1^{[k,1]}$ and $\Gamma_2^{[k,1]}$ are colored planar diagrams in Figure 5. In the case of a $[1,k]$-crossing, replace $\Gamma_1^{[k,1]}$ and $\Gamma_2^{[k,1]}$ with these symmetry diagrams.

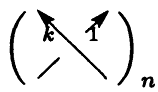
FIGURE 5. Colored planar diagrams in reduction of $[k,1]$-crossing

By Step 1, we have matrix factorizations $\mathcal{C}(\Gamma_1^{[k,1]})$ and $\mathcal{C}(\Gamma_2^{[k,1]})$ for $\Gamma_1^{[k,1]}$ and $\Gamma_2^{[k,1]}$. We concretely construct $\mathbb{Z}$-grading-preserving morphisms between the factorizations $\mathcal{C}(\Gamma_1^{[k,1]})_n$ and $\mathcal{C}(\Gamma_2^{[k,1]})_n$,

$$\chi_+^{[k,1]} : \mathcal{C}\left(\Gamma_2^{[k,1]}\right)_n \longrightarrow \mathcal{C}\left(\Gamma_1^{[k,1]}\right)_n,$$
$$\chi_-^{[k,1]} : \mathcal{C}\left(\Gamma_1^{[k,1]}\right)_n \longrightarrow \mathcal{C}\left(\Gamma_2^{[k,1]}\right)_n.$$

Using these morphisms, a complex for a single $[k,1]$-crossing is defined as follows.

(4) 
$$\mathcal{C}\left(\quad\right)_n = \left(0 \longrightarrow \mathcal{C}^{-k}\left(\Gamma_2^{[k,1]}\right)_n \xrightarrow{\chi_+^{[k,1]}} \mathcal{C}^{1-k}\left(\Gamma_1^{[k,1]}\right)_n \longrightarrow 0\right),$$

(5) 
$$\mathcal{C}\left(\quad\right)_n = \left(0 \longrightarrow \mathcal{C}^{k-1}\left(\Gamma_1^{[k,1]}\right)_n \xrightarrow{\chi_-^{[k,1]}} \mathcal{C}^{k}\left(\Gamma_2^{[k,1]}\right)_n \longrightarrow 0\right).$$

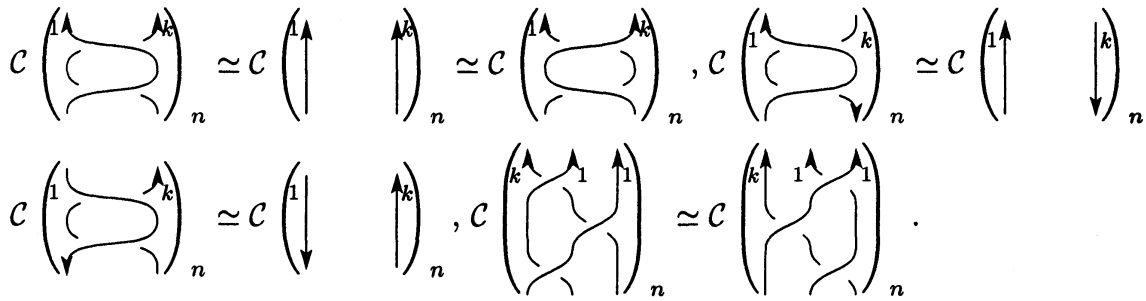
A complex for a $[1,k]$-crossing is defined by a similar way.

Remark 3.1. *This construction is a generalization of a complex for a* $[1,2]$*-crossing given by Rozansky* [18].

3.2. Step 3 in the case of $[k,1]$-crossing and $[1,k]$-crossing. To an oriented link diagram D with $[k,1]$-crossings and $[1,k]$-crossings only, we define a complex of matrix factorizations by decomposing D into single $[k,1]$-crossings and $[1,k]$-crossings and, then, taking the tensor product of complexes for all $[k,1]$-crossings and $[1,k]$-crossings in the decomposition.

First theorem in my thesis is that this complex has invariance under the Reidemeister moves composed of $[k,1]$-crossings and $[1,k]$-crossings up to homotopy equivalence.

Theorem 3.2 (Theorem 5.6[25] (In the case $k=1$, Khovanov-Rozansky[9])). *We consider tangle diagrams with $[k,1]$-crossings and $[1,k]$-crossings which are transformed into each other by colored Reidemeister moves composed of $[k,1]$-crossings and $[1,k]$-crossings. Complexes of factorizations for these tangle diagrams are isomorphic in $\mathcal{K}^b(\mathrm{HMF}^{gr})$:*



Remark 3.3. *For an oriented link diagram D with $[k,1]$-crossings and $[1,k]$-crossings, we explicitly describe the complex $C(D)_n$ and, then, calculate the $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$-graded link homology $\mathrm{H}^{i,j,k}(D)$ since the boundary maps $\chi_+^{[k,1]}$ and $\chi_+^{[k,1]}$ are concretely described.*

4. The rest of step 2: Complex for $[i,j]$-crossing

4.1. **Complex for approximate $[i,j]$-crossing.** We consider the case of general $[i,j]$-crossings. In the case of $[i,j]$-crossings, it is difficult both to define concrete boundary maps of a complex of matrix factorizations for the $[i,j]$-crossing and to show that there are isomorphisms between complexes for the colored tangle diagrams that are transformed into each other by colored Reidemeister moves in $\mathcal{K}^b(\mathrm{HMF}^{gr})$ if we define the complex[1].

Instead of constructing Step 2 in the case of $[i,j]$-crossing, we introduce an approximate $[i,j]$-crossing and define a complex for the approximate $[i,j]$-crossing in Figure 6.

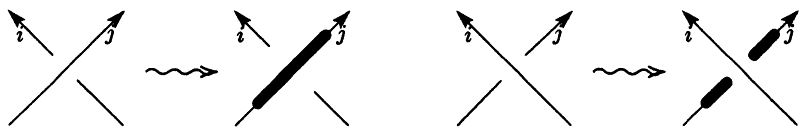


FIGURE 6. Approximate diagram of $[i,j]$-crossing

The wide edge of the approximate $[i,j]$-crossing represents a bundle of one-colored edges in Figure 7. We arrange an $[i,j]$-crossing in the orientation from bottom to up and change a colored edge from the left-bottom to the right-top into a wide edge at an over crossing or an under crossing (see Figure 6). Therefore, we obtain a complex for the approximate crossing using the definition of the complex for an $[i,1]$-crossing since every crossing of the approximate crossing is an $[i,1]$-crossing.

Second theorem in this paper is that we have the following isomorphisms in $\mathcal{K}^b(\mathrm{HMF}^{gr})$.

[1]B. Webster-G. Williamson and H. Wu claim that such a homology exists [21][23].

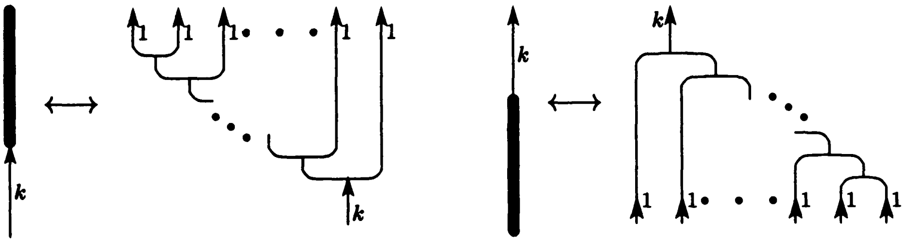
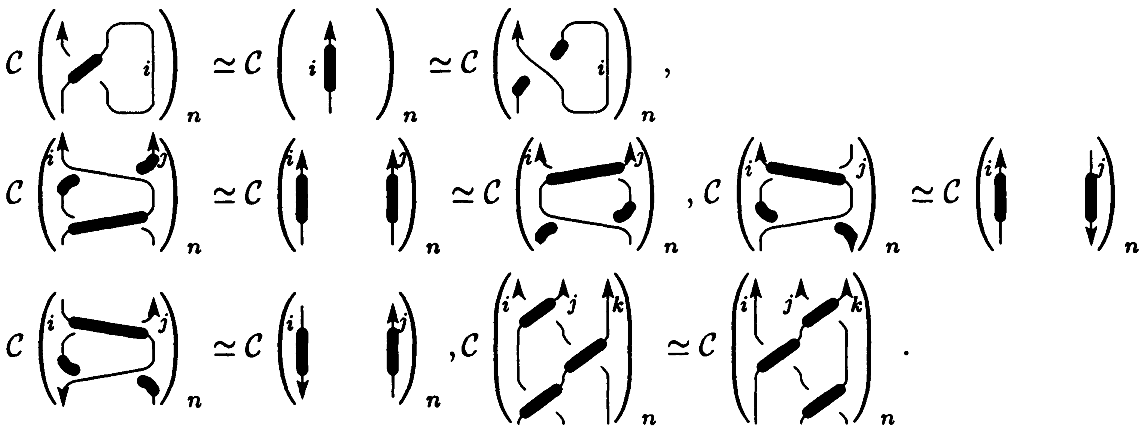
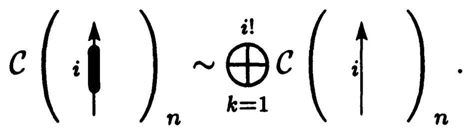


FIGURE 7. Wide edge and bundle of one-colored edges

Theorem 4.1 (Theorem 6.6[25]). *For approximate tangle diagrams which are transformed into each other by the Reidemeister moves composed of the approximate crossings, complexes of matrix factorizations for these approximate tangle diagrams are isomorphic in* $\mathcal{K}^b(\mathrm{HMF}^{gr})$*:*



We find that a matrix factorization for a wide edge consists of $i!$ copies of a matrix factorization for an original i-colored edge up to $\mathbb{Z}$-grading shift,



We have not got defined a complex for an $[i,j]$-crossing to give a homology whose Euler characteristic is the quantum ($\mathfrak{sl}_n$,$\wedge V_n$) link invariant, but we hope to return to this question in a future paper.

However, the information of Theorem 4.1 gives us a new link invariant for a colored oriented link diagram D. We consider a $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$-graded homology $H^{i,j,k}(D)$ through the complex for the approximate diagram of D. Then, we take the Poincaré polynomial of the homology $H^{i,j,k}(D)$, denoted by $\overline{P}(D)$,

$$\sum_{i,j,k} t^i q^j s^k \dim_{\mathbb{Q}} H^{i,j,k}(D) \in \mathbb{Q}[t^{\pm 1}, q^{\pm 1}, s]/\langle s^2-1\rangle.$$

A link invariant is obtained by normalizing the Poincaré polynomial $\overline{P}(D)$ as follows. For a colored oriented link diagram D, a function $\mathrm{Cr}_k(D)$ $(k=1,\ldots,n-1)$ is defined by

$$\mathrm{Cr}_k(D) := \text{the number of } [*,k]\text{-crossing of } D.$$

We define a rational function $P(D)$ to be

$$\overline{P}(D)\prod_{k=1}^{n-1}\frac{1}{([k]_q!)^{\mathrm{Cr}_k(D)}}.$$

By the construction and Theorem 4.1, we have third theorem in this paper.

Theorem 4.2 (Corollary 6.8[25]). *Two colored oriented link diagrams D and D' which are transformed into each other by colored Reidemeister moves have the same evaluation by P,*

$$P(D) = P(D').$$

$P(D)$ is a refined link invariant of the quantum $(\mathfrak{sl}_n, \wedge V_n)$ link invariant. We find that $P(D)$ specializing t to -1 and s to 1 is the quantum $(\mathfrak{sl}_n, \wedge V_n)$ link invariant by construction.

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