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On the universal sl_2 invariant of bottom tangles

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Abstract

A bottom tangle is a tangle in a cube consisting of arc components whose boundary points are placed on the bottom, and every link can be represented as the closure of a bottom tangle. The universal sl_2 invariant of n-component bottom tangles takes values in the n-fold completed tensor power of the quantized enveloping algebra $U_h(sl_2)$, and has a universality property over the colored Jones polynomials of n-component links via quantum traces in finite dimensional representations. In this note, we study the values of the universal sl_2 invariant of certain three types of bottom tangles which are called boundary, ribbon, and brunnian bottom tangles. For each types of bottom tangles, we give certain small subalgebras in which the universal sl_2 invariant of bottom tangles of the type takes values. As applications, it follows that each boundary, ribbon, and brunnian link has stronger divisibility by cyclotomic polynomials than algebraically split links for Habiro's reduced version of the colored Jones polynomials.

1 Introduction

First of all, we recall *tangles* and *bottom tangles.* Then we define the three types of bottom tangles, *boundary, ribbon*, and *brunnian* bottom tangles. After that, we will mention the background of my research.

1.1 Tangles and bottom tangles

A *tangle* is the image of an embedding

$$\coprod^{m}[0,1]\coprod^{n}S^1 \hookrightarrow S^3$$

for $m,n \geq 0$, whose boundary is on the two lines $[0,1]\times\{\frac{1}{2}\}\times\{0,1\}$ on the bottom and on the top of the cube. We equip the image of an embedding both orientation and framing. In this note, the image of $[0,1]$ (resp. S^1) is called an *arc* (resp. *cycle*) component, see Figure 1 for example, and a point in boundary of arc components is called *endpoint.*

A *bottom tangle* is a tangle satisfying

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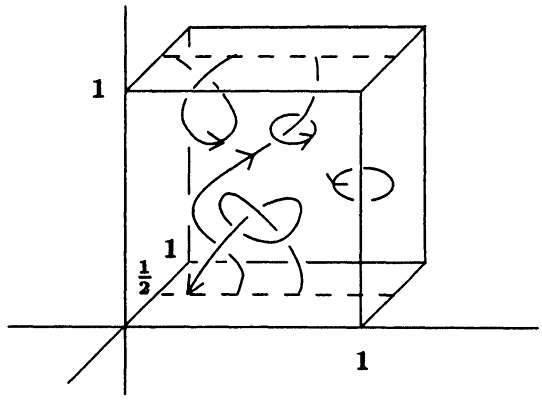
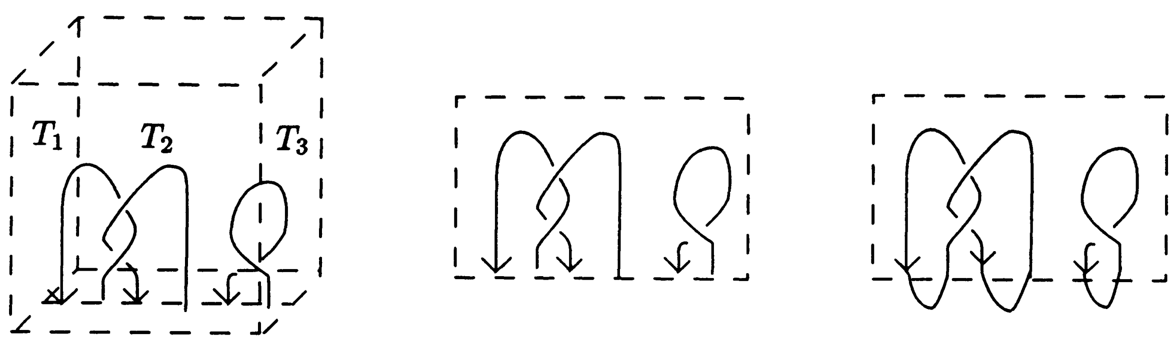


Figure 1: A tangle consisting of 3-arc components and 2-cycle components.



(a) (b) (c)

Figure 2: (a) A 3-component bottom tangle $T = T_1 \cup T_2 \cup T_3$. (b) A diagram of T in a rectangle. (c) A closure of T.

(1) there are no cycle components,

(2) every endpoint is on the line $[0, 1] \times \{\frac{1}{2}\} \times \{0\}$ on the bottom,

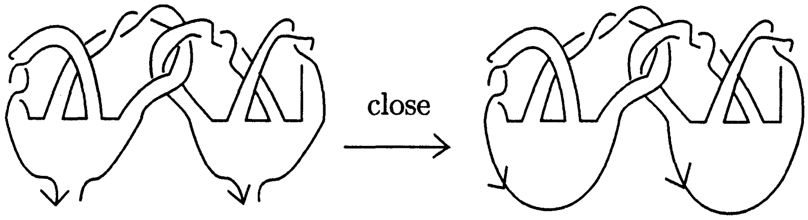
(3) two endpoints of each component are adjacent to each other, and

(4) each component runs from its right endpoint to its left endpoint.

For example, see Figure 2 (a). We draw a diagram of a bottom tangle in a rectangle, see Figure 2 (b). For each $n \geq 0$, let BT_n denote the set of the ambient isotopy classes, relative to endpoints, of n-component bottom tangles. The *closure link* $\mathrm{cl}(T)$ of T is defined as the unique isotopy class of links obtained from T by closing, see Figure 2 (c). For every n-component link L, there is an n-component bottom tangle whose closure is isotopic to L. For a bottom tangle, we can define the linking matrix as that of the closure link.

1.2 Boundary, ribbon, and brunnian bottom tangles

A *Seifert surface* of a knot K is a compact connected orientable surface F in S^3 bounded by K. An n-component link $L = L_1 \cup \cdots \cup L_n$ is called a *boundary link* if it bounds a disjoint union of n Seifert surfaces $F_1, \ldots, F_n$ in S^3 such that L_i bounds F_i for $i = 1, \ldots, n$. For a 1-component bottom tangle $T \in BT_1$, there is a knot $L_T = (T \cup \gamma) \subset [0,1]^3$ where γ is the line segment on the bottom $[0,1]^2 \times \{0\}$ such that $\partial\gamma = \partial T$. A *Seifert surface* of a 1-component bottom tangle T is a Seifert surface of the knot L_T in $[0,1]^3$. A bottom tangle $T = T_1 \cup \cdots \cup T_n$ is called a *boundary bottom tangle* if its components have disjoint Seifert surfaces $F_1, \ldots, F_n$ in $[0,1]^3$ such that L_{T_i} bounds F_i for $i = 1, \ldots, n$. For every boundary link L, there is a boundary bottom tangle whose closure is L.

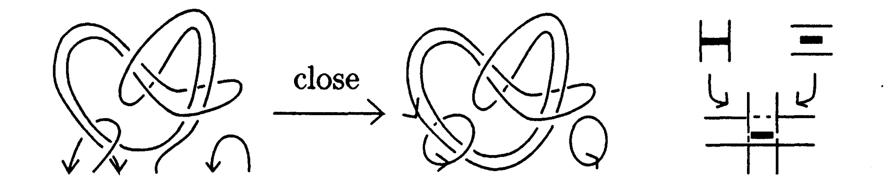


boundary bottom tangle boundary link

An n-component link L is called a *ribbon link* (cf. [1]) if it bounds the image of an immersion

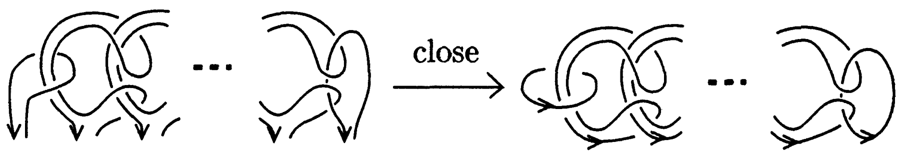
$$D \cup \cdots \cup D \to S^3$$

from a disjoint union of two dimensional disks into S^3 with only ribbon singularities. Here a ribbon singularity is a singularity whose preimage consists of two lines one of which is in the interior of the disks. A *ribbon bottom tangle* is defined as a bottom tangle whose closure is a ribbon link.



ribbon bottom tangle ribbon link ribbon singularity

A link is called *brunnian link* if its every proper sublink is trivial. Similarly, a bottom tangle is called *brunnian bottom tangle* if every proper subtangle is trivial, where a bottom tangle is said to be *trivial* if it has the trivial diagram that is copies of $\cap$. For each brunnian link L, there is a brunnian bottom tangle whose closure is L.



brunnian borrom tangle brunnian link

1.3 Back ground

In the 80's, Jones constructed a polynomial invariant of links by using von Neumann algebras. Shortly after, Reshetikhin and Turaev [8] defined invariants of framed links colored by finite dimensional representations of a ribbon Hopf algebra, which we call *colored link invariants.* The quantized enveloping algebra associated to a simple Lie algebra has a complete ribbon Hopf algebra structure, and Jones polynomial can be defined as the colored link invariant associated to the universal enveloping algebra $U_h := U_h(sl_2)$ and its 2-dimensional irreducible representation attached to all components of links. By a *colored Jones polynomial,* we mean a colored link invariant associated to U_h.

For a ribbon Hopf algebra, Lawrence [5, 4] and Ohtsuki [7] defined an invariant of framed tangle, which is called the *universal invariant.* By the *universal sl_2 invariant,* we mean the universal invariant associated to U_h. In [2], Habiro studied the universal invariant of bottom tangles associated to an arbitrary ribbon Hopf algebra, and in [3], he studied the universal sl_2 invariant of bottom tangles in detail. The universal sl_2 invariant of an n-component bottom tangle takes values in the n-fold completed tensor power $U_h^{\hat{\otimes} n}$ of U_h. The universal invariant of bottom tangles has a universality property such that the colored link invariants of a link L is obtained from the universal invariant of a bottom tangle T whose closure is isotopic to L, by taking the quantum trace in the representations attached to the components of the link L. In particular, one can obtain colored Jones polynomials of links from the universal sl_2 invariant of bottom tangles.

In this note, we study algebraic properties of the universal sl_2 invariant of boundary, ribbon, and of brunnian bottom tangles.

2 The quantized enveloping algebra U_h and its subalgebras

In this note, we use the following q-integer notations:

$$\{i\}_q = q^i - 1, \quad \{i\}_{q,n} = \{i\}_q\{i-1\}_q \cdots \{i-n+1\}_q, \quad \{n\}_q! = \{n\}_{q,n},$$
$$[i]_q = \{i\}_q/\{1\}_q, \quad [n]_q! = [n]_q[n-1]_q \cdots [1]_q, \quad \begin{bmatrix} i \\ n \end{bmatrix}_q = \{i\}_{q,n}/\{n\}_q!,$$

for $i \in \mathbb{Z}, n \geq 0$.

We denote by U_h the h-adically complete $\mathbb{Q}[[h]]$-algebra, topologically generated by the elements H, E, and F, satisfying the relations

$$HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}},$$

where we set

$$q = \exp h, \quad K = q^{H/2} = \exp \frac{hH}{2}.$$

We equip U_h with a topological $\mathbb{Z}$-graded algebra structure with $\deg E = 1$, $\deg F = -1$, and $\deg H = 0$. For a homogeneous element x of U_h, the degree of x is denoted by $|x|$.

There is a unique complete ribbon Hopf algebra structure on U_h as follows. The comultiplication $\Delta\colon U_h \to U_h \hat{\otimes} U_h$, the counit $\varepsilon\colon U_h \to \mathbb{Q}[[h]]$, and the antipode $S\colon U_h \to U_h$ are given by

$$\begin{aligned}
\Delta(H) &= H \otimes 1 + 1 \otimes H, \quad \varepsilon(H) = 0, \quad S(H) = -H,\\
\Delta(E) &= E \otimes 1 + K \otimes E, \quad \varepsilon(E) = 0, \quad S(E) = -K^{-1}E,\\
\Delta(F) &= F \otimes K^{-1} + 1 \otimes F, \quad \varepsilon(F) = 0, \quad S(F) = -FK.
\end{aligned}$$

The universal R-matrix $R \in U_h \hat{\otimes} U_h$ and its inverse are given by

$$R = D \sum_{n \geq 0} q^{\frac{1}{2}n(n-1)} \tilde{F}^{(n)} K^{-n} \otimes e^n, \tag{1}$$

$$R^{-1} = D^{-1} \sum_{n \geq 0} (-1)^n \tilde{F}^{(n)} \otimes K^{-n} e^n, \tag{2}$$

where we set

$$\begin{aligned}
D &= v^{\frac{1}{2}H \otimes H} = \exp\left(\frac{h}{4} H \otimes H\right) \in U_h^{\hat{\otimes} 2},\\
e &= (q^{1/2} - q^{-1/2})E, \quad \tilde{F}^{(n)} = F^n K^n / [n]_q!,
\end{aligned}$$

for $n \geq 0$.

The ribbon element $r \in U_h$ and its inverse are given by

$$r = \sum \bar{R}' K^{-1} \bar{R}'' = \sum \bar{R}'' K \bar{R}', \quad r^{-1} = \sum R' K R'' = \sum R'' K^{-1} R',$$

where we set $R = \sum R' \otimes R''$, and $R^{-1} = (S \otimes 1)R = \sum \bar{R}' \otimes \bar{R}''$.

2.1 Subalgebras of U_h and their completions

Let $U_{\mathbb{Z},q}$ denote the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_{\mathbb{Z}}$ generated by $K, K^{-1}, \tilde{E}^{(n)} = (v^{-1}E)^n/[n]_q!$, and $\tilde{F}^{(n)}$ for $n \geq 1$, and $U_{\mathbb{Z},q}^{\mathrm{ev}}$ the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_{\mathbb{Z},q}$ generated by the elements $K^2, K^{-2}, \tilde{E}^{(n)}$ and $\tilde{F}^{(n)}$ for $n \geq 1$.

Remark 2.1. Let $U_{\mathbb{Z}}$ denote Lusztig's integral form of U_h (cf. [6]), which is defined to be the $\mathbb{Z}[v, v^{-1}]$-subalgebra of U_h generated by K, K^{-1}, $E^{(n)} = E^n/[n]!$, and $F^{(n)} = F^n/[n]!$ for $n \geq 1$, where $[i] = \frac{q^{i/2} - q^{-i/2}}{q^{1/2} - q^{-1/2}}$ for $i \in \mathbb{Z}$ and $[n]! = [n] \cdots [1]$ for $n \geq 0$. We have

$$U_{\mathbb{Z}} = U_{\mathbb{Z},q} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}[v, v^{-1}].$$

Let $\bar{U}_q$ denote the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_{\mathbb{Z},q}$ generated by the elements K, K^{-1}, e and $f = (q-1)FK$, and $\bar{U}_q^{\mathrm{ev}}$ the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $\bar{U}_q$ generated by the elements K^2, K^{-2}, e and f.

Let $\mathcal{U}_q^{\mathrm{ev}}$ denote the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_{\mathbb{Z},q}^{\mathrm{ev}}$ generated by the elements K^2, K^{-2}, e and $\tilde{F}^{(n)}$ for $n \geq 1$.

We recall from [3] a filtration and a completion of $\mathcal{U}_q^{\mathrm{ev}}$. For $p \geq 0$, let $\mathcal{F}_p(\mathcal{U}_q^{\mathrm{ev}})$ be the two-sided ideal in $\mathcal{U}_q^{\mathrm{ev}}$ generated by e^p. We define $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$ as the completion in U_h of $\mathcal{U}_q^{\mathrm{ev}}$ with respect to the decreasing filtration $\{\mathcal{F}_p(\mathcal{U}_q^{\mathrm{ev}})\}_{p\geq 0}$, i.e., $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$ is the image of the homomorphism

$$\varprojlim_{p\geq 0} \left(\mathcal{U}_q^{\mathrm{ev}}/\mathcal{F}_p(\mathcal{U}_q^{\mathrm{ev}})\right) \to U_h$$

induced by $\mathcal{U}_q^{\mathrm{ev}} \subset U_h$. Then $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$ is a $\mathbb{Z}[q,q^{-1}]$-subalgebra of U_h.

For $n \geq 1$, let $(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$ be the completion of the n-fold tensor product $(\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}$ of $\mathcal{U}_q^{\mathrm{ev}}$ with respect to the decreasing filtration $\{\mathcal{F}_p\big((\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}\big)\}_{p\geq 0}$ such that

$$\mathcal{F}_p\big((\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}\big) = \sum_{i=1}^{n} (\mathcal{U}_q^{\mathrm{ev}})^{\otimes (i-1)} \otimes \mathcal{F}_p(\mathcal{U}_q^{\mathrm{ev}}) \otimes (\mathcal{U}_q^{\mathrm{ev}})^{\otimes (n-i)}.$$

It is natural to set

$$\mathcal{F}_p\big((\mathcal{U}_q^{\mathrm{ev}})^{\otimes 0}\big) = \mathcal{F}_p(\mathbb{Z}[q,q^{-1}]) = \begin{cases} \mathbb{Z}[q,q^{-1}] & \text{if } p=0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have

$$(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} 0} = \mathbb{Z}[q,q^{-1}].$$

For a $\mathbb{Z}[q,q^{-1}]$-subalgebra A of $(\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}$, we define the *closure* $(A)\tilde{}$ of A in $(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$ as the completion of A with respect to the decreasing filtration $\{\mathcal{F}_p\big((\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}\big) \cap A\}_{p\geq 0}$. Especially, we denote by $(\bar{U}_q^{\mathrm{ev}})^{\tilde{}\tilde{\otimes} n}$ the closure of $(\bar{U}_q^{\mathrm{ev}})^{\otimes n}$ in $(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$.

3 The universal sl_2 invariant of bottom tangles

In this section, we define the universal sl_2 invariant of bottom tangles (cf. [2]).

3.1 The universal sl_2 invariant of bottom tangles

In what follows, we write the R-matrix and its inverse as $R^{\pm 1} = \sum_{i\geq 0} R_i^{\pm}$, where we set

$$R_i = D(\alpha_i^+ \otimes \beta_i^+), \quad R_i^- = D^{-1}(\alpha_i^- \otimes \beta_i^-), \tag{3}$$

$$\alpha_i^+ \otimes \beta_i^+ = q^{\frac{1}{2}i(i-1)} \tilde{F}^{(i)} K^{-i} \otimes e^i, \quad \alpha_i^- \otimes \beta_i^- = (-1)^i \tilde{F}^{(i)} \otimes K^{-i} e^i. \tag{4}$$

(We cannot define $\alpha_i^+, \beta_i^+, \alpha_i^-$, or β_i^-, independently.)

Remark 3.1. In [9], we used different notations $R_i^+ = q^{\frac{1}{2}i(i-1)} \tilde{F}^{(i)} K^{-i} \otimes e^i$ and $R_i^- = (-1)^i \tilde{F}^{(i)} \otimes K^{-i} e^i$.

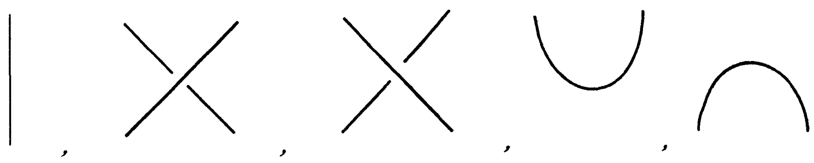


Figure 3: Fundamental tangles. The orientations of the strands are arbitrary.

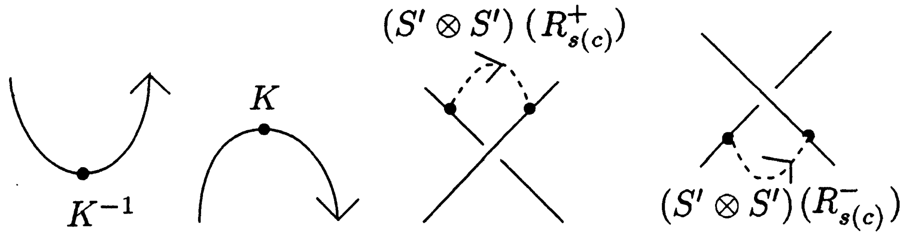


Figure 4: How to attach elements on the fundamental tangles.

We use diagrams of tangles obtained from copies of the fundamental tangles, as depicted in Figure 3, by pasting horizontally and vertically. For a bottom tangle $T = T_1 \cup \cdots \cup T_n$, we define the universal sl_2 invariant $J_T \in U_h^{\hat{\otimes} n}$ of T as follows. We choose a diagram P of T. We denote by $C(P)$ the set of the crossings of the diagram. We call a map

$$s\colon C(P) \quad \to \quad \{0, 1, 2, \ldots\}$$

a *state.* We denote by $\mathcal{S}(P)$ the set of states of the diagram P.

For each fundamental tangle in the diagram, we attach elements of U_h or of $U_h^{\otimes 2}$ associated to a state $s \in \mathcal{S}(P)$ following the rule described in Figure 4, where "S'" should be replaced with id if the string is oriented downward, and with S otherwise, see Figure 5. We define an element $J_{P,s} \in U_h^{\hat{\otimes} n}$ as follows. The ith component of $J_{P,s}$ is defined to be the product of the elements put on the component corresponding to T_i, where the elements are read off along each component reversing the orientation of P, and written from left to right. Here we read an element $y = \sum y_{[1]} \otimes y_{[2]} \in U_h^{\hat{\otimes} 2}$ on arrowed dashed line by assuming that the first tensorand is attached to the startpoint of the arrow and the second tensorand to the endpoint of the arrow, see Figure 6. (The result does not depend on how one expresses the element on each dashed line as a sum of tensors.)

Set

$$J_T = \sum_{s \in \mathcal{S}(P)} J_{P,s}.$$

As is well known [7], J_T does not depend on the choice of the diagram, and defines an isotopy invariant of bottom tangles.

For example, let us compute the universal sl_2 invariant J_C of a bottom tangle C with a diagram P as depicted in Figure 7 (a), where c_1 (resp. c_2) denotes the upper (resp. lower) crossing of P. The diagram attached the elements for a state $s \in \mathcal{S}(P)$ is depicted in Figure 7 (b), where we set $m = s(c_1), n = s(c_2)$. We have

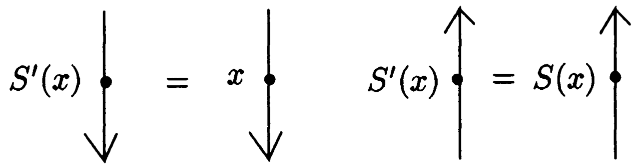


Figure 5: The definition of S'.

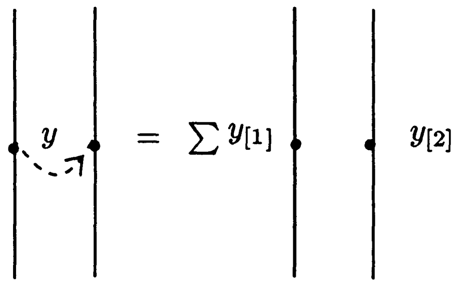


Figure 6: How we read an element $y = \sum y_{[1]} \otimes y_{[2]} \in U_h^{\hat{\otimes} 2}$.

$$
\begin{aligned}
J_C &= \sum_{s \in \mathcal{S}(P)} J_{P,s} \\
&= \sum_{m,n \geq 0} \sum S(D_1' \alpha_m^+) S(D_2' \beta_n^+) \otimes D_2'' \alpha_n^+ D'' \beta_m^+ \\
&= \sum_{m,n \geq 0} (-1)^{m+n} q^{-n+2mn} D^{-2} (\tilde{F}^{(m)} K^{-2n} e^n \otimes \tilde{F}^{(n)} K^{-2m} e^m).
\end{aligned}
$$

where we set $D = \sum D_1' \otimes D_1'' = \sum D_2' \otimes D_2''$.

The following propositions is fundamental.

Proposition 3.2 ([9]). *Let T be an n-component bottom tangle with 0-framing, and P a diagram of T. We have*

$$J_{P,s} \in (\mathcal{U}_q^{ev})^{\otimes n}.$$

Later, we use the following lemma.

Lemma 3.3. *Let T be an n-component bottom tangle with 0-framing, and P a diagram of T. Set $|s| = \max\{s(c) \mid c \in C(P)\}$. We have*

$$J_{P,s} \in F_{|s|}((\mathcal{U}_q^{ev})^{\otimes n}). \tag{5}$$

3.2 Colored Jones polynomials

If V is a finite dimensional representation of U_h, then the quantum trace $\mathrm{tr}_q^V(x)$ in V of an element $x \in U_h$ is defined by

$$\mathrm{tr}_q^V(x) = \mathrm{tr}^V(\rho_V(K^{-1}x)) \in \mathbb{Q}[[h]],$$

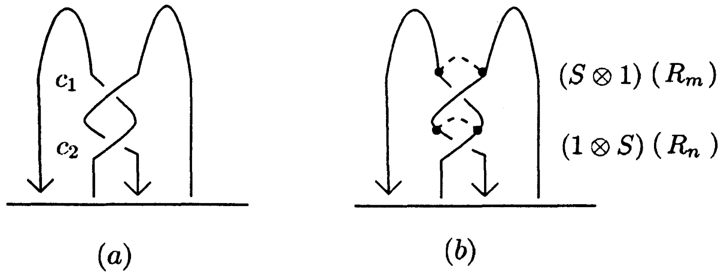


Figure 7: (a) A diagram P of $C \in BT_2$. (b) The diagram P attached elements.

where $\rho_V \colon U_h \to \mathrm{End}(V)$ denotes the left action of U_h on V, and $\mathrm{tr}^V \colon \mathrm{End}(V) \to \mathbb{Q}[[h]]$ denotes the trace in V. For every element $y = \sum_n a_n V_n \in \mathcal{R}$, $a_n \in \mathbb{Q}(v)$, we set

$$\mathrm{tr}_q^y(x) = \sum_n a_n\, \mathrm{tr}_q^{V_n}(x) \in \mathbb{Q}((h))$$

for $x \in U_h$. Here $\mathbb{Q}((v))$ denote the quotient field of $\mathbb{Q}[[h]]$.

The universal sl_2 invariant of bottom tangles has a universality property to the colored Jones polynomials of links as the following.

Proposition 3.4 (Habiro [3]). *Let $L = L_1 \cup \cdots \cup L_n$ be an n-component, ordered, oriented, framed link in S^3. Choose an n-component bottom tangle T whose closure is isotopic to L. For $y_1, \ldots, y_n \in \mathcal{R}$, the colored Jones polynomial $J_{L;y_1,\ldots,y_n}$ of L can be obtained from J_T by*

$$J_{L;y_1,\ldots,y_n} = (\mathrm{tr}_q^{y_1} \otimes \cdots \otimes \mathrm{tr}_q^{y_n})(J_T).$$

4 Main results

In this section, we give the main results. The results for boundary bottom tangles, which was conjectured by Habiro [3], and for ribbon bottom tangles are similar to each other as follows.

Theorem 4.1. *Let T be an n-component boundary bottom tangle with 0-framing. Then we have $J_T \in (\bar{U}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$.*

Theorem 4.2 ([9]). *Let T be an n-component ribbon bottom tangle with 0-framing. Then we have $J_T \in (\bar{U}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$.*

In fact, We have a refinement of each Theorem 4.1 and 4.2 with a smaller subalgebra $(\bar{U}_q^{\mathrm{ev}})^{\hat{\otimes} n} \subset (\bar{U}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$ in place of $(\bar{U}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$, see Section 8.2 for the definition of$(\bar{U}_q^{\mathrm{ev}})^{\hat{\otimes} n}$. Here, we do not know whether the inclusion is proper or not, but the definition of $(\bar{U}_q^{\mathrm{ev}})^{\hat{\otimes} n}$ is more natural than that of $(\bar{U}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$ in our setting.

The result for brunnian bottom tangles is as follows.

Theorem 4.3. *Let T be an n-component brunnian bottom tangle. Then we have*

$$J_T \in \bigcap_{i=0}^{n} \{(\bar{U}_q^{\mathrm{ev}})^{\otimes i-1} \otimes U_{\mathbb{Z},q}^{\mathrm{ev}} \otimes (\bar{U}_q^{\mathrm{ev}})^{\otimes n-i}\}^{\hat{}}.$$

For a bottom tangle $T = T_1 \cup \cdots \cup T_n$, let denote by $\check{T}_{i_1,\dots i_m}$ the subtangle obtained from T by removing its components $T_{i_1}, \dots, T_{i_m}$. In fact, Theorem 4.3 is a corollary of the following result.

Theorem 4.4. *Let T be an n-component bottom tangle with 0-framing whose subtangle $\check{T}_{i_1,\dots i_m}$ is trivial. Then we have*

$$J_T \in (A_1 \otimes A_2 \otimes \cdots \otimes A_n)^{\hat{}}.$$

where

$$A_i = \begin{cases} U_{\mathbb{Z},q}^{\mathrm{ev}} & i = i_1, \cdots, i_m \\ \bar{U}_q^{\mathrm{ev}} & \textit{other.} \end{cases}$$

5 Applications

Here, we give an application of each Theorem 4.1, 4.2, and 4.4. For $m \geq 1$, let V_m denote the m-dimensional irreducible representation of U_h. Let $\mathcal{R}$ denote the representation ring of U_h over $\mathbb{Q}(q^{\frac{1}{2}})$, i.e., $\mathcal{R}$ is the $\mathbb{Q}(q^{\frac{1}{2}})$-algebra

$$\mathcal{R} = \mathrm{Span}_{\mathbb{Q}(q^{\frac{1}{2}})}\{V_m \mid m \geq 1\}$$

with the multiplication induced by the tensor product. It is well known that $\mathcal{R} = \mathbb{Q}(q^{\frac{1}{2}})[V_2]$.

Habiro [3] studied the following elements in $\mathcal{R}$

$$\tilde{P}_l' = \frac{q^{\frac{1}{2}l}}{\{l\}_q!} \prod_{i=0}^{l-1} (V_2 - q^{i+\frac{1}{2}} - q^{-i-\frac{1}{2}})$$

for $l \geq 0$, which are used in an important technical step in his construction of the unified Witten-Reshetikhin-Turaev invariants for integral homology spheres. He proved the following.

Theorem 5.1 (Habiro [3]). *Let L be an n-component, algebraically-split link with 0-framing. We have*

$$J_{L;\tilde{P}_{l_1}',\dots,\tilde{P}_{l_n}'} \in \frac{\{2l_j+1\}_{q,l_j+1}}{\{1\}_q}\mathbb{Z}[q, q^{-1}],$$

for $l_1, \dots, l_n \geq 0$, where j is an integer such that $l_j = \max\{l_i\}_{1 \leq i \leq n}$.

Habiro [3] proved that Theorem 4.1 implies the following result.

Theorem 5.2. *Let L be an n-component boundary link with 0-framing. We have*

$$J_{L;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}} \in \frac{\{2l_j+1\}_{q,l_j+1}}{\{1\}_q} \prod_{1\le i\le n, i\neq j} I_{l_i}$$

for $l_1,\dots,l_n \ge 0$, where j is an integer such that $l_j = \max\{l_i\}_{1\le i\le n}$. Here, for $l\ge 0$, I_l is the ideal in $\mathbb{Z}[q,q^{-1}]$ generated by the elements $\{l-k\}_q!\{k\}_q!$ for $k=0,\dots,l$.

Remark 5.3. For $m\ge 1$, let $\Phi_m(q)\in\mathbb{Z}[q]$ denote the mth cyclotomic polynomial. It is not difficult to prove that $I_l, l\ge 0$, is contained in the principle ideal generated by $\prod_m \Phi_m(q)^{f(l,m)}$, where $f(l,m)=\max\{0,\lfloor\frac{l+1}{m}\rfloor-1\}$. Here for $r\in\mathbb{Q}$, we denote by $\lfloor r\rfloor$ the largest integer smaller than or equal to r.

Similarly, we have the following.

Theorem 5.4. *Let L be an n-component ribbon link with 0-framing. We have*

$$J_{L;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}} \in \frac{\{2l_j+1\}_{q,l_j+1}}{\{1\}_q} \prod_{1\le i\le n, i\neq j} I_{l_i},$$

for $l_1,\dots,l_n\ge 0$, where j is an integer such that $l_j=\max\{l_i\}_{1\le i\le n}$.

For a link $L=L_1\cup\cdots\cup L_n$, we denote by $\check{L}_{i_1,\dots i_m}$ the sublink obtained from L by removing its components $L_{i_1},\dots,L_{i_m}$. In a similar way in which Habiro proved Theorem 5.2 by assuming Theorem 4.1, we can prove the following.

Theorem 5.5. *Let $L=L_1\cup\cdots\cup L_n$ be a link with 0-framing whose sublink $\check{L}_{i_1,\dots i_m}$ is trivial. We have*

$$J_{L;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}} \in \frac{\{2l_j+1\}_q!}{\{1\}_q\{l_{i_1}\}_q!\cdots\{l_{i_m}\}_q!} \prod_{1\le i\le n, i\neq j, i_1,\dots,i_m} I_{l_i},$$

for $l_1,\dots,l_n\ge 0$, where j is an integer such that $l_j=\max\{l_i \mid 1\le i\le n, i\neq i_1,\dots i_m\}$.

Corollary 5.6. *Let L be an n-component brunnian link with 0-framing. We have*

$$J_{L;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}} \in \frac{\{2l_j+1\}_q!}{\{1\}_q\{l_{i_k}\}_q!} \prod_{1\le i\le n, i\neq j,k} I_{l_i},$$

for $l_1,\dots,l_n\ge 0$, where j is an integer such that $l_j=\max\{l_i\mid 1\le i\le n\}$ and k is an integer such that $l_k=\min\{l_i\mid 1\le i\le n\}$.

6 The universal sl_2 invariant of boundary, ribbon, and of brunnian bottom tangles

In this section, we study the universal sl_2 invariant of boundary, ribbon, and of brunnian bottom tangles. We recall Habiro's formulas for the universal invariant of boundary bottom tangles and of ribbon bottom tangles, which we used in a proof of Theorem 4.1 and 4.2. (We do not write the proofs in this note.) For brunnian bottom tangles, we prove Theorem 4.4.

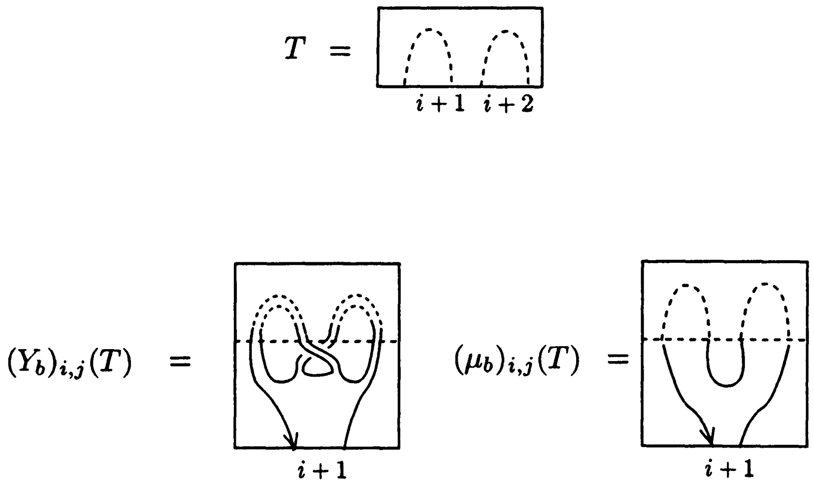


Figure 8: A bottom tangle $T \in BT_{i+j+2}$ and the bottom tangles $(Y_b)_{(i,j)}(T)$, $(\mu_b)_{(i,j)}(T) \in BT_{i+j+1}$. We depict only the $(i+1), (i+2)$th components of T, and the $(i+1)$th components of $(Y_b)_{(i,j)}(T)$, $(\mu_b)_{(i,j)}(T)$.

6.1 The universal sl_2 invariant of boundary bottom tangles

Let $Y\colon U_h \hat{\otimes} U_h \to U_h$ be the U_h-module homomorphism defined by

$$Y(x \otimes y) = \sum x_{(1)} \beta_k S((\alpha_k \triangleright y)_{(1)}) S(x_{(2)}) (\alpha_k \triangleright y)_{(2)}$$

for $x, y \in U_h$.

Remark 6.1. The morphism Y is equal to $Y_{\underline{H}}$ for $H = U_h$ in [2, Section 9.3].

For $T \in BT_{i+j+2}$, $i, j \geq 0$, let $(Y_b)_{i,j}(T) \in BT_{i+j+1}$ and $(\mu_b)_{(i,j)}(T) \in BT_{i+j+1}$ denote the bottom tangles as depicted in Figure 8.

In what follows, we use a notation

$$f_{i,j} = \mathrm{id}^{\otimes i} \otimes f \otimes \mathrm{id}^{\otimes j}\colon U_h^{\hat{\otimes} i+j+k} \to U_h^{\hat{\otimes} i+j+l}$$

for $f\colon U_h^{\hat{\otimes} k} \to U_h^{\hat{\otimes} l}$.

Lemma 6.2 (Habiro [2]). *For a bottom tangle $T \in BT_{i+j+2}$, $i, j \geq 0$, we have*

$$J_{(Y_b)_{i,j}(T)} = Y_{i,j}(J_T), \tag{6}$$

$$J_{(\mu_b)_{i,j}(T)} = \mu_{i,j}(J_T). \tag{7}$$

where $\mu\colon U_h \hat{\otimes} U_h \to U_h$ is the multiplication of U_h.

Let $T = T_1 \cup \cdots \cup T_n$ be a boundary bottom tangle and $F_1, \ldots, F_n$ a disjoint compact, oriented surfaces such that $\partial F_i = T_i$ for $i = 1, \ldots, n$. We can arrange the surfaces $F_1, \ldots, F_n$ as depicted in Figure 9, where *Double*(T') is the tangle obtained from a bottom tangle T' by duplicating and then reversing the orientation of the inner component of each duplicated components. This implies the following proposition, which is implicit in [2, Theorem 9.9].

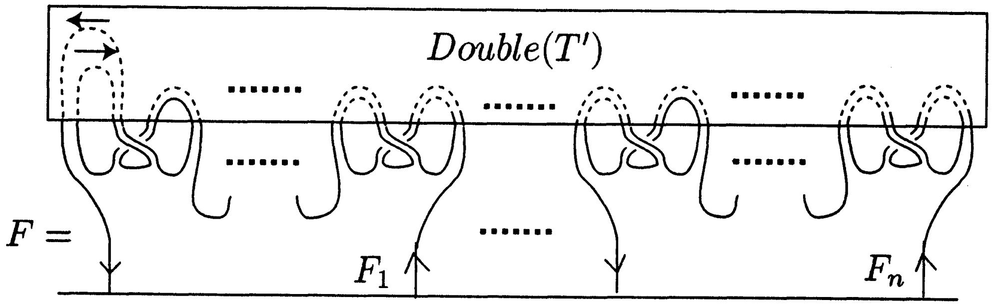
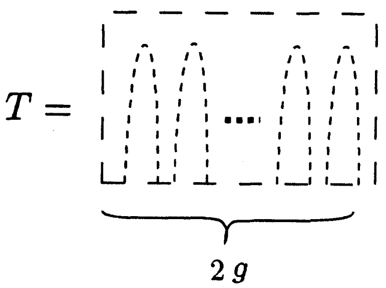


Figure 9: An arranged Seifert surfaces of the bottom tangle T.



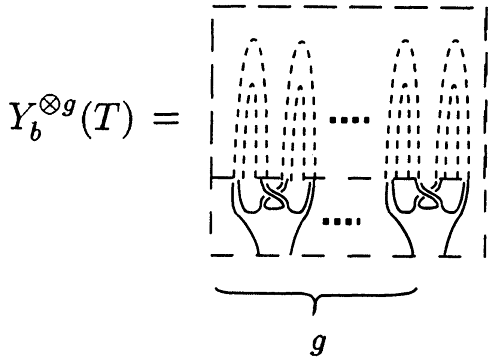


Figure 10: A bottom tangle $T \in BT_{2g}$ and the bottom tangle $Y_b^{\otimes g}(T) \in BT_g$.

Proposition 6.3. *For an n-component bottom tangle T, the following conditions are equivalent.*

(1) *T is a boundary bottom tangle.*

(2) *There is a bottom tangle $T' \in BT_{2g}, g \geq 0$, and there are integers $g_1, \dots, g_n \geq 0$ satisfying $g_1 + \cdots + g_n = g$, such that*

$$T = \mu_b^{[g_1,\dots,g_n]} Y_b^{\otimes g}(T'), \tag{8}$$

where

$$Y_b^{\otimes g}\colon\ BT_{2g} \to BT_g$$

is as depicted in Figure 10, and

$$\mu_b^{[g_1,\dots,g_n]}\colon\ BT_{g_1+\cdots+g_n} \to BT_n$$

is as depicted in Figure 11.

If (8) holds, then we call $(T'; g_1, \dots, g_n)$ a *boundary data* for T.
For $n \geq 1$, let

$$\mu^{[n]}\colon\ U_h^{\hat{\otimes} n} \to U_h, \quad x_1 \otimes \cdots \otimes x_n \mapsto x_1 x_2 \cdots x_n$$

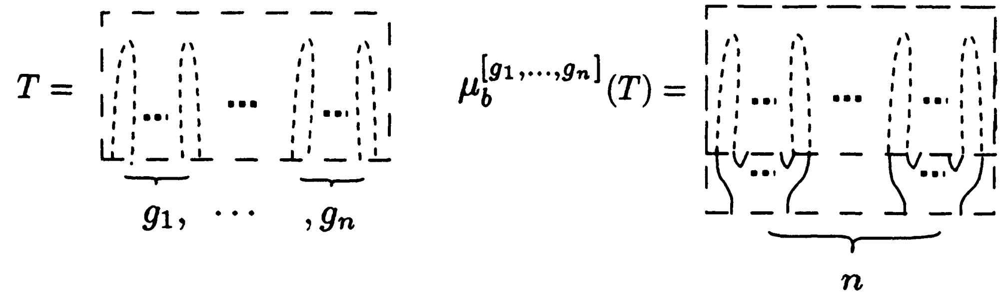


Figure 11: A bottom tangle $T \in BT_k$ and the bottom tangle $\mu_b^{[g_1,\dots,g_n]}(T) \in BT_n$.

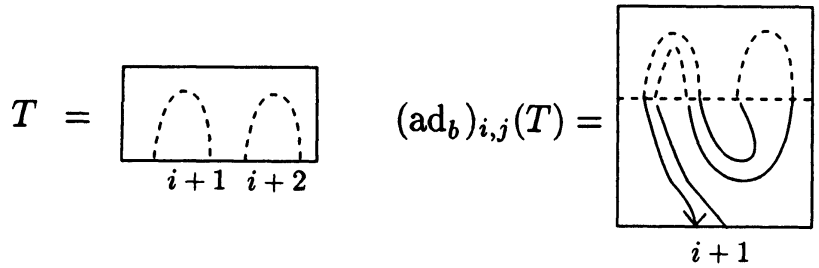


Figure 12: A bottom tangle $T \in BT_{i+j+2}$ and the bottom tangles $(\mathrm{ad}_b)_{(i,j)}(T)$. We depict only the $(i+1), (i+2)$th components of T, and the $(i+1)$th components of $(\mathrm{ad}_b)_{(i,j)}(T)$.

denote the n-input multiplication. For integers $g_1, \dots, g_n \geq 0$, $g_1 + \cdots + g_n = g$, set

$$\mu^{[g_1,\dots,g_n]} = \mu^{[g_1]} \otimes \cdots \otimes \mu^{[g_n]} \colon U_h^{\hat{\otimes} k} \to U_h^{\hat{\otimes} n}.$$

Lemma 6.2 and Proposition 6.3 imply the following.

Proposition 6.4 (Habiro [2]). *Let T be an n-component boundary bottom tangle and $(T' \in BT_{2g}; g_1, \dots, g_n)$ a boundary data for T. Then we have*

$$J_T = \mu^{[g_1,\dots,g_n]} Y^{\otimes g}(J_{T'}).$$

6.2 The universal sl_2 invariant of ribbon bottom tangles

Habiro [3] studied the universal sl_2 invariant of 1-component ribbon bottom tangles. We generalize those to n-component ribbon bottom tangles for $n \geq 1$.

We use the left adjoint action $\mathrm{ad} \colon U_h \otimes U_h \to U_h$ defined by

$$\mathrm{ad}(a \otimes b) = \sum a' b S(a''),$$

for $a, b \in U_h$, where we set $\Delta(a) = \sum a' \otimes a''$. We also use the notation $a \triangleright b = \mathrm{ad}(a \otimes b)$.

For $T \in BT_{i+j+2}$, $i, j \geq 0$, let $(\mathrm{ad}_b)_{i,j}(T) \in BT_{i+j+1}$ denote the bottom tangle as depicted in Figure 12. We use the following lemma.

Lemma 6.5 (Habiro [2]). *For a bottom tangle $T \in BT_{i+j+2}$, $i, j \geq 0$, we have*

$$J_{(\mathrm{ad}_b)_{i,j}(T)} = \mathrm{ad}_{i,j}(J_T).$$

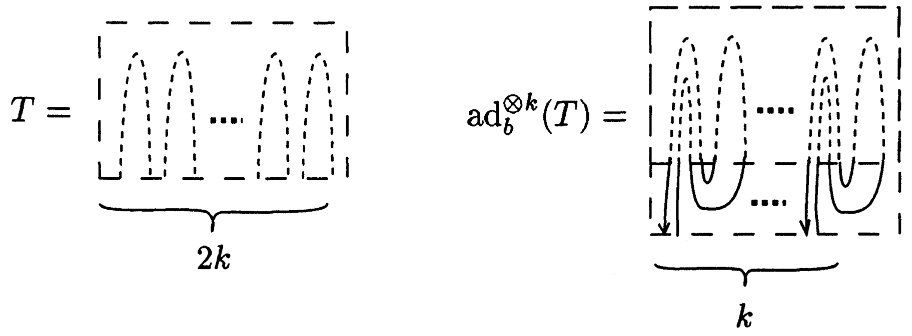


Figure 13: A bottom tangle $T \in BT_{2k}$ and the bottom tangle $\mathrm{ad}_b^{\otimes k}(T) \in BT_k$.

For a $2k$-component bottom tangle $W = W_1 \cup \cdots \cup W_{2k} \in BT_{2k}, k \geq 0$, set

$$W^{\mathrm{ev}} = \bigcup_{i=1}^{k} W_{2i} \in BT_k, \quad \text{and} \quad W^{\mathrm{odd}} = \bigcup_{i=1}^{k} W_{2i-1} \in BT_k.$$

For a diagram P of W, let P^{ev} (resp. P^{odd}) denote the part of the diagram P corresponding to W^{ev} (resp. W^{odd}). We say a bottom tangle $W \in BT_{2k}$ is *even-trivial* if W^{ev} is a trivial bottom tangle. For example, see Figure 14. We also say a diagram P of W is *even-trivial* if and only if P^{ev} has no self crossings. Note that a bottom tangle W has an even-trivial diagram if and only if W is even-trivial.

The following Proposition is almost the same as [2, Theorem 11.5].

Proposition 6.6. *For an n-component bottom tangle T, the following conditions are equivalent.*

(1) *T is a ribbon bottom tangle.*

(2) *There is an even-trivial bottom tangle $W \in BT_{2k}, k \geq 0$, and there are integers $N_1, \ldots, N_n \geq 0$ satisfying $N_1 + \cdots + N_n = k$, such that*

$$T = \mu_b^{[N_1,\ldots,N_n]}\, \mathrm{ad}_b^{\otimes k}(W), \tag{9}$$

where

$$\mathrm{ad}_b^{\otimes k}\colon\ BT_{2k} \to BT_k$$

is as depicted in Figure 13.

If (9) holds, then we call $(W; N_1, \ldots, N_n)$ a *ribbon data* for T. For example, the ribbon bottom tangle $\mu^{[1,2,0]}(ad_b)^{\otimes 3}(W) \in BT_3$ with the ribbon data $(W \in BT_3; 1, 2, 0)$, where W is the bottom tangle in Figure 14, is as depicted in Figure 15.

Lemma 6.5 and Proposition 6.6 imply the following.

Proposition 6.7. *Let T be an n-component ribbon bottom tangle and $(W \in BT_{2k}; N_1, \ldots, N_n)$ a ribbon data for T. Then we have*

$$J_T = \mu^{[N_1,\ldots,N_n]}\, \mathrm{ad}^{\otimes k}(J_W),$$

where $\mathrm{ad}^{\otimes k}\colon\ U_h^{\hat{\otimes} 2k} \to U_h^{\hat{\otimes} k}$ is the k-fold tensor power of the adjoint action.

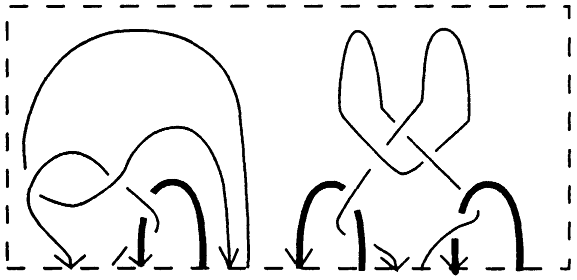
$W =$ 

Figure 14: An even-trivial bottom tangle $W \in BT_6$. Here W^{ev} is depicted with thick lines.

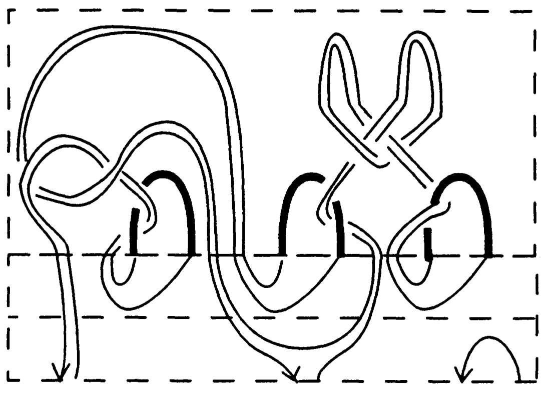
$\mu^{[1,2,0]}(\mathrm{ad}_b)^{\otimes 3}(W) =$ 

Figure 15: The ribbon bottom tangle $\mu^{[1,2,0]}(ad_b)^{\otimes 3}(W) \in BT_3$ for the even-trivial bottom tangle $W \in BT_3$ in Figure 14.

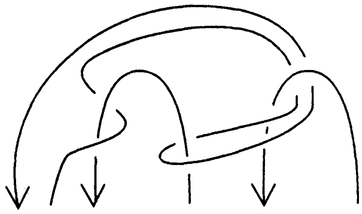


Figure 16: A diagram of the Borromean tangle $P = P_1 \cup P_2 \cup P_3$, where $P_2 \cup P_3$ is the trivial diagram.

6.3 The universal sl_2 invariant of brunnian bottom tangles

We prove Theorem 4.4. We only have to prove the following claim.

Claim: There is a diagram P of T, such that every state $s \in \mathcal{S}(P)$, we have

$$J_{P,s} \in A_1 \otimes \cdots A_n. \tag{10}$$

By Lemma 3.3 and (10), we will have

$$J_{P,s} \in (A_1 \otimes \cdots A_n) \cap F_{|s|}((\mathcal{U}_q^{ev})^{\otimes n}).$$

It will imply that

$$J_T = \sum_{p \geq 0} \sum_{s \in \mathcal{S}(P), |s|=p} J_{P,s} \in (A_1 \otimes \cdots A_n)\hat{}.$$

We prove (10). By definition, the subtangle $T_{i_1,\ldots,i_m}$ has the trivial diagram, hence T has a diagram $P = P_1 \cup \cdots \cup P_n$ whose subdiagram $P_{i_1,\ldots,i_m}$ corresponding to $T_{i_1,\ldots,i_m}$ is the trivial diagram. Figure 16 is an example with the Borromean tangle that is a 3-component brunnian bottom tangle, whose closure is Borromean rings. Note that P has two kinds of crossings:

- Crossings between $P_{i_1,\ldots,i_m}$ and $P_j, j \neq i_1, \ldots, i_m$
- Self crossings of $P_{i_1,\ldots,i_m}$

Let calculate $J_{P,s}$ for a state $s \in \mathcal{S}(P)$. We modify the elements attached to crossings as follows. Let c be a crossing of the diagram with strands oriented downward, and set $m = s(c)$. As depicted in Figure 17, we replace the two dots labeled by $R^{\pm}_m$ with two black dots labeled by $D^{\pm 1}$ and two white dots labeled by $\alpha^{\pm}_m \otimes \beta^{\pm}_m$. Similarly, we modify the dots on the other crossings. We have completed the modification. We have

$$\begin{aligned}
R =& D \sum_{n \geq 0} q^{\frac{1}{2}n(n-1)} \tilde{F}^{(n)} K^{-n} \otimes e^n \\
=& D \sum_{n \geq 0} q^{n(n-1)} f^n K^{-n} \otimes \tilde{E}^{(n)}, \\
R^{-1} =& D^{-1} \sum_{n \geq 0} (-1)^n \tilde{F}^{(n)} \otimes K^{-n} e^n \\
=& D^{-1} \sum_{n \geq 0} (-1)^n q^{\frac{1}{2}n(n-1)} f^n \otimes K^{-n} \tilde{E}^{(n)}.
\end{aligned}$$

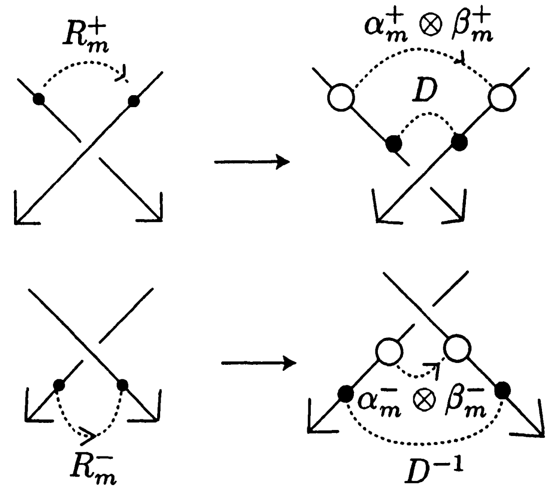


Figure 17: The modification process of elements on positive and negative crossings.

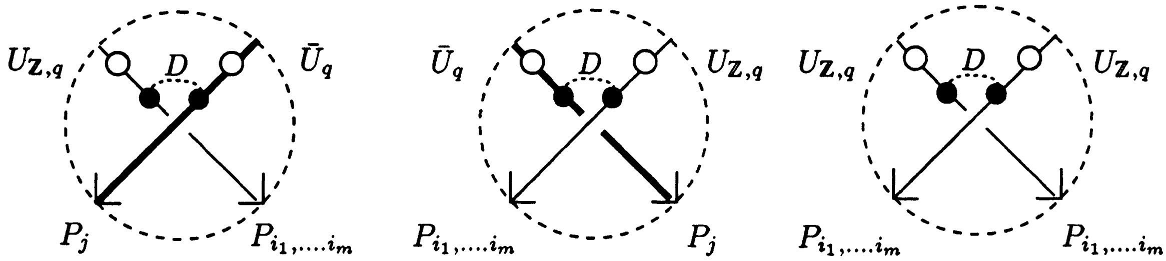


Figure 18: How we treat $\alpha_m \otimes \beta_m$.

Hence we have

$$\alpha_m^{\pm} \otimes \beta_m^{\pm} \in (U_{\mathbb{Z},q} \otimes \bar{U}_q) \cap (\bar{U}_q \otimes U_{\mathbb{Z},q}) \subset U_{\mathbb{Z},q} \otimes U_{\mathbb{Z},q}.$$

Hence for a crossings between $P_{i_1\dots,i_m}$ and P_j, $j \neq i_1, \dots, i_m$, we can assume that the element on the white dot on $P_{i_1\dots,i_m}$ is in $U_{\mathbb{Z},q}$ and that on P_j is in $\bar{U}_q$, and for a self crossing of $P_{i_1\dots,i_m}$, we can assume the element on the white dot is in $U_{\mathbb{Z},q}$, see Figure 18. We slide the elements $D^{\pm 1}$ on the black dots to the heads of tensorands of $J_{P,s}$ by using the formula

$$(1 \otimes x)D = D(K^{|-x|} \otimes x) \tag{11}$$

where x is a homogeneous element of U_h, see Figure 19. Since T is with 0-framing, those $D^{\pm 1}$s are cancelled. Hence, $i_1, \dots, i_m$th tensorands of $J_{P,s}$ are contained in $U_{\mathbb{Z},q}$ and others in $\bar{U}_q$. In the view of Proposition 3.2, $J_{P,s}$ is contained in even part of the subalgebra, hence we have the assertion.

7 Examples

The Borromean tangle $B \in BT_3$ is the bottom tangle depicted in Figure 16, which we can depict as in Figure 20 as well. Note that B is a 3-component, algebraically-split,

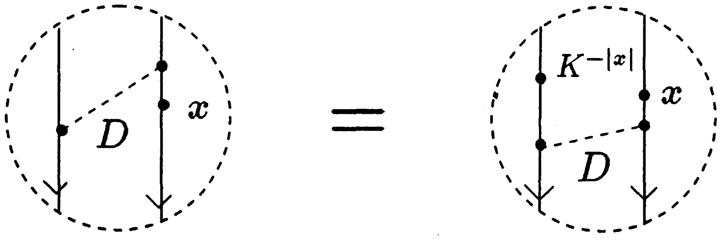


Figure 19: The sliding process of D.

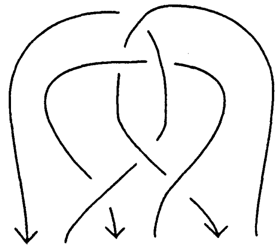


Figure 20: The Borromean tangle $B \in BT_3$.

0-framed bottom tangle, and the closure of B is the Borromean rings L_B. It is well known that L_B is not a ribbon link. In [3], the formulas of the universal sl_2 invariant of B is observed:

$$\begin{aligned} J_B = & \sum_{m_1,m_2,m_3,n_1,n_2,n_3 \geq 0} q^{m_3+n_3}(-1)^{n_1+n_2+n_3} q^{\sum_{i=1}^{3}\left(-\frac{1}{2}m_i(m_i+1)-n_i+m_im_{i+1}-2m_in_{i-1}\right)} \\ & \tilde{F}^{(n_3)}e^{m_1}\tilde{F}^{(m_3)}e^{n_1}K^{-2m_2} \otimes \tilde{F}^{(n_1)}e^{m_2}\tilde{F}^{(m_1)}e^{n_2}K^{-2m_3} \otimes \tilde{F}^{(n_2)}e^{m_3}\tilde{F}^{(m_2)}e^{n_3}K^{-2m_1} \\ & \notin (\bar{U}_q^{\mathrm{ev}})^{\hat{\otimes} 3}, \end{aligned} \tag{12}$$

where the index i should be considered modulo 3. The following is also observed in [3];

$$J_{L_B;\tilde{P}'_i,\tilde{P}'_j,\tilde{P}'_k} = \begin{cases} (-1)^i q^{-i(3i-1)}\{2i+1\}_{q,i+1}/\{1\}_q & \text{if } \ i=j=k, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Since $\frac{\{2i+1\}_{q,i+1}}{\{1\}_q} \notin \frac{\{2i+1\}_{q,i+1}}{\{1\}_q} I_i I_i$ for $i \geq 1$, each of (12) and (13) implies that the Borromean rings L_B is not a boundary or a ribbon link.

For $n \geq 3$, Milnor's link $L_{M,n}$ is the n-component brunnian link as depicted in Figure 21. Note that $L_{M,3}$ is the Borromean rings L_B. For $m \geq 1$, recall that $\Phi_m(q)$ is the mth cyclotomic polynomial in q. We have

$$J_{L_{M,n};\tilde{P}'_1,\ldots,\tilde{P}'_1} = (-1)^{n-2}q^{-2n+4}\Phi_4(q)^{n-3}\Phi_3(q)\Phi_2(q)^{n-2}\Phi_1(q)^{n-2} \notin \mathbb{Z}[q,q^{-1}]\Phi_1(q)^n.$$

Hence, for all $n \geq 3$, $L_{M,n}$ is not a boundary or a ribbon link.

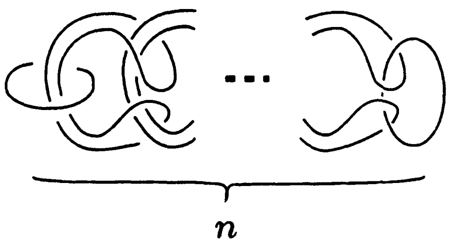


Figure 21: Milnor's link $L_{M.n}$.

8 Completion for $\bar{U}_q^{\mathrm{ev}}$

8.1 Filtrations of $\bar{U}_q^{\mathrm{ev}}$

In this subsection, we define two filtrations $\{A_p\}_{p\geq 0}$ and $\{C_p\}_{p\geq 0}$ of $\bar{U}_q^{\mathrm{ev}}$, which are cofinal with each other. We give four equivalent definitions for $\{A_p\}_{p\geq 0}$, and two for $\{C_p\}_{p\geq 0}$.

For a subset $X \subset \bar{U}_q^{\mathrm{ev}}$, let $\langle X\rangle_{\mathrm{ideal}}$ denote the two-sided ideal of $\bar{U}_q^{\mathrm{ev}}$ generated by X. For $p \geq 0$, set

$$
\begin{aligned}
A_p &= \langle U_{\mathbf{Z},q} \triangleright e^p\rangle_{\mathrm{ideal}}, \quad A'_p = \langle U_{\mathbf{Z},q} \triangleright f^p\rangle_{\mathrm{ideal}},\\
B_p &= \langle K^p(U_{\mathbf{Z},q} \triangleright K^{-p}e^p)\rangle_{\mathrm{ideal}}, \quad B'_p = \langle K^p(U_{\mathbf{Z},q} \triangleright f^pK^{-p})\rangle_{\mathrm{ideal}},\\
C_p &= \langle \sum_{p'\geq p} (U_{\mathbf{Z},q}\tilde{E}^{(p')} \triangleright \bar{U}_q^{\mathrm{ev}})\rangle_{\mathrm{ideal}}, \quad C'_p = \langle \sum_{p'\geq p} (U_{\mathbf{Z},q}\tilde{F}^{(p')} \triangleright \bar{U}_q^{\mathrm{ev}})\rangle_{\mathrm{ideal}}.
\end{aligned}
$$

Proposition 8.1 ([9]). (i) *$\{A_p\}_{p\geq 0}$ is a decreasing filtration.*

(ii) *For $p \geq 0$, we have*

$$A_p = A'_p = B_p = B'_p.$$

Proposition 8.2 ([9]). (i) *For $p \geq 0$, we have $C_p = C'_p$.*

(ii) *For $p \geq 0$, we have $C_{2p} \subset A_p$.*

(iii) *If $p \geq 0$ is even, then we have $C_{2p} = A_p$.*

Corollary 8.3. *For $p \geq 0$, we have*

$$C_{2p} \subset h^pU_h.$$

Proof. Since $e^p \in h^pU_h$, we have $A_p \subset h^pU_h$. Then the assertion follows from Proposition 8.2 (*iii*). □

8.2 The completion $(\bar{U}_q^{\mathrm{ev}})^{\hat{\otimes} n}$ of $(\bar{U}_q^{\mathrm{ev}})^{\otimes n}$

In this subsection we define the completion $(\bar{U}_q^{\mathrm{ev}})^{\hat{\otimes} n}$ of $(\bar{U}_q^{\mathrm{ev}})^{\otimes n}$. Let $(\bar{U}_q^{\mathrm{ev}})\hat{}$ denote the completion in U_h of $\bar{U}_q^{\mathrm{ev}}$ with respect to the decreasing filtration $\{C_p\}_{p\geq 0}$, i.e., $(\bar{U}_q^{\mathrm{ev}})\hat{}$ is the image of the homomorphism

$$\varprojlim_p (\bar{U}_q^{\mathrm{ev}}/C_p) \to U_h.$$

induced by the inclusion $\bar{U}_q^{\mathrm{ev}} \subset U_h$, which is well defined since $C_{2p} \subset h^p U_h$ for $p \geq 0$. For $n \geq 1$, we define a filtration $\{C_p^{(n)}\}_{p\geq 0}$ for $(\bar{U}_q^{\mathrm{ev}})^{\otimes n}$ by

$$C_p^{(n)} = \sum_{j=1}^{n} \bar{U}_q^{\mathrm{ev}} \otimes \cdots \otimes \bar{U}_q^{\mathrm{ev}} \otimes C_p \otimes \bar{U}_q^{\mathrm{ev}} \otimes \cdots \otimes \bar{U}_q^{\mathrm{ev}},$$

where C_p is at the jth position. Define the completion $(\bar{U}_q^{\mathrm{ev}})^{\hat{}\hat{\otimes} n}$ of $(\bar{U}_q^{\mathrm{ev}})^{\otimes n}$ as the image of the homomorphism

$$\varprojlim_p \left((\bar{U}_q^{\mathrm{ev}})^{\otimes n}/C_p^{(n)}\right) \to U_h^{\hat{\otimes} n}.$$

For $n = 0$, it is natural to set

$$C_p^{(0)} = \begin{cases} \mathbb{Z}[q, q^{-1}] & \text{if } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have

$$(\bar{U}_q^{\mathrm{ev}})^{\hat{}\hat{\otimes} 0} = \mathbb{Z}[q, q^{-1}].$$

References

[1] C. McA. Gordon, Ribbon concordance of knots in 3-sphere. Math. Ann. **257** (1981), no. 2, 157–170.

[2] K. Habiro, Bottom tangles and universal invariants. Alg. Geom. Topol. **6** (2006), 1113–1214.

[3] K. Habiro, A unified Witten-Reshetikhin-Turaev invariants for integral homology spheres. Invent. Math. **171** (2008), no. 1, 1–81.

[4] R. J. Lawrence, A universal link invariant. in: The interface of mathematics and particle physics (Oxford, 1988), 151–156, Inst. Math. Appl. Conf. Ser. New Ser., vol. 24, Oxford Univ. Press, New York, 1990.

[5] R. J. Lawrence, A universal link invariant using quantum groups. in: Differential geometric methods in theoretical physics (Chester, 1989), 55–63, World Sci. Publishing, Teaneck, NJ, 1989.

[6] G. Lusztig, Introduction to quantum groups. Progress in Mathematics 110, Birkhäuser, Boston, 1993.

[7] T. Ohtsuki, Colored ribbon Hopf algebras and universal invariants of framed links. J. Knot Theory Ramifications **2** (1993), no. 2, 211–232.

[8] N. Y. Reshetikhin, V. G. Turaev, Ribbon graphs and their invariants derived from quantum groups. Comm. Math. Phys. **127** (1990), no. 1, 1–26.

[9] S. Suzuki, On the universal sl_2 invariant of ribbon bottom tangles. Alg. Geom. Topol. **10** (2010), 1027-1061.