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Borchers Lifts, Symmetry Relations, and Applications

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Abstract. This paper is related to the authors' talk at the RIMS conference 2010 on: *Automorphic forms, automorphic representations and related topics* in Tokyo. We mainly study holomorphic Siegel modular forms on $\mathrm{Sp}_2(\mathbb{Z})$ obtained as Borcherds lifts and the connection with the Witt and Siegel Φ-operator. As a direct consequence we obtain for example that Siegel Eisenstein series are not Borcherds lifts.

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1 Introduction and the main results

1.1 Introduction

In this note we mainly summarize the results presented at the RIMS conference 2010 on: *Automorphic forms, automorphic representations and related topics* in Tokyo. A Borcherds lift ([Bo1],[Bo2],[Bo3]) on $\Gamma_2 = \mathrm{Sp}_2(\mathbb{Z})$ is a meromorphic automorphic form F on Γ_2 (with a multiplier system of finite order) whose divisor is of the form $\sum_d A(d)H_d$, where d runs over the positive integers congruent to 0 or 1 modulo 4, $A(d) \in \mathbb{Z}$ ($A(d) = 0$ except for a finite number of d) and H_d is the Humbert surface of discriminant d. Since every Borcherds lift is a quotient of holomorphic Borcherds lifts, we mainly consider the holomorphic case in this paper.

We employ our previous result on the multiplicative symmetries for Borcherds lifts ([HM]; see Theorem 3.1). We obtain that the image of a holomorphic Borcherds lift on Γ_2 under the Siegel operator is proportional to a power of Δ, the Ramanujan discriminant function. This implies that the Siegel Eisenstein series is never a Borcherds lift. Then we show that a holomorphic Borcherds lift on Γ_2 with trivial character is proportional to $\chi_{10}^a \chi_{35}^b F'$, where χ_{10} and χ_{35} are Borcherds lifts of weight 10 and 35, respectively, $a \in \mathbb{Z}_{\geq 0}, b \in \{0,1\}$ and F' is a Borcherds lift of weight divisible by 12 such that the image of F' under the Witt operator is nonzero (Corollary 1.5).

1.2 Siegel modular forms

To explain our results more precisely, let

$$\Gamma_n := \left\{ \gamma \in \mathrm{GL}_{2n}(\mathbb{Z}) \mid {}^t\gamma \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} \gamma = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} \right\}$$

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be the Siegel modular group of degree n and $\mathfrak{H}_n := \{Z \in \mathrm{M}_n(\mathbb{C}) \mid {}^tZ = Z, \mathrm{Im}(Z) > 0\}$ be the upper half space of degree n, where $\mathbf{0}_n$ (respectively $\mathbf{1}_n$) is the zero (respectively identity) matrix of degree n.

Let $M_k(\Gamma_n)$ denote the space of holomorphic automorphic forms of weight k on Γ_n and $S_k(\Gamma_n)$ be the subspace of cuspforms.

In the case $n = 2$ which we are mainly interested in we often write (τ_1, z, τ_2) for a point

$$\begin{pmatrix} \tau_1 & z \\ z & \tau_2 \end{pmatrix} \in \mathfrak{H}_2.$$

For $F \in M_k(\Gamma_2)$, we put

$$\Phi(F)(\tau) := \lim_{y\to\infty} F(\tau, 0, iy) \qquad (\tau \in \mathfrak{H}_1),$$
$$\mathcal{W}(F)(\tau_1, \tau_2) := F(\tau_1, 0, \tau_2) \qquad (\tau_1, \tau_2 \in \mathfrak{H}_1).$$

Then $\Phi(F) \in M_k(\Gamma_1)$ and $\mathcal{W}(F) \in \mathrm{Sym}^2(M_k(\Gamma_1))$. The operator Φ (respectively $\mathcal{W}$) is called the *Siegel* (respectively *Witt*) operator. Then $S_k(\Gamma_2) = \{F \in M_k(\Gamma_2) \mid \Phi(F) = 0\}$ is the space of cusp forms. A Siegel modular form $F \in M_k(\Gamma_2)$ admits the Fourier expansion

$$F(\tau_1, z, \tau_2) = \sum_{n,r,m\in\mathbb{Z}} A_F(n, r, m)\ \mathbf{e}(n\tau_1 + rz + m\tau_2),$$

where we put $\mathbf{e}(z) = \exp(2\pi i z)$ for $z \in \mathbb{C}$. Note that $A_F(n, r, m) = 0$ unless $n, m, 4nm - r^2 \geq 0$.

For $k \geq 4$ let $E_k(Z)$ denote the Siegel Eisenstein series on Γ_2 of weight k. Due to Igusa ([Ig]), the graded ring $\bigoplus_{k\geq 0} M_k(\Gamma_2)$ is generated by $E_4, E_6, \chi_{10}, \chi_{12}$ and χ_{35}, where

$$\chi_{10} := -43867 \cdot 2^{-12} \cdot 3^{-5} \cdot 5^{-2} \cdot 7^{-1} \cdot 53^{-1}(E_4E_6 - E_{10}) \in S_{10}(\Gamma_2),$$

$$\chi_{12} := 131 \cdot 593 \cdot 2^{-13} \cdot 3^{-7} \cdot 5^{-3} \cdot 7^{-2} \cdot 337^{-1} \left(3^2 \cdot 7^2\, E_4^3 + 2 \cdot 5^3\, E_6^2 - 691\, E_{12}\right) \in S_{12}(\Gamma_2)$$

and χ_{35} is a unique element of $S_{35}(\Gamma_2)$ up to constant multiples. Note that we follow Igusa's normalizations of χ_{10} and χ_{12} so that

$$A_{\chi_{10}}(1, 1, 1) = -1/4,$$
$$A_{\chi_{12}}(1, 1, 1) = 1/12.$$

We also recall that van der Geer ([Ge1]) defined a Siegel modular form

$$G_{24} := \left(\chi_{12} - 2^{-12} \cdot 3^{-6}(E_6^2 + E_4^3)\right)^2 - E_4\left(2 \cdot 3^{-1}\chi_{10} - 2^{-11} \cdot 3^{-6}E_4E_6\right)^2 \in M_{24}(\Gamma_2),$$

whose divisor is the Humbert surface of discriminant 5 (for the definition of Humbert surfaces, see **2.2**). It is known that χ_{10}, χ_{35} and G_{24} are Borcherds lifts (see [GN1] and [GN2]), but χ_{12} is not a Borcherds lift (see [HM]).

1.3 Main results

Employing our previous result on the multiplicative symmetries for Borcherds lifts ([HM]; see Theorem 3.1), we give several necessary conditions for $F \in M_k(\Gamma_2)$ to be a Borcherds lift.

Theorem 1.1. *Assume that $F \in M_k(\Gamma_2)$ is a Borcherds lift. Then $\Phi(F)$ is proportional to a power Δ^r of the modular discriminant Δ with $r \geq 0$.*

Corollary 1.2. *If $F \in M_k(\Gamma_2) \setminus S_k(\Gamma_2)$ is a Borcherds lift, then the weight k is divisible by 12.*

We note that $\chi_{10} \in S_{10}(\Gamma_2)$ is a Borcherds lift, and hence that the assumption of noncuspidality is necessary.

Corollary 1.3. *The Siegel Eisenstein series E_k is not a Borcherds lift.*

Moreover we have the following result:

Theorem 1.4. *If $F \in M_k(\Gamma_2)$ is a Borcherds lift and $\mathcal{W}(F) \neq 0$, then the weight k is divisible by 12 and greater than 12.*

Corollary 1.5. *Let $F \in M_k(\Gamma_2)$ be a Borcherds lift. We let $b = 0$ if k is even and $b = 1$ otherwise. Define $a \in \mathbb{Z}_{\geq 0}$ such that the coefficient of H_1 in the divisor of F is equal to $2a + b$. Then there exists a Borcherds lift $F' \in M_{k'}(\Gamma_2)$ with $\mathcal{W}(F') \neq 0$ such that F is proportional to $\chi_{10}^a \chi_{35}^b F'$. In particular, the weight k of F is of the form*

$$10a + 35b + 12c \ \ (a \in \mathbb{Z}_{\geq 0}, b \in \{0, 1\}, c \in \mathbb{Z}_{\geq 0}, c \neq 1).$$

2 Borcherds lifts

2.1 Jacobi forms

For $k \in \mathbb{Z}$, let $J_{k,1}^{wh}$ denote the space of holomorphic functions on $\mathfrak{H} \times \mathbb{C}$ satisfying the following conditions:

(i) $$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k \mathbf{e}\left(\frac{cz^2}{c\tau + d}\right) \phi(\tau, z) \qquad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1, \tau \in \mathfrak{H}, z \in \mathbb{C}\right).$$

(ii) $$\phi(\tau, z + \lambda\tau + \mu) = \mathbf{e}(-\lambda^2\tau - 2\lambda z)\phi(\tau, z) \qquad (\lambda, \mu \in \mathbb{Z}).$$

(iii) Let $\phi(\tau, z) = \sum_{n,r \in \mathbb{Z}} a_\phi(n, r)\mathbf{e}(n\tau + rz)$ be the Fourier expansion of ϕ. Then $a_\phi(n, r) = 0$ if $4n - r^2$ is sufficiently small.

We call $J_{k,1}^{wh}$ the space of *weakly holomorphic Jacobi forms* of weight k and index 1. The Fourier coefficient $a_\phi(n, r)$ depends only on $N = 4n - r^2$ and is often denoted by $a_\phi(N)$. We put $a_\phi(N) = 0$ if $N \equiv 1$ or $2 \pmod 4$. We then have

$$\phi(\tau, z) = \sum_{N \in \mathbb{Z}} a_\phi(N) \sum_{r \in \mathbb{Z},\, r^2 \equiv -N \bmod 4} \mathbf{e}\left(\frac{N + r^2}{4}\tau + rz\right).$$

For $\phi \in J_{0,1}^{wh}$, we call $\{a_\phi(N) \mid N < 0\}$ the *principal part* of ϕ, which determines ϕ since the space of holomorphic Jacobi forms of weight 0 and index 1 vanishes.

2.2 Humbert surfaces

Let

$$Q := \begin{pmatrix} &&&&1\\ &&&1&\\ &&-2&&\\ &1&&&\\ 1&&&& \end{pmatrix}.$$

Put $Q(X,Y) := {}^tXQY$ and $Q[X] := Q(X,X)$ for $X, Y \in \mathbb{C}^5$. For $Z = (\tau_1, z, \tau_2) \in \mathfrak{H}_2$ put $\widetilde{Z} := {}^t(-\tau_1\tau_2 + z^2, \tau_1, z, \tau_2, 1) \in \mathbb{C}^5$. Note that $Q[\widetilde{Z}] = 0$ and $Q(\widetilde{Z}, \overline{\widetilde{Z}}) = 4\det(\mathrm{Im}(Z)) > 0$. There exists a homomorphism $\iota\colon \mathrm{Sp}_2(\mathbb{R}) \to O(Q)_{\mathbb{R}}$ such that $\widetilde{g\langle Z\rangle} = j(g,Z)^{-1}\iota(g)\widetilde{Z}$ for $g \in \mathrm{Sp}_2(\mathbb{R})$ and $Z \in \mathfrak{H}_2$.

Let $L := \mathbb{Z}^5, L^* := Q^{-1}L$ and $L^*_{\text{prim}} := \{\lambda \in L^* \mid n^{-1}\lambda \notin L^* \text{ for any integer } n > 1\}$. For an integer $d \in \mathbb{Z}$, let

$$\mathcal{H}_d := \sum_{X \in \mathcal{L}_d} \left\{ Z \in \mathfrak{H}_2 \mid Q(X, \widetilde{Z}) = 0 \right\},$$

where $\mathcal{L}_d := \{X \in L^*_{\text{prim}} \mid Q[X] = -d/2\}$. Note that $\mathcal{H}_d = 0$ unless $d > 0$ and $d \equiv 0$ or $1 \pmod 4$. Since L^*_d is $\iota(\Gamma_2)$-invariant, $\mathcal{H}_d$ is Γ_2-invariant. Denote by H_d the image of $\mathcal{H}_d$ in $\Gamma_2\backslash\mathfrak{H}_2$ by the natural projection $\mathfrak{H}_2 \to \Gamma_2\backslash\mathfrak{H}_2$. The divisor H_d of $\Gamma_2\backslash\mathfrak{H}_2$ is called the *Humbert surface* of discriminant d. It is known that H_d is nonzero and irreducible if $d \equiv 0$ or $1 \pmod 4$ (see [Ge2], page 212, Theorem 2.4; see also [GH], Section 3). Note that

$$\mathcal{H}_1 = \bigcup_{\gamma \in \Gamma_2} \gamma\{(\tau_1, 0, \tau_2) \mid \tau_1, \tau_2 \in \mathfrak{H}\}$$

$$\mathcal{H}_4 = \bigcup_{\gamma \in \Gamma_2} \gamma\{(\tau, z, \tau) \mid \tau \in \mathfrak{H}, z \in \mathbb{C}\}.$$

Let v be the unique nontrivial quadratic character of Γ_2 and $M_k(\Gamma_2, v)$ the space of Siegel modular forms on Γ_2 of weight k with character v. The following result of Igusa is quite useful (see [GN1], Corollary 1.4).

Lemma 2.1. *Let $F \in M_k(\Gamma_2, v)$. If k is odd, $\chi_5^{-1}F \in M_{k-5}(\Gamma_2)$. If k is even, $\chi_{30}^{-1}F \in M_{k-30}(\Gamma_2)$.*

2.3 Borcherds lifts on Γ_2

As a special case of Borcherds theory ([Bo1] and [Bo2]; see also [GN3], §2.1), we have the following result:

Theorem 2.2. *Let $\phi \in J_{0,1}^{wh}$ and write $a(N)$ for $a_\phi(N)$. Assume that $a(N) \in \mathbb{Z}$ if $N < 0$.*

(i) Set

$$\delta := \sum_{r \in \mathbb{Z}} a(-r^2),$$
$$\rho := \frac{1}{2} \sum_{r \in \mathbb{Z}, r>0} a(-r^2)r,$$
$$\nu := \frac{1}{4} \sum_{r \in \mathbb{Z}} a(-r^2)r^2$$

and

$$\Lambda := \left\{(m,r,n) \in \mathbb{Z}^3 \mid m > 0 \text{ or } m = 0, n > 0 \text{ or } m = n = 0, r > 0\right\}.$$

Then

$$\Psi_\phi(\tau_1, z, \tau_2) := \mathbf{e}\left(\frac{\delta}{24}\tau_2 - \rho z + \nu \tau_1\right) \prod_{(m,r,n) \in \Lambda} (1 - \mathbf{e}(m\tau_1 + rz + n\tau_2))^{a(4mn-r^2)}$$

converges absolutely if $\det(\mathrm{Im}(Z))$ *is sufficiently large, and is meromorphically continued to $\mathfrak{H}_2$.*

(ii) The function Ψ_ϕ is a meromorphic modular form on Γ_2 of weight $k_\phi = a(0)/2$ and character v^α $(\alpha \in \{0,1\})$.

(iii) The divisor of Ψ_ϕ is

$$\sum_d a(-d) H_d^*,$$

where d runs over the positive integers congruent to 0 or 1 modulo 4 and

$$H_d^* := \sum_{f>0,\, f^2 \mid d} H_{f^{-2}d}.$$

The meromorphic modular form Ψ_ϕ is called the *Borcherds lift* of ϕ.

Remark 2.3. It is well-known that the weight of Borcherds lifts is related to the Cohen numbers $H(N) = H(2,N)$. These are the coefficients of the Cohen Eisenstein series

$$\sum_{N \geq 0} H(2,N)\mathbf{e}(N\tau),$$

of weight $5/2$. For convenience we put $h(N) = \sum_{f^2 \mid N} \mu(f)\, H(f^{-2}N)$, where μ is the Möbius function. Moreover put $\hat{H}(N) = -60H(N)$ and $\hat{h}(N) = -60h(N)$. Then we have

Theorem 2.4.

(i) For each positive integer d with $d \equiv 0$ or $1 \pmod 4$, there exists an $F_d \in M_{k_d}(\Gamma_2, v^{\alpha_d})$ with $\alpha_d \in \{0,1\}$ satisfying $\mathrm{div}(F_d) = H_d$.

(ii) We have $k_d = \widehat{h}(d)$.

(iii) We have $F_1 \in M_5(\Gamma_2, v), F_4 \in M_{30}(\Gamma_2, v)$ and $F_d \in M_{k_d}(\Gamma_2)$ if $d > 4$.

(iv) A Borcherds lift $F \in M_k(\Gamma_2, v^\alpha)$ $(\alpha \in \{0,1\})$ is a constant multiple of $\prod_d F_d^{A(d)}$, where d runs over the positive integers with $d \equiv 0$ or $1 \pmod 4$, and $A(d)$ is a nonnegative integer ($A(d) = 0$ except for a finite number of d) satisfying $A(1) + A(4) \equiv \alpha \pmod 2$. Furthermore we have

$$k = \sum_{d>0} A(d)\widehat{h}(d).$$

Moreover we have

Theorem 2.5. *The weight k_d of F_d is divisible by 24 if and only if $d > 4$ and $d \neq 8$.*

Remark 2.6. The Borcherds lifts in $M_k(\Gamma_2)$ with $k \leq 60$ are listed as follows:

Borcherds lift	weight	divisor
F_1^{2a} $(1 \leq a \leq 6)$	$10a$	$2aH_1$
$F_1^{2a+1}F_4$ $(1 \leq a \leq 2)$	$10a + 35$	$(2a+1)H_1 + H_4$
$F_1^{2a}F_5$ $(1 \leq a \leq 3)$	$10a + 24$	$2aH_1 + H_5$
F_4^2	60	$2H_4$
F_5^2	48	$2H_5$
F_8	60	H_8

The table shows that every Borcherds lift of weight less than or equal to 60 is a monomial of F_1, F_4, F_5 and F_8. We also see that there is no holomorphic Borcherds lift of weight 12. This gives another proof of the fact that χ_{12} is not a Borcherds lift, which was proved in [HM] in a different way.

2.4 The image of Ψ_ϕ under the Witt operator

For $m \in \mathbb{Z}_{>0}$, let $\mathcal{M}_m$ be the set of matrices in $\mathrm{M}_2(\mathbb{Z})$ of determinant m. As is well-known, there exists a polynomial Φ_m in $\mathbb{Z}[X, Y]$, called the modular polynomial of degree m, such that

$$\prod_{M \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{M}_m} (X - j(M\langle\tau\rangle)) = \Phi_m(X, j(\tau)).$$

The degree of $\Phi_m(X,Y)$ in X is equal to $\sigma_1(m) = \sum_{0<d|m} d$. Let

$$\eta(\tau) := \mathbf{e}(\tau/24) \prod_{n=1}^{\infty} (1 - \mathbf{e}(n\tau)) \qquad (\tau \in \mathfrak{H})$$

be the Dedekind's eta function.

Theorem 2.7. *Let $\phi \in J_{0,1}^{wh}$ and suppose that $a(N) := a_\phi(N) \in \mathbb{Z}$ if $N < 0$. Assume that the Borcherds lift Ψ_ϕ of ϕ is holomorphic.*

(i) We have $\mathcal{W}(\Psi_\phi) = 0$ if and only if $\sum_{r>0} a(-r^2) > 0$.

(ii) Assume that $\sum_{r>0} a(-r^2) = 0$. Then

$$\mathcal{W}(\Psi_\phi) = c\,(\eta(\tau_1)\eta(\tau_2))^{b(0)} \prod_{n>0} \Phi_n(j(\tau_1), j(\tau_2))^{b(-n)}, \tag{2.1}$$

where $c \in \mathbb{C}^\times$ and

$$b(n) := \sum_{r\in\mathbb{Z}} a(4n - r^2).$$

(iii) Assume that $\sum_{r>0} a(-r^2) = 0$. The automorphic form $\mathcal{W}(\Psi_\phi)$ belongs to $\mathrm{Sym}^2(S_{b(0)/2}(\Gamma_1))$ if and only if $\sum_{r\in\mathbb{Z}} a(-r^2)r^2 > 0$.

Remark 2.8. The degree of $\mathcal{W}(\Psi_\phi)$ in $q_1 = \mathbf{e}[\tau_1]$ is equal to

$$b(0)/24 - \sum_{n>0} \sigma_1(n) b(-n).$$

Corollary 2.9. *Let $d > 4$. Then $F_d \in S_{k_d}(\Gamma_2)$ if and only if $d = \square$.*

3 Multiplicative symmetries and the main theorems

3.1 The multiplicative symmetries

For $F \in M_k(\Gamma_2)$ and a prime number p, we put

$$F|\mathcal{T}_p^{\uparrow}(\tau_1, z, \tau_2) = F(p\tau_1, pz, \tau_2) \prod_{a=0}^{p-1} F\left(\frac{\tau_1 + a}{p}, z, \tau_2\right),$$

$$F|\mathcal{T}_p^{\downarrow}(\tau_1, z, \tau_2) = F(\tau_1, pz, p\tau_2) \prod_{a=0}^{p-1} F\left(\tau_1, z, \frac{\tau_2 + a}{p}\right).$$

We say that F satisfies the *multiplicative symmetries* if the condition

$$F|\mathcal{T}_p^{\uparrow} = \epsilon_p(F)\, F|\mathcal{T}_p^{\downarrow} \tag{MS$_p$}$$

holds with $\epsilon_p(F) \in \mathbb{C}^\times, |\epsilon_p(F)| = 1$ for any prime number p. Denote by $A_{F,p}^{\uparrow}(n,r,m)$ (respectively $A_{F,p}^{\downarrow}(n,r,m)$) the coefficient of $\mathbf{e}(n\tau_1 + rz + m\tau_2)$ in the Fourier expansion of $F|\mathcal{T}_p^{\uparrow}(\tau_1, z, \tau_2)$ (respectively $F|\mathcal{T}_p^{\downarrow}(\tau_1, z, \tau_2)$). If F satisfies $(\mathrm{MS})_p$, then we have

$$A_{F,p}^{\uparrow}(m,r,n) = \epsilon_p(F) A_{F,p}^{\downarrow}(m,r,n)$$

for any (m,n,r). As a special case of [HM], we have the following result.

Theorem 3.1. *Suppose that $F \in M_k(\Gamma_2)$ is a Borcherds lift. Then F satisfies the multiplicative symmetries.*

3.2 A characterization of powers of the modular discriminant

Let k be a positive integer greater than or equal to 4. Denote by $M_k(\Gamma_1)$ (respectively $S_k(\Gamma_1)$) the space of holomorphic automorphic (respectively cusp) forms on $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$ of weight k. Recall that $S_{12}(\Gamma_1) = \mathbb{C} \cdot \Delta$ and that Δ has no zeros in $\mathfrak{H}$.

For $f \in M_k(\Gamma_1)$ and a prime number p, we define the multiplicative Hecke operator by

$$(f|\mathcal{T}_p)(\tau) = f(p\tau) \prod_{c=0}^{p-1} f\left(\frac{\tau+c}{p}\right).$$

Then $f|\mathcal{T}_p \in M_{(p+1)k}(\Gamma_1)$. The following property plays a crucial role in the proof of Theorem 1.1.

Proposition 3.2. *Let f be a nonzero element of $M_k(\Gamma_1)$. Then f satisfies*

$$(*)_p \qquad f|\mathcal{T}_p(\tau) = \epsilon_p(f) f(\tau)^{p+1} \qquad (\tau \in \mathfrak{H}).$$

for any prime number p with $\epsilon_p(f) \in \mathbb{C}^\times, |\epsilon_p(f)| = 1$ if and only if f is a constant multiple of Δ^r $(r \in \mathbb{Z}_{\geq 0})$.

Remark 3.3. If $f \in M_k(\Gamma_1)$ satisfies $(*)_2$, f is a constant multiple of Δ^r.

3.3 Multiplicative symmetries for $\mathrm{Sym}^2(M_k(\Gamma_1))$

For $\varphi \in \mathrm{Sym}^2(M_k(\Gamma_1))$ and a prime number p, we define the multiplicative Hecke operators by

$$(\varphi|\mathcal{T}_p^{\uparrow})(\tau_1,\tau_2) = \varphi(p\tau_1,\tau_2) \prod_{c=0}^{p-1} \varphi\left(\frac{\tau_1+c}{p},\tau_2\right),$$

$$(\varphi|\mathcal{T}_p^{\downarrow})(\tau_1,\tau_2) = \varphi(\tau_1,p\tau_2) \prod_{c=0}^{p-1} \varphi\left(\tau_1,\frac{\tau_2+c}{p}\right)$$

We say that φ satisfies the multiplicative symmetry for p if there exists an $\epsilon_p(\varphi) \in \mathbb{C}^\times, |\epsilon_p(\varphi)| = 1$ such that

$$(\mathrm{ms})_p \qquad \varphi|\mathcal{T}_p^{\uparrow} = \epsilon_p(\varphi)\varphi|\mathcal{T}_p^{\downarrow}$$

holds. For $\varphi \in \mathrm{Sym}^2(M_k(\Gamma_1))$, put $\Phi'(\varphi)(\tau) = \lim_{y\to\infty} \varphi(\tau, iy)$. Then $\Phi'(\varphi) \in M_k(\Gamma_1)$. The following facts can be verified.

Lemma 3.4. *If $\varphi \in \mathrm{Sym}^2(M_k(\Gamma_1))$ satisfies $(\mathrm{ms})_p$ and $f = \Phi'(\varphi) \neq 0$, then f satisfies $(*)_p$. In particular, f is a constant multiple of Δ^r and k is divisible by 12.*

Proposition 3.5. *If $\varphi \in \mathrm{Sym}^2(M_k(\Gamma_1)) \setminus \{0\}$ satisfies $(\mathrm{ms})_2$, k is divisible by 12.*

Proposition 3.6. *Suppose that $F \in M_k(\Gamma_2)$ satisfies $(\mathrm{MS})_p$ for a prime p. Put $f = \Phi(F)$ and $\varphi = \mathcal{W}(F)$. Then, for any prime number p, f (respectively φ) satisfies $(*)_p$ (respectively $(\mathrm{ms})_p$) and $\epsilon_p(F) = \epsilon_p(f) = \epsilon_p(\varphi)$.*

3.4 Proof of Theorem 1.1

By Proposition 3.6 and Proposition 3.3, we obtain the following result, from which Theorem 1.1 follows.

Proposition 3.7. *Assume that* $F \in M_k(\Gamma_2)$ *satisfies* $(\mathrm{MS})_2$ *and* $f = \Phi(F) \neq 0$. *Then* $f = c\,\Delta^r$ $(c \in \mathbb{C}^\times, r \in \mathbb{Z}_{\geq 0})$. *In particular, the weight* k *is divisible by* 12.

3.5 Proof of Theorem 1.4

Theorem 1.4 is a direct consequence of Theorem 3.1 and the following result.

Proposition 3.8. *If* $F \in M_k(\Gamma_2)$ *satisfies* $(\mathrm{MS})_2$ *and* $\mathcal{W}(F) \neq 0$, *then* k *is divisible by* 12.

PROOF. Let $\varphi = \mathcal{W}(F)$. Then $\varphi \neq 0$ and φ satisfies $(\mathrm{ms})_2$. The proposition now follows from Proposition 3.5. □

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