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ANALYTIC PROPERTIES OF SHINTANI ZETA FUNCTIONS

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ABSTRACT. In this note, we describe various theoretical results, numerical computations, and speculations concerning the analytic properties of the *Shintani zeta functions* associated to the space of binary cubic forms. We describe how these zeta functions *almost* fit into the general analytic theory of zeta and L-functions, and we discuss the relationship between this analytic theory and counting problems involving cubic rings and fields.

1. INTRODUCTION

In this note we will discuss the analytic theory of the *Shintani zeta functions* associated to the space of binary cubic forms. These zeta functions are defined by the equation

$$\xi^{\pm}(s) := \sum_{x \in \mathrm{SL}_2(\mathbb{Z}) \backslash V_{\mathbb{Z}}} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s}, \tag{1.1}$$

where the sum is over equivalence classes of *integral binary cubic forms* (to be described in Section 2). As we will describe, these zeta functions are unique and therefore interesting from an analytic point of view. Simultaneously, work of Davenport-Heilbronn [10] and Delone-Faddeev [11] establishes that $\xi^{\pm}(s)$ are essentially the generating functions for *cubic rings*, lending arithmetic interest to these zeta functions as well.

The subject begins with the pioneering work of Shintani [27], who proved that these zeta functions enjoy an analytic continuation and a functional equation. The shape of this functional equation (see (3.4)) is a bit unusual, and in particular involves a matrix, so that $\xi^{+}(s)$ and $\xi^{-}(s)$ are not independent. However, followup work by Datskovsky and Wright [36, 12], Ohno [22], and Nakagawa [21] illustrated how these zeta functions may be very nearly brought into the existing analytic framework (see (3.13)), although with a couple of interesting anomalies. In particular, they do not appear to be related to L-functions with Euler products in any simple way, and, curiously, their analytic continuations have poles at $s = 5/6$ (as well as at $s = 1$).

Therefore, from an analytic perspective, the Shintani zeta functions might be regarded as "black sheep" in the family of zeta functions, which motivated us to further study their analytic properties. The main objective of this paper is to discuss these zeta functions from an analytic point of view. In one sense, our investigations were less successful than we hoped: much of the existing analytic machinery is not sensitive to any particular information about the Shintani zeta function, and so our answers to some questions will be limited to numerical computations and speculations. However, we did obtain a couple of interesting theoretical results in this direction, and we will discuss these as well.

We will also be interested in the relationship between the analytic theory and arithmetic applications. This subject starts with a famous result of Davenport and Heilbronn [10]. Let $N_3^{\pm}(X)$ count the number of cubic fields K with $\pm\mathrm{Disc}(K) < X$. Davenport and Heilbronn proved that

$$N_3^{+}(X) = \frac{1}{12\zeta(3)} X + o(X), \qquad N_3^{-}(X) = \frac{1}{4\zeta(3)} X + o(X). \tag{1.2}$$

Although Davenport and Heilbronn did not use Shintani zeta functions in their proof, Datskovsky and Wright gave such a proof [13], and it extended to counting cubic extensions of any global field. In the context of their proof, the main terms in (1.2) correspond to the poles of $\xi^{\pm}(s)$ at $s = 1$.

After Davenport and Heilbronn published their results, numerical computations were performed, and it turned out that the asymptotics in (1.2) were an extremely poor match for the data. Many speculated that the proof of (1.2) was wrong. However, Datskovsky-Wright [13] and Roberts [24] observed that this discrepancy is naturally explained by the theory of Shintani zeta functions. As these zeta functions have poles at $s = 5/6$, these authors conjectured that the counting functions in (1.2) have secondary terms of order $X^{5/6}$, where the constants are given by appropriate limits of residues of adelic Shintani zeta functions.

Our conference talk in Tokyo consisted largely of vague and optimistic speculation. However, after some very productive conversations with Takashi Taniguchi, whom we met at the conference, we were able to push our ideas further and convert their heuristic into a proof. In particular, we obtained the conjectured secondary terms of order $X^{5/6}$ in (1.2), with error terms of $O(X^{19/24+\epsilon})$. Because this work will appear in [33], we will use this paper to concentrate largely on our earlier computations and speculations. However, we summarize our proof, and some related results, in Section 5.

Remark. Roberts' conjecture was also proved independently by Bhargava, Shankar, and Tsimerman [5]. Their proof is similar in spirit to Davenport and Heilbronn's original proof, and does not use the theory of Shintani zeta functions.

Encouraged by our experience in Tokyo, we will engage in some further speculation up front. Let $a^{\pm}(n)$ denote the nth coefficient in (1.1). As we will see, $a^{\pm}(n)$ is essentially a bound for the number of cubic fields of discriminant n. (It is also related to the amount of 3-torsion in the class group $\mathrm{Cl}(\mathbb{Q}(\sqrt{n}))$.) What bounds can we prove for $a^{\pm}(n)$?

The best known bound, due to Ellenberg and Venkatesh [16], is

$$a^{\pm}(n) \ll n^{1/3+\epsilon}. \tag{1.3}$$

However, one expects the true upper bound to be on the order of n^{ϵ}, or perhaps still smaller. We therefore naturally ask: Can one prove a better result using the theory of Shintani zeta functions?

Our question was met with some skepticism, and was the subject of several failed attempts by the author. Nevertheless, we cautiously hope that such a proof may be possible.

Organization of this paper. We begin in Section 2 by defining the representation V of $\mathrm{SL}_2(\mathbb{Z})$ referred to in (1.1). We also describe how this representation is related to the problem of counting cubic rings and fields. In Section 3 we give an introduction to the analytic theory of Shintani zeta functions, drawing on work of Shintani, Datskovsky-Wright, Ohno, and Nakagawa. In Section 4 we consider problems involving the distribution of the zeroes. This involves a variety of theoretical results, numerical computations, speculations, and descriptions of failed approaches. Finally, in Section 5 we discuss how analytic methods can be used to prove results about the distribution of cubic fields. In particular, we discuss our proof of Roberts' conjecture, as additional results which motivated our approach. We also discuss recent work of Taniguchi [30] which is needed in our proof. As the details will appear elsewhere, we will be brief.

Notation. For the most part our choice of notation is standard. However, we have adopted the notation $\xi^{\pm}(s)$ for the basic Shintani zeta functions, although $\xi_1(s)$ and $\xi_2(s)$ seem to be more common. We have written $\xi^{\mathrm{add}}(s)$ and $\xi^{\mathrm{sub}}(s)$ for the diagonalized Shintani zeta functions defined in (3.10) and (3.11).

Acknowledgments

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2. Binary cubic forms and the Delone-Faddeev correspondence

In this section, we define the notation used in (1.1), and describe how this lattice is related to counting problems involving cubic rings and fields. We refer to Bhargava's paper [3] (see also [5]) for an elegant summary and reformulation of this theory, and give only a brief summary.

The lattice $V_{\mathbb{Z}}$ of *integral binary cubic forms* is defined by

$$V_{\mathbb{Z}} := \{au^3 + bu^2v + cuv^2 + dv^3 \ : a, b, c, d \in \mathbb{Z}\}, \tag{2.1}$$

and the *discriminant* of a cubic form is given by the usual equation

$$\mathrm{Disc}(f) = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd. \tag{2.2}$$

Furthermore, there is a natural action of $\mathrm{GL}_2(\mathbb{Z})$ (as well as $\mathrm{SL}_2(\mathbb{Z})$) on $V_{\mathbb{Z}}$, given by

$$(g \cdot f)(u, v) = \frac{1}{\det g} f((u, v) \cdot g). \tag{2.3}$$

A cubic form f is *irreducible* if $f(u, v)$ is irreducible as a polynomial over $\mathbb{Q}$, and it is *nondegenerate* if $\mathrm{Disc}(x) \neq 0$. It is proved in [27] that $\mathrm{Stab}(x)$, the stabilizer of x in $\mathrm{SL}_2(\mathbb{Z})$, is an abelian group of order one or three for any nondegenerate x.

We note that if we consider this group action over $\mathbb{R}$ or $\mathbb{C}$ instead of Z, then V becomes a *prehomogeneous vector space.* Namely, the group action is almost transitive, having only finitely many Zariski open orbits. The theory of general prehomogeneous vector spaces and their associated zeta functions has been studied by many authors, notably Sato and Shintani [26]. It would be very interesting to extend the analysis described here to other prehomogeneous vector spaces, such as those appearing in Bhargava's recent work [4]. One should see the book of Yukie [37] for some results in the quartic case.

The space of cubic forms V is lent interest by its relation to cubic rings and fields. This was established by the work of Delone-Faddeev [11] and Davenport-Heilbronn [10]. We define a *cubic ring* to be any ring which is free of rank 3 as a $\mathbb{Z}$-module. The following result was proved by Delone and Faddeev [11], with an extension to the degenerate case by Gan, Gross, and Savin [17]:

Theorem 2.1 (Delone-Faddeev, 1964). *There is a canonical, explicit, discriminant-preserving bijection between the set of cubic rings up to isomorphism and the set of* $\mathrm{GL}_2(\mathbb{Z})$*-equivalence classes of integral binary cubic forms. Furthermore, under this correspondence, irreducible cubic forms correspond to orders in cubic fields.*

See [3] (among other sources) for an explicit and simple description of the bijection.

The Delone-Faddeev correspondence was also essentially shown by Davenport and Heilbronn [10] in their proof of (1.2). Their proof, as simplified and streamlined by Bhargava [3], is as follows.

First of all, one chooses a fundamental domain for the action of $\mathrm{GL}_2(\mathbb{Z})$ on V with the property that nearly all of the reducible points are in the cusp. Cutting off the cusp, one proves that the number of lattice points remaining is asymptotically equal to the volume of the fundamental domain. One therefore obtains asymptotics for the number of cubic *orders* of bounded discriminant.

To restrict the count to *maximal* orders only, one observes that a cubic order is maximal if and only if it satisfies a maximality condition at each prime p. This condition, in turn, may be checked by reducing the corresponding cubic form modulo p^2. Therefore, for each p, the set of cubic orders which are maximal at p has a density in the set of all cubic orders, and the product of all these densities converges to a positive limit. It then follows, at least heuristically, that the number of maximal cubic orders of bounded discriminant is equal to the product of the previous asymptotic and this limit, and Davenport-Heilbronn use a sieve to make this argument rigorous.

Remark. The Shintani zeta function counts $\mathrm{SL}_2(\mathbb{Z})$-orbits of cubic forms rather than $\mathrm{GL}_2(\mathbb{Z})$-orbits, and it weights some of them by a factor of $1/3$, so it is not exactly the generating series for cubic rings. The discrepancies depend on the Galois group of the splitting field of the cubic form. If we hope to count fields, then the counting functions for all Galois groups other than Sym(3) are extremely well understood, and so these discrepancies will not impede our analysis.

3. Shintani zeta functions

Recall that the Shintani zeta functions are defined by the Dirichlet series

$$\xi^{\pm}(s) := \sum_{x \in \mathrm{SL}_2(\mathbb{Z}) \backslash V_{\mathbb{Z}}} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s}, \tag{3.1}$$

where the count is over points of positive or negative discriminant respectively, and the lattice $V_{\mathbb{Z}}$ is defined by (2.1). The functional equation will relate $\xi^{\pm}(s)$ to *dual* zeta functions, defined as follows: The dual lattice to $V_{\mathbb{Z}}$ is

$$\widehat{V}_{\mathbb{Z}} := \{au^3 + bu^2v + cuv^2 + dv^3 \ : a, d \in \mathbb{Z}, \quad b, c \in 3\mathbb{Z}\}, \tag{3.2}$$

and one checks that $\mathrm{SL}_2(\mathbb{Z})$ acts on $\widehat{V}_{\mathbb{Z}}$ as well as $V_{\mathbb{Z}}$. The dual Shintani zeta functions are defined by

$$\widehat{\xi}^{\pm}(s) := \sum_{x \in \mathrm{SL}_2(\mathbb{Z}) \backslash \widehat{V}_{\mathbb{Z}}} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s}. \tag{3.3}$$

Shintani proved [27] that all of these Dirichlet series converge absolutely for $\Re(s) > 1$, enjoy analytic continuation to all of $\mathbb{C}$ with poles only at $s = 1$ and $s = 5/6$, and satisfy the matrix functional equation

(3.4)
$$\begin{pmatrix} \xi^+(1-s) \\ \xi^-(1-s) \end{pmatrix} = \Gamma\Big(s - \frac{1}{6}\Big)\Gamma(s)^2\Gamma\Big(s + \frac{1}{6}\Big) 2^{-1} 3^{6s-2} \pi^{-4s} \times \begin{pmatrix} \sin 2\pi s & \sin \pi s \\ 3 \sin \pi s & \sin 2\pi s \end{pmatrix} \begin{pmatrix} \widehat{\xi}^+(s) \\ \widehat{\xi}^-(s) \end{pmatrix}.$$

He also explicitly computed all of the residues.

We will give a brief summary of the proof. Shintani defined the *completed zeta function*

$$Z(f, s) := \int_{\mathrm{GL}_2^+(\mathbb{R})/SL_2(\mathbb{Z})} (\det\ g)^{6s} \sum_{x \in V'_{\mathbb{Z}}} f(gx) dg, \tag{3.5}$$

where f is a suitable test function, and $V_{\mathbb{Z}}'$ consists of those $x \in V_{\mathbb{Z}}$ with nonzero discriminant.[1] (The exponent 6 arises because $\mathrm{Disc}(gx) = (\det g)^6 \mathrm{Disc}(x)$.) One defines $\widehat{Z}(f,s)$ analogously by summing over $\widehat{V}_{\mathbb{Z}}$.

It is then readily shown that

$$Z(f,s) = \frac{1}{4\pi}\xi^+(s)\int_{V^+} |P(x)|^{s-1} f(x)dx + \frac{1}{12\pi}\xi^-(s)\int_{V^-} |P(x)|^{s-1} f(x)dx, \tag{3.6}$$

where V^+ and V^- denote those portions of $V_{\mathbb{R}} = V \otimes \mathbb{R}$ with positive and negative discriminant respectively.

Shintani then proves the functional equation

$$Z(f,s) = \widehat{Z}(\widehat{f}, 1-s). \tag{3.7}$$

This is proved by Poisson summation, as for the the Riemann zeta function. However, in this case the zero locus consists of an infinite number of SL_2-orbits rather than a single point. Shintani evaluates the appropriate integrals by introducing an Eisenstein series, and proving that one may recover the original integrals by taking an appropriate limit. The Eisenstein series, in turn, incorporates extra averaging which allows for the evaluation of the modified integrals.

Shintani also proves that the integrals occuring in (3.6) have analytic continuations and a functional equation similar to (3.4). Put together, these functional equations allow him to prove (3.4) and the rest of his theorem.

Shintani's work was followed up by Datskovsky and Wright [36, 12, 13], Ohno [22], Nakagawa [21], and Taniguchi [30], among many others. In Section 5 we will discuss Datskovsky-Wright's theory of the *adelic* Shintani zeta function. Here we will discuss how the functional equation for the Shintani zeta functions may be brought in line with the general analytic theory of zeta functions.

We begin with an observation of Datskovsky and Wright in [12], that the matrix occuring in (3.4) has a simple diagonalization. In view of this diagonalization, Ohno [22] performed numerical computations which led him to conjecture that the standard and dual Shintani zeta functions are related by the simple equations

$$\widehat{\xi}^+(s) = 3^{-3s}\xi^-(s), \tag{3.8}$$

$$\widehat{\xi}^-(s) = 3^{1-3s}\xi^+(s). \tag{3.9}$$

Shortly thereafter, Ohno's conjecture was proved by Nakagawa [21].

Combining all of these results yields an elegant reformulation of Shintani's functional equation. Define *diagonalized Shintani zeta functions*

$$\xi^{\mathrm{add}}(s) := 3^{1/2}\xi^+(s) + \xi^-(s), \tag{3.10}$$

$$\xi^{\mathrm{sub}}(s) := 3^{1/2}\xi^+(s) - \xi^-(s), \tag{3.11}$$

and *completed zeta functions*

$$\Lambda^{\mathrm{add}}(s) := \left(\frac{432}{\pi^4}\right)^{s/2} \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{12}\right)\Gamma\left(\frac{s}{2}-\frac{1}{12}\right)\xi^{\mathrm{add}}(s), \tag{3.12}$$

$$\Lambda^{\mathrm{sub}}(s) := \left(\frac{432}{\pi^4}\right)^{s/2} \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}+\frac{5}{12}\right)\Gamma\left(\frac{s}{2}+\frac{7}{12}\right)\xi^{\mathrm{sub}}(s).$$

[1] A technical point: Shintani omits the factor of $\det g$ from his definition of (2.3), in contrast to Bhargava and Datskovsky-Wright. This choice of normalization is reflected in the functional equation for the completed zeta function, but not in the final results.

Then, we have

$$\Lambda^{\mathrm{add}}(1-s) = \Lambda^{\mathrm{add}}(s), \tag{3.13}$$

$$\Lambda^{\mathrm{sub}}(1-s) = \Lambda^{\mathrm{sub}}(s). \tag{3.14}$$

One may compare these zeta functions with the general formalism in, say, Chapter 5 of Iwaniec and Kowalski's book [20]. For the most part, these diagonalized zeta functions fit this framework. However, there are a couple of differences. These zeta functions don't have Euler products, or any obvious simple relation to Euler products.[2] It was suggested to the author that the presence of the negative number $-1/12$ in (3.12) is perhaps a bit unusual. And, perhaps most significantly, there is the pole at $s = 5/6$, which is retained in $\xi^{\mathrm{add}}(s)$ but disappears in $\xi^{\mathrm{sub}}(s)$.

The analytic properties of these zeta functions invite a variety of philosophical questions. For example, is some conceptual way of predicting that the linear combinations in (3.10) and (3.11) are those which should enjoy simple functional equations? And is there some reason that $\xi^{\mathrm{add}}(s)$ has a pole at $s = 5/6$ but $\xi^{\mathrm{sub}}(s)$ does not? Still more questions are posed in Datskovsky and Wright's papers. Unfortunately, we are currently unable to offer any answers.

4. The distribution of the zeroes

Now that we have described four interesting zeta functions (Shintani's original zeta functions, and the diagonalized functions), we decided to investigate them further from an analytic point of view. In particular, we investigated the distribution of their zeroes. A standard formula establishes that the number of zeroes with $|\Im(s)| < T$ is given by

$$N(T) = \frac{T}{\pi} \log\left(\frac{432\, T^4}{(2\pi e)^4}\right) + O(\log T). \tag{4.1}$$

Should these zeroes all lie on the half line, or even within the critical strip?

4.1. Epstein zeta functions and their relatives.

We began by looking for some reasonable basis for making guesses. Most interesting examples of zeta and L-functions have Euler products, and the existence of an Euler product is generally expected to affect the distribution of the zeroes. Accordingly we turned to the theory of *Epstein zeta functions*, which don't have Euler products.

The *Epstein zeta functions* are defined by

$$\zeta_Q(s) = \sum_{(u,v)\neq(0,0)} (au^2 + buv + cv^2)^{-s},$$

where $Q(u,v) = au^2 + buv + cv^2$ is a positive definite quadratic form. These enjoy analytic continuation to the whole complex plane, with the functional equation

$$\left(\frac{\sqrt{|D|}}{2\pi}\right)^s \Gamma(s)\zeta_Q(s) = \left(\frac{\sqrt{|D|}}{2\pi}\right)^{1-s} \Gamma(1-s)\zeta_Q(1-s),$$

where $D = b^2 - 4ac < 0$ is the discriminant of Q. Furthermore, if $a, b, c \in \mathbb{Z}$, then $\zeta_Q(s)$ constitutes one piece of the Dedekind zeta function $\zeta_{\mathbb{Q}(\sqrt{D})}(s)$, corresponding to an element of the class group, and $\zeta_Q(s)$ only has an Euler product if $h(D) = 1$. However, $\zeta_Q(s)$ does have a representation as a finite linear combination of Hecke L-functions. So we should expect any analogy with Shintani zeta functions to be inexact.

[2] However, see [12] for an interesting expression for these zeta functions as infinite sums of Euler products, which is valid for $\Re(s) > 1$.

However, the zeroes of Epstein zeta functions have been studied by a variety of authors, and we hoped that the analogy might prove fruitful. Our basic reference is the excellent survey article by Hejhal [19]. (We also recommend Hejhal's article for an interesting foray into the history of computational number theory: the remarkable CRAY-1 supercomputers which they used had *over seven million bytes* of memory.)

Some of the results discussed in [19] are as follows. All asymptotics refer to the number of zeroes of a fixed Epstein zeta function $\zeta_Q(s)$ with integral coefficients, such that $|\Im(s)| < T$.

- (Potter and Titchmarsh [23]) At least $\gg T$ zeros lie on the critical line.
- (Voronin [35]) At most $O(T)$ zeros lie to the right of any fixed line $Re(s) = \sigma > 1/2$.
- (Davenport and Heilbronn [9]) Unless $\zeta_Q(s)$ is the Dedekind zeta function of a quadratic field, at least $\gg T$ nontrivial zeros are outside the critical strip.
- (Bombieri and Hejhal [6]) Under certain hypotheses on the Hecke L-functions associated to $\mathbb{Q}(\sqrt{D})$ (including, but not limited to, GRH), almost all zeroes of $\zeta_Q(s)$ lie on the critical line.

These sorts of results are interesting in their own right, and they are also related to interesting arithmetic questions. For example, in [29] Stark discusses the relationship between these results and the class number one problem. If D is a negative integer, then [9] implies that $\zeta_Q(s)$ has zeroes outside the critical strip if and only if $h(D) = 1$. Therefore, an independent characterization of those Q for which $\zeta_Q(s)$ has such zeroes would lead to a new solution of the class number one problem.

Stark proposes that it would be desirable to FIND A PURELY ANALYTIC PROOF[3] of this characterization. He suggests that such a proof might extend to other fixed class numbers, and ("if we were really lucky") perhaps even effectively approach the strength of Siegel's theorem.

Although we have not answered Stark's challenge, it did add motivation to our related investigations. We began with some numerical experiments. We reconstructed a list of the first million coefficients of the Shintani zeta functions from a table of cubic fields computed by Belabas [1]. We then availed ourselves of the ComputeL and L computational packages, by Dokchitser [14, 15] and Rubinstein [25] respectively. These packages implement algorithms to compute L-functions and their derivatives inside or outside the critical strip. They are quite effective near the real axis, even if only a few hundred Dirichlet coefficients are known. As one moves away from the real axis, the computational complexity of these algorithms grows quickly, both in terms of running time and number of Dirichlet coefficients required. Using a typical desktop computer, Rubinstein's software allowed us to run computations up to approximately $\Im(s) = 1000$ in a reasonable amount of time.

4.2. **Zeroes inside the critical strip.** Using this software, we were able to find the low-lying zeroes of each of the Shintani zeta functions by computing contour integrals of $\xi'(s)/\xi(s)$ over appropriate rectangles. We observed that none of the Shintani zeta functions satisfy the Riemann hypothesis, but that all of them had zeroes on the critical line.[4]

For example, the first few zeroes of $\xi^+(s)$ on the critical line are at

$$\begin{array}{l} 0.5 + 4.745125599327\cdots i \\ 0.5 + 6.962286575567\cdots i \\ 0.5 + 8.4742944491274\cdots i \\ 0.5 + 10.152261066735\cdots i, \end{array}$$

[3]Indeed, Stark proposes this problem in all capital letters.

[4]In the case of $\xi^{\pm}(s)$, we did not check that these zeroes are not simply *close* to the critical line. In the case of $\xi^{\mathrm{add}}(s)$ and $\xi^{\mathrm{sub}}(s)$, the functional equation forces any zeroes off the line to occur in pairs, so we can be sure that these zeroes are exactly on the line.

and $\xi^+(s)$ also has a pair of zeroes at

$$0.18579\cdots + 7.05984\cdots i, \quad 0.81420\cdots + 7.05984\cdots i.$$

The distribution of the zeroes of $\xi^-(s)$ is roughly similar, although there is a lower zero at $0.5 + 1.32\cdots i$, and the first exception to RH is higher. Soundararajan remarked to the author that the ordinates of the zeroes of $\xi^+(s)$ are very close to one third those of the Riemann zeta function. However, we have no way of predicting this phenomenon, and it did not seem to hold up as we computed more zeroes.

We chose to investigate $\xi^+(s)$ in more detail, and we used Rubinstein's software to compute the first 1788 zeroes of $\xi^+(s)$ on the critical line, up to a height of $\Im(s) \approx 953$. We predict that we have missed roughly 1570 zeroes off the critical line, based on the classical formula

$$N(T) = \frac{T}{\pi} \log\left(\frac{432\, T^4}{(2\pi e)^4}\right) + O(\log T) \tag{4.2}$$

for the number of zeroes with $|\Im(s)| < T$. In particular, up to $\Im(s) = 953$, roughly 53% of the zeroes lie on the critical line. Moreover, our data suggests that this percentage is roughly consistent for $\Re(s) < 953$ as well.

Based on Bombieri and Hejhal's work, we are inclined to guess that almost all of the zeroes of $\xi^{\pm}(s)$ should lie on the critical line as $\Im(s) \to \infty$. However, we could easily be wrong. The zeta functions considered by Bombieri and Hejhal are finite sums of L-functions with Euler products, and so their analysis does not apply, even heuristically.

With more numerical data, we could conduct a variety of further experiments. For example, this would allow us to investigate our guess above. To give another example, we could compute the pair correlation of the zeroes. In the classical setting, work of Montgomery and many other authors predicts that such pair correlations should be related to distributions of eigenvalues of random matrices. In the setting of Epstein zeta functions, Farmer and Koutsoliotas (unpublished, in progress) numerically observed that the zeroes may instead be modeled by those of random self-reciprocal polynomials. With more data in hand, it would be interesting to see if their observations carry over to Shintani zeta functions.

It would also be of interest to prove the existence of infinitely many zeroes on the critical line. However, our attempts were immediately stymied: this has not yet been proven for any zeta or L-function of degree greater than two, regardless of the existence of an Euler product.

We briefly mention a couple of other unsuccessful investigations. We used ComputeL to compute a variety of possibly "special" values of the Shintani zeta functions, such as at $s = 2$ and $s = 1/2$. However, we did not find any obviously interesting behavior. We also tried to a prove zero density estimate, along the lines of Voronin's theorem, but the methods we tried did not yield any nontrivial results.

4.3. **Zeroes outside the critical strip.** We also looked for zeroes outside the critical strip. As there is no obvious reason for them not to exist (i.e., an Euler product), we expected to find them. For $\xi^{\text{add}}(s)$ and $\xi^{\text{sub}}(s)$, this was quickly accomplished. However, we did not find zeroes of $\xi^+(s)$ and $\xi^-(s)$ outside the strip.[5]

However, we still believed they should exist. Motivated by Davenport and Heilbronn's work [9], Soundararajan and the author [28] developed a method which allows us to numerically prove the existence of zeroes, even if we cannot find them. Although we expect the method to almost

[5]We did not push our computational tools to their limits, as we found it more interesing to develop a theoretical approach.

always work in principle, we cannot always reduce the problem to a manageable computation. In particular, we proved the existence of zeroes for $\xi^-(s)$ but not for $\xi^+(s)$.

Our method is quite general. Suppose we are given a Dirichlet series $A(s) := \sum_n a(n)n^{-s}$ with real coefficients and abscissa of absolute convergence $\Re(s) = 1$. (Our theorem assumes nothing about analytic continuation or a functional equation, but these will be needed in our application.) Our main result is the following:

Theorem 4.1. [28] *Suppose that there is a completely multiplicative function $\chi(n)$, taking values in ± 1, such that*

$$A(\sigma, \chi) := \sum_n a(n)\chi(n)n^{-\sigma} < 0 \tag{4.3}$$

for some real $\sigma > 1$. Assume further that if n_0 is the smallest integer for which $a(n_0) \neq 0$, then $a(n_0)\chi(n_0) > 0$.

Then $A(s)$ has infinitely many zeroes outside the critical strip.

Sketch proof. First of all, an "almost periodicity" argument using Rouché's theorem shows that it is enough to prove that $A(s)$ has *one* zero outside the critical strip. Secondly, we show that the function $\chi(n)$ can be well approximated, uniformly for $n < N$ for any N, by the function n^{it} for some choice of t.

Assuming (4.3), a continuity argument implies that $A(\sigma, \chi) = 0$ for some σ (hence the requirement that χ and the $a(n)$ be real valued). It then follows that $A(\sigma - it)$ is very close to zero, and Rouché's theorem implies that $A(s)$ has a zero near $\sigma - it$. □

The Shintani zeta functions $\xi^{\pm}(s)$ illustrate a typical application of our result. We will show that the negative discriminant Shintani zeta function

$$\xi^-(s) = \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{7^s} + \cdots \tag{4.4}$$

has zeroes outside the critical strip. Define the function $\chi(n)$ by $\chi(3) = 1$, and $\chi(p) = -1$ for $p \neq 3$. We compute that

$$\sum_{n \le 10^6} a(n)\chi(n)n^{-1.3} = -0.162\ldots$$

and

$$\sum_{n > 10^6} |a(n)\chi(n)n^{-1.3}| < \sum_{n > 10^6} a(n)n^{-1.3} = 0.06\ldots \tag{4.5}$$

It follows that $\sum_n a(n)\chi(n)n^{-1.3} < 0$, and we're done.

Notice that we implicitly used the analytic continuation and functional equation for $\xi^-(s)$. To compute the tail of the Dirichlet series in (4.5), we needed to compute $\xi^-(1.3)$ to very high accuracy. We did this using Dokchitser's ComputeL [15], which uses the analytic continuation and functional equation in an essential way.

For the positive discriminant series

$$\xi^+(s) = 1/3 + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{8^s} + \frac{1}{9^s} + \cdots, \tag{4.6}$$

we were not able to prove a similar result. Experimenting, we found a choice of $\chi(n)$ for which

$$\sum_{n \le 10^6} a(n)\chi(n)n^{-1.1} = -0.11\cdots$$

However, we were unable to obtain a negative value for any substantially larger value of σ. We computed that

$$\sum_{n>10^6} a(n)n^{-1.1} > 7,$$

and although this constitutes some evidence for the existence of zeroes, it falls well fell short of a proof. By computing, say, the first trillion $a(n)$, we might be able to produce a proof, but this seems a bit unreasonable. We hope instead to improve our criterion, perhaps to somehow allow the use of complex-valued χ.

5. Sieve methods, almost prime discriminants, and Roberts' conjecture

In this last section, we develop a method to obtain explicit functional equations for variants of the Shintani zeta function. We also explain how to use these in combination with sieve methods to obtain a variety of interesting results (including Roberts' conjecture). This work was carried out in parallel by Taniguchi [30] and the present author [32, 33], and our manuscripts are currently in preparation. We hasten to mention our gratitude to Taniguchi for his many comments and suggestions.

The method begins with the work of Datskovsky and Wright [36, 12, 13], who developed an adelic version of the Shintani zeta function. Their work was continued by Taniguchi [30], who made much of Datskovsky and Wright's work explicit and computed many of the quantities which appear in the resulting functional equations. The idea is that one may insert a variety of conditions into the definition of the Shintani zeta function, and still obtain analytic continuation and explicit functional equations. We will describe the general formalism first, but the reader may wish to skip to the examples.

To explain our approach, we recall Tate's work in the classical setting, given in his thesis [31]. Let $f \in \mathcal{S}(\mathbb{A}_{\mathbb{Q}})$ be a Schwartz function, and define a zeta function

$$\zeta(f, s) := \int_{\mathbb{A}_{\mathbb{Q}}^\times} f(\mathfrak{a})|\mathfrak{a}|^{-s} d^\times \mathfrak{a}. \tag{5.1}$$

Then $\zeta(f, s)$ has analytic continuation with a functional equation

$$\zeta(f, s) = \zeta(\widehat{f}, 1 - s), \tag{5.2}$$

where $\widehat{f}$ is the Fourier transform of f. One recovers the Riemann zeta function, its analytic continuation, and the functional equation by choosing f to be the characteristic function of $\mathbb{Z}_p$ at all p-adic places, and $f(x) = e^{-\pi\xi^2}$ at the infinite place. In particular, with this choice $\widehat{f} = f$ and $\zeta(f, s)$ is the usual (completed!) Riemann zeta function.

However, (5.2) continues to hold for other choices of f. For example, for any integers a and q, one obtains analytic continuation and a functional equation for the Dirichlet series $\sum_{n\equiv a \pmod q} n^{-s}$, by making a different choice of f at those p-adic places dividing q. The functional equation will no longer be self-dual, but will involve an exponential sum over $\mathbb{Z}/n\mathbb{Z}$.

These facts may be of course proved by other means. But all of this discussion generalizes to Shintani zeta functions, where other proofs of these facts are not known. This is the work of Datskovsky-Wright and Taniguchi. The *adelic Shintani zeta function* (over $\mathbb{Q}$) is defined by

$$Z(f, s) := \int_{\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})/\mathrm{GL}_2(\mathbb{Q})} |\det g|^{-s} \Big(\sum_{x \in V'_{\mathbb{Q}}} f(gx) \Big) dg. \tag{5.3}$$

In [36], Wright proves the functional equation

$$Z(f,s)=Z(\widehat{f},2-s), \tag{5.4}$$

and in [12] Datskovsky and Wright prove that a standard choice of f recovers the original Shintani zeta functions.[6] However, we may obtain variants of the Shintani zeta function by making other choices of f.

In particular, for any integer d, suppose $\mathcal{D}$ is any $\mathrm{GL}_2(\mathbb{Z}/d\mathbb{Z})$-invariant subset of $V_{\mathbb{Z}/d\mathbb{Z}}$. We define a restricted Shintani zeta function

$$\xi^{\pm}_{\mathcal{D}}(s)=\sum_{n\geq 1}a^{\pm}_{\mathcal{D}}(n)n^{-s}:=\sum_{\substack{x\in \mathrm{SL}_2(\mathbb{Z})\backslash V_{\mathbb{Z}}\\ x \text{ (mod } d)\in\mathcal{D}}}\frac{1}{|\mathrm{Stab}(x)|}|\mathrm{Disc}(x)|^{-s}. \tag{5.5}$$

As Datskovsky-Wright and Taniguchi proved, these Shintani zeta functions also enjoy analytic continuation and a functional equation. To describe the functional equation, let $\Phi_{\mathcal{D}}(x)$ be the characteristic function of $\mathcal{D}$. Its *dual* is defined by the equation

$$\widehat{\Phi}_{\mathcal{D}}(x):=\frac{1}{d^4}\sum_{y\in V_{\mathbb{Z}/d\mathbb{Z}}}\Phi_{\mathcal{D}}(y)\exp(2\pi i[x,y]/d), \tag{5.6}$$

where

$$[x,y]:=x_4y_1-\frac{1}{3}x_3y_2+\frac{1}{3}x_2y_3-x_1y_4, \tag{5.7}$$

is the alternating bilinear form used to identify V with $\widehat{V}$, and x_i and y_j are the coordinates of x and y respectively.

This dual may be regarded as a *cubic Gauss sum*, and originates as a product of p-adic Fourier transforms of the function $\Phi_{\mathcal{D}}$. This integral reduces naturally to the finite sum above. We also note that $\mathcal{D}$ is multiplicative in the natural sense.

Fortuitously, these Gauss sums are typically quite small. Moreover, they typically enjoy nice formulas. These sums are analyzed in Taniguchi's work [30]. Thus far, Taniguchi has computed these sums in the cases of interest to us, and it appears that his method will allow us to compute $\widehat{\Phi}_{\mathcal{D}}$ for a variety of choices of $\mathcal{D}$.

We are now prepared to state the functional equation, essentially following [30]:

$$\begin{pmatrix}\xi^{+}_{\mathcal{D}}(1-s)\\ \xi^{-}_{\mathcal{D}}(1-s)\end{pmatrix}=\Gamma\left(s-\frac{1}{6}\right)\Gamma(s)^2\Gamma\left(s+\frac{1}{6}\right)2^{-1}3^{6s-2}\pi^{-4s}\begin{pmatrix}\sin 2\pi s & \sin\pi s\\ 3\sin\pi s & \sin 2\pi s\end{pmatrix}\begin{pmatrix}\widehat{\xi}^{+}_{\mathcal{D}}(s)\\ \widehat{\xi}^{-}_{\mathcal{D}}(s)\end{pmatrix}, \tag{5.8}$$

where

$$\widehat{\xi}^{\pm}_{\mathcal{D}}(s):=\sum_{x\in\mathrm{SL}_2(\mathbb{Z})\backslash\widehat{V}_{\mathbb{Z}}}\frac{1}{|\mathrm{Stab}(x)|}\widehat{\Phi}_{\mathcal{D}}(x)\left(|\mathrm{Disc}(x)|/d^4\right)^{-s}. \tag{5.9}$$

Notice that the shape of the functional equation is completely uniform, all $\mathcal{D}$-dependence having been incorporated into the definition (5.9). This means in particular that $\xi^{\pm}_{\mathcal{D}}$ may be diagonalized in exactly the same way as described before.

[6] Datskovsky and Wright relate $Z(f,s)$ to $\xi^{\pm}(s/2)$.

We can apply this to counting problems using standard analytic methods. We begin with *Perron's formula*, which establishes that

$$\sum_{n<X} a_{\mathcal{D}}^{\pm}(n) = \int_{2-i\infty}^{2+i\infty} \xi_{\mathcal{D}}^{\pm}(s) X^s \frac{ds}{s}. \tag{5.10}$$

To estimate this, *in principle* we shift to the contour to the left, pick up main terms from the residues at $s = 1$ and $s = 5/6$, use the functional equation to rewrite the integrand in terms of (5.9), and estimate the resulting error. In practice, the integral will not converge at infinity, so one must either truncate the integral or incorporate a smoothing and unsmoothing process. We will use the latter approach, following work of Chandrasekharan and Narasimhan [8].

One expects, and in typical cases can prove,

$$\sum_{n<X} a_{\mathcal{D}}^{\pm}(n) = \mathrm{Res}_{s=1}\xi_{\mathcal{D}}^{\pm}(s)X + \frac{6}{5}\mathrm{Res}_{s=5/6}\xi_{\mathcal{D}}^{\pm}(s)X^{5/6} + O\Big(\max(S^{2/5}X^{3/5}, SX^{3/8})\Big), \tag{5.11}$$

where

$$S = \sum_{x \in V_{\mathbf{Z}/d\mathbf{Z}}} |\widehat{\Phi}_{\mathcal{D}}(x)|. \tag{5.12}$$

A straightforward reading of [8] leads one to expect an error term of $SX^{3/5}$, but a closer inspection reveals that one can do better. The residues in (5.11) can be computed from Taniguchi's tables [30]. This kind of estimate is already interesting, and we may combine it further with sieve methods to study the distribution of cubic rings.

We now illustrate our construction with two examples.

5.1. **The d-divisible Shintani zeta function.** We define the *d-divisible Shintani zeta function* by

$$\xi_d^{\pm}(s) := \sum_{\substack{x \in \mathrm{SL}_2(\mathbf{Z}) \backslash V_{\mathbf{Z}} \\ d \mid \mathrm{Disc}(x)}} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s}, \tag{5.13}$$

which is the usual Shintani zeta function with the additional divisibility condition. The discussion above establishes analytic continuation and the functional equation for $\xi_d^{\pm}(s)$.

As we claimed in our Tokyo lecture, we can use this zeta function to prove statements about prime and almost-prime cubic field discriminants. We have not yet finished this (and the current paper was subject to a deadline). Accordingly, this section will describe what we are reasonably confident that we can prove; the details will appear soon.

First of all, for squarefree d we have the bound [30]

$$\sum_{x \in V_{\mathbf{Z}/d\mathbf{Z}}} |\widehat{\Phi}(x)| \ll d^{1+\epsilon}. \tag{5.14}$$

By (5.11), the number of (irreducible) cubic orders with $\pm\mathrm{Disc}(x) < X$ and $d|\mathrm{Disc}(x)$ is, for squarefree $d \ll X^{3/8}$,

$$\frac{1}{2}\alpha^{\pm} X \prod_{p|d} \left(\frac{1}{p} + \frac{1}{p^2} - \frac{1}{p^3}\right) + \frac{3}{5}\gamma^{\pm} X^{5/6} \prod_{p|d} \left(\frac{1}{p} + \frac{1}{p^{4/3}} - \frac{1}{p^{7/3}}\right) + O(d^{2/5+\epsilon}X^{3/5}), \tag{5.15}$$

where

$$\alpha^+ = \pi^2/36, \quad \alpha^- = \pi^2/12, \quad \gamma^+ = \frac{\Gamma(1/3)^3\zeta(1/3)}{4\sqrt{3}\pi}, \quad \gamma^- = \sqrt{3}\gamma^+. \tag{5.16}$$

Later, we may also be able to handle the case where d is not squarefree.

The formula (5.15) takes the shape of a common assumption in the theory of sieve methods (see, e.g., [18] or [20]). This allows us to now prove results on prime and almost prime cubic discriminants using standard methods. It follows by the theory of the *Selberg sieve* that the number of cubic fields of *prime* discriminant $< X$ is $\ll X/\log X$, where the implied constant can be made explicit. It follows by Brun's theory of the *combinatorial sieve* that that the number of cubic fields of discriminant $< X$, whose prime factors are all greater than X^α, is equal to $(C^\pm/\alpha + o_\alpha(1))\frac{X}{\log X}$, where $C^\pm$ is an explicit constant, and the error term $o(1)$ can be made more precise. In particular, this implies the existence of infinitely many cubic fields whose discriminants have a bounded number of prime factors.

The precise details will be worked out in a forthcoming paper.

5.2. **Nonmaximal rings and Roberts' conjecture.** We conclude by returning to the Davenport-Heilbronn theorem. We will sketch our proof of the following conjecture of Datskovsky-Wright [13] and Roberts [24]:

Theorem 5.1. *We have*

$$N_3^\pm(X) = C_\pm \frac{1}{12\zeta(3)} X + K_\pm \frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)} X^{5/6} + O(X^{19/24+\epsilon}), \tag{5.17}$$

where $C_- = 3$, $C_+ = 1$, $K_- = \sqrt{3}$, *and* $K_+ = 1$.

Roberts' conjecture was also proved independently, with an error term of $O(X^{39/48+\epsilon})$, by Bhargava, Shankar, and Tsimerman [5]. Their proof uses a very geometric approach, and does not use the theory of Shintani zeta functions. In contrast, our proof is similar to the heuristic argument of Datskovsky-Wright and Roberts, which we now review.

We may count cubic fields by counting their corresponding maximal orders. We recall that Davenport and Heilbronn proved that a cubic order is maximal if it satisfies a maximality condition at each prime p, which may be detected by reducing the coefficients of the corresponding cubic form modulo p^2.

These maximality conditions may be detected by appropriate adelic test functions. It then follows by (5.11), and Taniguchi's formulas for the relevant residues and cubic Gauss sums, that the number of cubic orders which are maximal at all primes less than P is equal to

$$\frac{1}{2}\alpha^\pm X \prod_{p<P}\left(1-\frac{1}{p^2}\right)\left(1-\frac{1}{p^3}\right)+\frac{3}{5}\gamma^\pm X^{5/6}\prod_{p<P}\left(1-\frac{1}{p^{5/3}}\right)\left(1-\frac{1}{p^2}\right)+O\left(\Big(\prod_{p<P}p\Big)^{1+\epsilon}X^{3/8}\right), \tag{5.18}$$

where the constants are as in (5.16). Formally taking a limit as $P \to \infty$, and ignoring the nightmare error term[7], we obtain the two main terms of (5.17).

To prove Roberts' conjecture, we adapt an observation of Belabas, Bhargava, and Pomerance [2]. They obtained the main term in (5.17) with an error term of $X^{7/8+\epsilon}$, by writing

$$N_3^\pm(X) = \sum_{q\geq 1}\mu(q)N^\pm(q,X), \tag{5.19}$$

[7]Recall that the error term is in fact the larger of the term listed and $O\Big((\prod_{p<P}p)^{2/5+\epsilon}X^{3/5}\Big)$, but the error term above will be larger except for extremely small P.

where $N^{\pm}(q,X)$ counts the number of cubic orders which are not maximal at any prime dividing q. As $N^{\pm}(q,X) \ll X/q^{2-\epsilon}$ (an indispensibly useful fact), we may truncate the sum in (5.19) to $q \leq Q$ with error $O(X/Q^{1-\epsilon})$. They then estimated $N(q,X)$ using geometric methods.

We departed from their approach by estimating $N(q,X)$ using adelic Shintani zeta functions, as described above. However, the observant reader will note that this approach will not work exactly as stated. There are two reasons for this. The first is that the Shintani zeta function is not exactly the generating function for cubic orders. To deal with this problem, we replaced the quantity $N^{\pm}(q,X)$ with the analogous partial sum of the Shintani zeta functions, and at the end corrected for the final contribution from the various discrepancies. In particular, our estimates included the contribution from quadratic fields, which we could then simply subtract.

The second reason is that the approach described above only yields an error term of $O(X^{5/6+\epsilon})$! (One takes $Q = X^{1/6}$ to equalize the errors coming from the tail of (5.19) and from (5.11).) We improved this error term by incorporating the summation in (5.19) into our analytic method. Following [8], we estimated weighted versions of the quantities $N^{\pm}(q,X)$ (or, more precisely, the analogous quantities described above), where a discriminant n has the weight $(X-n)^3$. With this weighting, the application of Perron's formula in (5.10) is much smoother and yields better error terms. It is then proved in [8] that the original quantities $N^{\pm}(q,X)$ may be recovered as finite differences of the weighted quantities, within manageable error terms.

We improved our error term by using this weighting to our advantage. In particular, the error made in unweighting one term $N^{\pm}(q,X)$ is comparable to that made in unweighting the entire sum in (5.19). Accordingly, we postponed this unweighting until the very end, allowing us to prove a better error term than the sum of the error terms in (5.11) over $q \leq Q$.

We conclude by remarking that our methods seem likely to have further applications. For example, we may be able to count quartic or quintic extensions, or extensions over base fields other than $\mathbb{Q}$. Furthermore, if we are content with formulas for smoothed versions of these counting functions, we may be able to obtain secondary terms in these counting functions as well. We look forward to investigating these and other applications in the near future.

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