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DIFFERENTIAL OPERATORS AND SIEGEL-MAASS FORMS

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1. Introduction

The theory of non-analytic modular forms has a rich history and is closely connected with Siegel's theory of indefinite quadratic forms (see [9], [10], and [11]).

For a discrete subgroup Γ of $SL(2,\mathbb{R})$, a non-analytic modular form f of weight (α,β) is a real analytic function which satisfies the following conditions:

$$\Omega_{\alpha,\beta} f = 0, \tag{1}$$

$$f\,|_{(\alpha,\beta)}\, M = f \quad \text{for all } M \in \Gamma, \tag{2}$$

$$\int_{\Gamma\backslash\mathbb{H}} f(z)\overline{f(z)} y^{(\alpha+\beta)-2} dxdy < \infty. \tag{3}$$

Here the slash operator is defined as

$$f\,\Big|_{(\alpha,\beta)} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) := f\left(\frac{az+b}{cz+d}\right)(cz+d)^{-\alpha}(c\overline{z}+d)^{-\beta}$$

and

$$\Omega_{\alpha,\beta} := (z-\overline{z})^2 \frac{\partial^2}{\partial z \partial \overline{z}} - \beta(z-\overline{z})\frac{\partial}{\partial z} + \alpha(z-\overline{z})\frac{\partial}{\partial \overline{z}}.$$

For $\alpha+\beta>2$, a prototype of such a form is given by the Eisenstein series

$$E_{\alpha,\beta}(z) := \sum_{M\in\Gamma_\infty\backslash\Gamma} 1\,\Big|_{(\alpha,\beta)} M.$$

We let

$$M_\alpha := \alpha + (z-\overline{z})\frac{\partial}{\partial z}$$

and

$$N_\beta := -\beta + (z-\overline{z})\frac{\partial}{\partial \overline{z}}$$

be the Maass operators. They commute with the slash operator and shift the weight (α,β) to $(\alpha+1,\beta-1)$ and $(\alpha-1,\beta+1)$, respectively. In the case of the Eisenstein series, it follows from the definitions that

$$M_\alpha E_{\alpha,\beta}(z) = \alpha E_{\alpha+1,\beta-1}(z)$$

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and

$$N_\beta E_{\alpha,\beta}(z) = -\beta E_{\alpha-1,\beta+1}(z).$$

We note that for $\alpha = 0$ (resp. $\beta = 0$) the operator M_0 (resp. N_0) annihilates the Eisenstein series $E_{0,\beta}$ (resp. $E_{\alpha,0}$). Moreover, the space of holomorphic modular forms of weight α is contained in the space of non-analytic modular forms of weight $(\alpha, 0)$ and is *distinguished* in this space by the condition that $N_0 f = 0$. (Similarly, the space of anti-holomorphic forms can be distinguished by the condition $M_0 f = 0$.) Indeed, the classical modular forms of weight α are typically introduced without any reference to the operator $\Omega_{\alpha,0}$ and instead by the conditions

$$\frac{\partial}{\partial \bar{z}} f = 0, \tag{4}$$

$$f\,|_\alpha M = f \quad \text{for all } M \in \Gamma, \tag{5}$$

$$\int_{\Gamma\backslash\mathbb{H}} f(z)\overline{f(z)}y^{\alpha-2}dxdy < \infty. \tag{6}$$

The purpose of this note is to point out a simple observation and use it to define subspaces of non-analytic Siegel modular forms in terms of a Maass operator.

For $Z \in \mathbb{H}_n$, the Siegel upper half plane of degree n, let $E_{\alpha,\beta}(Z)$ be the non-analytic Siegel Eisenstein series of weight (α, β) and M_α be the corresponding Maass operator. It follows from the definitions (for details, see section §19 of [8]) that

$$M_\alpha E_{\alpha,\beta}(Z) = \alpha\left(\alpha - \frac{1}{2}\right)\cdots\left(\alpha - \frac{n-1}{2}\right) E_{\alpha+1,\beta-1}(Z).$$

We explore the following observation alluded to above: In each degree n, the Eisenstein series $E_{\alpha,\beta}(Z)$ with $\alpha = 0, \frac{1}{2}, \ldots, \frac{n-1}{2}$ are annihilated by the Maass operator M_α. Hence, in analogy with the case of holomorphic modular forms in degree 1, each subspace of non-holomorphic modular forms of weights $(0,\beta), \left(\frac{1}{2},\beta\right), \ldots, \left(\frac{n-1}{2},\beta\right)$ that are annihilated by the Maass operator is *distinguished.* These are the subspaces whose further study we would like to motivate with this note.

In the next section, we introduce the notation and give the definition of the space of non-analytic Siegel modular forms. The rest of the paper is devoted to examples of forms in these spaces.

2. Notation

Let $\mathbb{A}$ be a commutative ring with unity and $M_{m,n}(\mathbb{A})$ be the set of $m \times n$ matrices with entries in $\mathbb{A}$. For any matrices $U, V \in M_{m,n}(\mathbb{A})$, set $U[V] := {}^tVUV$ where ${}^tV$ is the transpose of V. If $U \in M_{n,n}(\mathbb{A})$, let $tr(U)$

be the trace of U. We denote the symplectic group of degree n over the integers by $\Gamma_n := \mathrm{Sp}_n(\mathbb{Z})$ and for $1 \leq j < n$, we define the subgroups

$$\Gamma_{n,j} := \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_n \mid A = \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right), C = \left(\begin{smallmatrix} * & 0 \\ 0 & 0 \end{smallmatrix}\right), D = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \right\}$$

and

$$A_{n,j} := \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_n \mid A = \left(\begin{smallmatrix} \pm I_j & 0 \\ * & * \end{smallmatrix}\right), C = 0, D = \left(\begin{smallmatrix} \pm I_j & * \\ 0 & * \end{smallmatrix}\right) \right\}.$$

The subgroups $\Gamma_{n,j}$ (denoted by $C_{n,j}$ in [6]) and $A_{n,j}$ play an important role in the theory of Siegel modular forms. For more details, see Chapter I, §5 and Chapter II, §2 of Freitag [3] and Chapter II, §5 and Chapter III, §7 of Klingen [6]. Let Γ be a subgroup of Γ_n, $\mathbb{H}_n$ be the Siegel upper half plane of degree n, and $Z = (z_{ij}) = X + iY \in \mathbb{H}_n$ be a typical variable. As usual, if $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$ and $Z \in \mathbb{H}_n$, then we set

$$M \circ Z := (AZ + B)(CZ + D)^{-1}.$$

Furthermore, for functions $G : \mathbb{H}_n \to \mathbb{C}$ and for fixed α, $\beta \in \mathbb{C}$ such that $\alpha - \beta \in \mathbb{Z}$, we define the slash operator

$$G\Big|_{(\alpha,\beta)} M := \det(CZ + D)^{-\alpha} \det(C\overline{Z} + D)^{-\beta} G(M \circ Z) \tag{7}$$

for all $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$. Finally, let $\mathcal{M}_{\frac{n-1}{2}} := \det(Z - \overline{Z}) \det(\partial_Z)$ where $\partial_Z := (1 + \delta_{ij}) \frac{\partial}{\partial z_{ij}}$. Note that $\mathcal{M}_{\frac{n-1}{2}}$ is the Maass operator (as in section §19 of [8]) corresponding to the weight $\left(\frac{n-1}{2}, \beta\right)$.

Definition 1. *Let α and β be half-integers such that $\alpha - \beta \in \mathbb{Z}$, $0 \leq \alpha \leq \frac{n-1}{2}$, and $\beta \geq 0$. A Siegel-Maass form of weight (α, β) on Γ is a real-analytic function $F : \mathbb{H}_n \to \mathbb{C}$ satisfying the following conditions:*

$$F \,|_{(\alpha,\beta)}\, M = F \quad \textit{for all } M \in \Gamma, \tag{8}$$

$$\mathcal{M}_{\frac{n-1}{2}}(F) = 0, \tag{9}$$

$$f \textit{ is bounded on domains of type } Y \geq Y_0, Y_0 > 0. \tag{10}$$

Remarks:
1) If F is holomorphic on $\mathbb{H}_n$ (hence $\beta = 0$), then F is a singular Siegel modular form of weight α. In particular, if $F \not\equiv 0$, then F is not a cusp form. On the other hand, if G is a holomorphic Siegel modular form of weight β, then $F = \overline{G}$ is a Siegel-Maass form of weight $(0, \beta)$. Hence the space of holomorphic Siegel modular forms can be viewed as a subspace of Siegel-Maass forms.
2) Of particular interest is the case where $\alpha = \frac{n-1}{2}$. In this special case, the space of functions $\left\{F\colon\ F\,|_{\left(\frac{n-1}{2},\beta\right)} M = F \quad \forall M \in \Gamma \text{ and } \mathcal{M}_{\frac{n-1}{2}}(F) = 0\right\}$ is

invariant under the action of the Hecke operators (for details on Hecke operators, see Chapter IV of Freitag [3]): The definition of the Hecke operators implies that $G = F\,|\,T$ has the correct transformation property whenever T is a Hecke operator, and, in addition, $\mathcal{M}_{\frac{n-1}{2}}(G) = 0$, since the Maass operator $\mathcal{M}_{\frac{n-1}{2}}$ commutes with the slash action in (7) when (and only when) $\alpha = \frac{n-1}{2}$.

3. Examples

In this section we present examples of non-holomorphic Siegel-Maass forms.

Eisenstein series

Let α and β be half-integers such that $\alpha - \beta \in \mathbb{Z}$, $\alpha + \beta > n+1$, and $0 \le \alpha \le \frac{n-1}{2}$. Then (see also [8]) the non-analytic Eisenstein series

$$E_{\alpha,\beta}(Z) := \sum_{M \in \Gamma_{n,0} \backslash \Gamma_n} 1 \Big|_{(\alpha,\beta)} M$$

satisfies (9), since $\det(\partial_Z)\det(CZ+D)^{-\alpha}\det(C\overline{Z}+D)^{-\beta} = 0$ (see also Anhang IV of [3]) implies that each term of $E_{\alpha,\beta}$ is annihilated by $\mathcal{M}_{\frac{n-1}{2}}$. Hence $E_{\alpha,\beta}$ is a Siegel-Maass form of weight (α, β) on Γ_n.

Theta series

Let $Q \in M_{m,m}(\mathbb{Z})$ be symmetric, even, and unimodular of type (k,l) and let R be a majorant of Q, i.e., $RQ^{-1}R = Q$ and ${}^tR = R > 0$. Furthermore, let $\Phi(N) := \det\left({}^tNQ\zeta_+\right)^{\kappa}\det\left({}^tNQ\zeta_-\right)^{\lambda}$ be a spherical function of weight $(\kappa,\lambda) \in \mathbb{N}_0^2$ relative to the pair (Q,R), i.e, $Q\zeta_+ = R\zeta_+$, $Q\zeta_- = -R\zeta_-$, and $R[\zeta_+] = R[\zeta_-] = 0$, where $\zeta_+, \zeta_- \in M_{m,n}(\mathbb{C})$ (with $m > n$). Note that if $\kappa \neq 0 \neq \lambda$ and if $n \ge k$ or if $n \ge l$, then $\Phi(N) \equiv 0$. Andrianov and Maloletkin [1] define the theta series

$$\begin{aligned}\theta_{Q,R,\Phi}(Z) &:= \sum_{N \in M_{m,n}(\mathbb{Z})} \Phi(N)\exp\{\pi i\, tr\,(Q[N]\,X + iR[N]\,Y)\} \\ &= \sum_{N \in M_{m,n}(\mathbb{Z})} \Phi(N)\exp\left\{\frac{\pi i}{2}\, tr\left((Q+R)[N]\,Z + i(Q-R)[N]\,\overline{Z}\right)\right\},\end{aligned}$$

and they show that for all $M \in \Gamma_n$,

$$\theta_{Q,R,\Phi}\Big|_{(\frac{k}{2}+\kappa,\,\frac{l}{2}+\lambda)} M = \theta_{Q,R,\Phi}. \tag{11}$$

If $k < n$, then one can check that $\det((Q+R)[N]) = 0$ for all $N \in M_{m,n}(\mathbb{Z})$. Note that $\det(\partial_Z)e^{\pi i\, tr(AZ)} = (2\pi i)^n\det(A)e^{\pi i\, tr(AZ)}$ for all $A \in M_{n,n}(\mathbb{Z})$, implying that $\mathcal{M}_{\frac{n-1}{2}}(\theta_{Q,R,\Phi}) = 0$. Hence if $k < n$ and if $\kappa = \lambda = 0$, i.e., $\Phi(N) \equiv 1$, then $\theta_{Q,R,\Phi}$ is a Siegel-Maass form of weight $(\frac{k}{2}, \frac{l}{2})$. Also, if $k < n < l$, $\lambda > 0$, and $\kappa = 0$ (otherwise $\Phi(N) \equiv 0$), then $\theta_{Q,R,\Phi}$ is a Siegel-Maass form of weight $(\frac{k}{2}, \frac{l}{2}+\lambda)$.

Poincaré series

1) Let $V^* \in M_{j,j}(\mathbb{Z})$ be symmetric, even, and positive definite, and set $V = \left(\begin{smallmatrix} V^* & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_{n,n}(\mathbb{Z})$. Let $\alpha = \frac{n-1}{2}$ and let β be a half-integer such that $\alpha - \beta \in \mathbb{Z}$ and $\alpha + \beta > n + j + 1$. Then (generalizing our first example) the Poincaré series

$$P_{V,\beta}(Z) := \sum_{M \in A_{n,j} \backslash \Gamma_n} e^{\pi i \, tr(VZ)} \Big|_{(\frac{n-1}{2}, \beta)} M$$

is a Siegel-Maass form of weight $(\frac{n-1}{2}, \beta)$ on Γ_n. We already pointed out that the Maass operator $\mathcal{M}_{\frac{n-1}{2}}$ commutes with the action in (7) and shifts the weight (α, β) to $(\alpha+1, \beta-1)$ when $\alpha = \frac{n-1}{2}$. Moreover, $\det(\partial_Z) e^{\pi i \, tr(VZ)} = 0$, and, consequently, each term of $P_{V,\beta}$ is annihilated by $\mathcal{M}_{\frac{n-1}{2}}$. Hence $P_{V,\beta}$ satisfies (9).

2) We use a different type of Poincaré series to construct another explicit example of a Siegel-Maass form. Let $\mathcal{M} = (m_{st}) \in M_{n-1,n-1}(\mathbb{Z})$ be of rank $n-1$, symmetric, positive definite, and even. Let $\tilde{\phi}_{\mathcal{M}} : \mathbb{H} \times \mathbb{C}^{n-1} \to \mathbb{C}$ be a skew-holomorphic Jacobi cusp form of weight $\tilde{k}$ and index $\mathcal{M}$ in the sense of Arakawa [2] and Hayashida [4] (see also Skoruppa [12] for the case $n = 2$). Hence there exists a $C > 0$ such that

$$(12) \qquad \left|\tilde{\phi}_{\mathcal{M}}(\tau, z)\right| y_1^{\tilde{k}/2} e^{-\frac{\pi}{y_1} tr(\mathcal{M}\, {}^t vv)} < C \quad \text{for all} \quad (\tau, z) \in \mathbb{H} \times \mathbb{C}^{n-1},$$

where $y_1 = \mathrm{Im}(\tau)$ and $v = \mathrm{Im}(z)$. In addition, $\tilde{\phi}_{\mathcal{M}}$ is in the kernel of the heat operator

$$(13) \qquad L_{\mathcal{M}} := 4\pi i \det(\mathcal{M}) \frac{\partial}{\partial_\tau} - \sum_{1 \le s, t \le n-1} \mathcal{M}_{st} \frac{\partial^2}{\partial_{z_s} \partial_{z_t}},$$

where $\mathcal{M}_{st}$ is the cofactor of the entry m_{st}.

Let $f_1 : \mathbb{H} \to \mathbb{C}$ be a holomorphic elliptic cusp form of weight k_1. Then $\phi_{\mathcal{M}}(\tau, z) := \tilde{\phi}_{\mathcal{M}}(\tau, z) \overline{f_1(\tau)}$ is a skew-holomorphic Jacobi cusp form of weight $k = \tilde{k} + k_1$ and index $\mathcal{M}$ and $\phi_{\mathcal{M}[U]}(\tau, z) := \phi_{\mathcal{M}}(\tau, z\, {}^t U)$ (where $U \in \mathrm{GL}_{n-1}(\mathbb{Z})$) is a skew-holomorphic Jacobi cusp form of weight k and index $\mathcal{M}[U]$. Write $Z = \left(\begin{smallmatrix} \tau & z \\ {}^t z & W \end{smallmatrix}\right) \in \mathbb{H}_n$, where $\tau \in \mathbb{H}_1$, $z \in \mathbb{C}^{n-1}$, $W \in \mathbb{H}_{n-1}$, and set

$$F(Z) := \sum_{\substack{\mathcal{M}' = \mathcal{M}[U] \\ U \in \mathrm{GL}_{n-1}(\mathbb{Z})}} \phi_{\mathcal{M}'}(\tau, z) \exp\left\{\pi i \, \mathrm{tr}(\mathcal{M}' W)\right\},$$

where the sum is over all symmetric, positive definite, and even matrices $\mathcal{M}'$ such that $\mathcal{M}' = \mathcal{M}[U]$ for some $U \in \mathrm{GL}_{n-1}(\mathbb{Z})$. In particular, if $n = 2$, then $F(Z) = \phi_{\mathcal{M}}(\tau, z) \exp\{\pi i \, \mathrm{tr}(\mathcal{M} W)\}$. In general, F does not satisfy (8) for all $M \in \Gamma_n$. However, the following proposition constructs a Poincaré series (depending on F) which is a Siegel-Maass form on Γ_n.

PROPOSITION 1. *If $k > 2n^2 - n + 6$, then*

$$P_F(Z) := \sum_{M \in \Gamma_{n,1} \backslash \Gamma_n} F \Big|_{\left(\frac{n-1}{2}, k - \frac{n-1}{2}\right)} M \tag{14}$$

is a Siegel-Maass form of weight $\left(\frac{n-1}{2}, k - \frac{n-1}{2}\right)$ on Γ_n.

PROOF: It is easy to check that (see also [5])

$$F \Big|_{\left(\frac{n-1}{2}, k - \frac{n-1}{2}\right)} M = F \quad \text{for all} \quad M \in \Gamma_{n,1}.$$

Furthermore,

$$\det(\partial_Z) \left(\phi_{\mathcal{M}'}(\tau, z)\, e^{\pi i \, tr(\mathcal{M}'W)} \right) = (2\pi i)^{n-2} L_{\mathcal{M}'}(\phi_{\mathcal{M}'}(\tau, z))\, e^{\pi i \, tr(\mathcal{M}'W)} = 0,$$

i.e., $\mathcal{M}_{\frac{n-1}{2}}(F) = 0$.

We proceed as in [5] (see also Freitag [3]) to lift F to a Siegel-Maass form on Γ_n. Clearly, if $P_F(Z)$ converges, then $P_F(Z)$ satisfies (8) and (9) with $\alpha = \frac{n-1}{2}$ and $\beta = k - \frac{n-1}{2}$. It remains to show that $P_F(Z)$ converges.

Set $G(Z) := y_1^{\left(\frac{k}{2} - N\right)} \det(Y)^N |F(Z)|$, where $N < k/2$. With the help of (12), one can show that $G(Z)$ is bounded on $\mathbb{H}_n$ if $N > (n-1)^2 + 1$ (see also [5]). Let $h(Z) := y_1^{-\left(\frac{k}{2} - N\right)}$ and $H(Z) := \det(Y)^{-N}$. If $k - 2N > n + 2$, then

$$E_{hH}(Z) := \sum_{M \in \Gamma_{n,1} \backslash \Gamma_n} \left| (hH) \Big|_{(k,0)} M \right| = H(Z) \sum_{M \in \Gamma_{n,1} \backslash \Gamma_n} \left| h \Big|_{(k-2N,0)} M \right|$$

converges absolutely (see I. 5.4_1 in [3]). Hence $P_F(Z)$ converges if $k > 2n^2 - n + 6$. We conclude that if $k > 2n^2 - n + 6$, then P_F is a Siegel-Maass form of weight $\left(\frac{n-1}{2}, k - \frac{n-1}{2}\right)$ on Γ_n. □

Remarks:
1) Note that the Poincaré series P_F discussed above, as well as $\theta_{Q,R,\Phi}$ in the case where $k < n < l$, $\lambda > 0$, and $\kappa = 0$, are in the kernel of the Siegel ϕ-operator.
2) Note that in degree 1, the corresponding examples of weight $(0, \beta)$, namely the Eisenstein series, the Poincare series, and the theta series attached to quadratic forms of signature $(0, 2\beta)$, are indeed all anti-holomorphic and annihilated by the Maass operator M_0.

4. CONCLUSION

In this note, we have introduced and tried to motivate the further study of a subspace of non-holomorphic Siegel forms. Some natural questions present themselves:
1) What is the dimension of this distinguished subspace?
2) What can be said about the subspace spanned by the theta series? Is there a basis theorem?

3) Can one characterize the Fourier coefficients of Siegel-Maass forms (at least when $n = 2$)?
4) It would be interesting to find out if the construction of Poincaré series in our last example can shed some light on Kohnen's question (see [7]) on how skew-holomorphic Jacobi forms are related to Siegel modular forms. Is there a link between an arbitrary Siegel-Maass form and skew-holomorphic Jacobi forms?

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