数理解析研究所講究録
第 1715 巻 2010 年 143-158

Fourier expansion of Arakawa lifting and central L-values

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Abstract

This note is a write-up of our talk at the RIMS-conference on automorphic forms held at the University of Tokyo in January 2010. Our results deal with the Fourier coefficients of Arakawa lifts and central values of some automorphic L-functions. In our previous paper [M-N-2] we provide an explicit formula for the Fourier coefficients in terms of toral integrals of some automorphic forms with respect to Hecke characters of imaginary quadratic fields. After studying explicit relations between the toral integrals and the central L-values, we explicitly determine the constant of proportionality relating the square norm of a Fourier coefficient of an Arakawa lift with the central L-value. In some case we can relate such square norm with the central value of some degree eight L-function of convolution type attached to the lift and the character. We also discuss the existence of strictly positive central values of the L-functions in our concern.

1 Reviews on Arakawa lifting and its Fourier expansion.

1.1

In this section we review our results in [M-N-2]. Let B be a definite quaternion algebra over $\mathbb{Q}$ and denote by n its reduced norm. Let $G = GSp(1,1)$ be the $\mathbb{Q}$-algebraic group defined by

$$G_{\mathbb{Q}} := \{g \in M_2(B) \mid {}^t\bar{g}Qg = \nu(g)Q,\ \nu(g) \in \mathbb{Q}^\times\},$$

where $Q := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By Z_G we denote the center of G.

We put $G^1_\infty := \{g \in M_2(\mathbb{H}) \mid {}^t\bar{g}Qg = Q\}$, where $\mathbb{H} := B \otimes_{\mathbb{Q}} \mathbb{R}$ is the Hamilton quaternion algebra. Then

$$K_\infty := \{\begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a \pm b \in \mathbb{H}^1\}$$

forms a maximal compact subgroup of G^1_∞, where $\mathbb{H}^1 := \{u \in \mathbb{H} \mid n(u) = 1\}$. For a non-negative integer κ we let $(\sigma'_\kappa, V_\kappa)$ be the κ-th symmetric tensor representation of $GL_2(\mathbb{C})$ and σ_κ the pull-back of σ'_κ to $\mathbb{H}^\times$ via the standard embedding $\mathbb{H} \subset M_2(\mathbb{C})$ (cf. [M-N-2, (1.4)]). This induces an irreducible representation (τ_κ, V_κ) of K by

$$\tau_\kappa(\begin{pmatrix} a & b \\ b & a \end{pmatrix}) := \sigma_\kappa(a+b),\ (\begin{pmatrix} a & b \\ b & a \end{pmatrix} \in K_\infty).$$

*Partially supported by Grand-in-Aid for Scientific Research C-20540031, JSPS
†Partially supported by Grand-in-Aid for Young Scientists (B) 21740025, JSPS

Let H and H' be $\mathbb{Q}$-algebraic groups defined by

$$H_{\mathbb{Q}} = GL_2(\mathbb{Q}), \; H'_{\mathbb{Q}} := B^{\times}$$

respectively.

Fix a maximal order $\mathfrak{O}$ of B. We denote by d_B the discriminant of B and fix a divisor D of d_B. For $p|d_B$ let $\mathfrak{P}_p$ be the maximal ideal of the p-adic completion $\mathfrak{O}_p$ of $\mathfrak{O}$ and let

$$L_p := \begin{cases} {}^t(\mathfrak{O}_p \oplus \mathfrak{O}_p) & (p \not| d_B \text{ or } p|D), \\ {}^t(\mathfrak{O}_p \oplus \mathfrak{P}_p^{-1}) & (p|\frac{d_B}{D}). \end{cases}$$

We put $K_p := \{k \in G_p \mid kL_p = L_p\}$ for each finite prime p and $K_f := \prod_{p<\infty} K_p$. It is known that, up to G_p-conjugate, the two groups K_p for the two L_p's exhaust maximal compact subgroups of G_p. For $\kappa > 4$ we then introduce the space $\mathcal{S}_\kappa$ of V_κ-valued cusp forms F on $G_{\mathbb{A}_{\mathbb{Q}}}$ satisfying the following:

1. $F(z\gamma g k_f k_\infty) = \tau_\kappa(k_\infty)^{-1}F(g)$ for $(z,\gamma,g,k_f,k_\infty) \in Z_{G,\mathbb{A}_{\mathbb{Q}}} \times G_{\mathbb{Q}} \times G_{\mathbb{A}_{\mathbb{Q}}} \times K_f \times K_\infty$,

2. for each fixed $g_f \in G_{\mathbb{A}_f}$, the right translations of the coefficients of $F|_{G^1_\infty}(g_f *)$ by $g_\infty \in G^1_\infty$ generate, as a $(\mathfrak{g}, K_\infty)$-module, quaternionic discrete series with the minimal K_∞-type τ_κ (cf. [Gr-W]). Here $\mathfrak{g}$ denotes the Lie algebra of G^1_∞.

For a positive integer κ we let $S_\kappa(D)$ be the space of elliptic cusp forms of weight κ with level D (cf. [M-N-2, §3.1]) and $\mathcal{A}_\kappa$ be the space of automorphic forms of weight σ_κ with respect to $\prod_{v<\infty} \mathfrak{O}_v^\times$ (cf. [M-N-2, §3.2]), where $\mathfrak{O}_v^\times$ denotes the multiplicative group of $\mathfrak{O}_v$. Now we can review the definition of Arakawa lifting. By a metaplectic representation of $GSp(1,1)_{\mathbb{A}_{\mathbb{Q}}} \times H_{\mathbb{A}_{\mathbb{Q}}} \times H'_{\mathbb{A}_{\mathbb{Q}}}$, we define the $\mathrm{End}(V_\kappa)$-valued theta function $\theta_\kappa(g,h,h')$ with some specified $\mathrm{End}(V_\kappa)$-valued Schwartz-Bruhat function on $B_{\mathbb{A}}^{\oplus 2} \times \mathbb{A}_{\mathbb{Q}}^\times$ (cf. [M-N-1, §3]). Then, for $\kappa > 4$, we define the Arakawa lifting by

$$S_\kappa(D) \times \mathcal{A}_\kappa \ni (f, f') \mapsto \mathcal{L}(f,f')(g) \in \mathcal{S}_\kappa$$

with

$$\mathcal{L}(f,f')(g) := \int_{\mathbb{R}^2_+(H\times H')_{\mathbb{Q}} \backslash (H \times H')_{\mathbb{A}_{\mathbb{Q}}}} \overline{f(h)}\theta_\kappa(g,h,h')f'(h')dhdh'.$$

1.2

We now review the Fourier expansion of $\mathcal{L}(f,f')$ described in [M-N-2, §1.3]. We let $B^- := \{x \in B \mid \mathrm{tr}(x) = 0\}$ and have

$$\mathcal{L}(f,f')(g) = \sum_{\xi \in B^- \backslash \{0\}} \mathcal{L}(f,f')_\xi(g),$$

where

$$\mathcal{L}(f,f')_\xi(g) := \int_{B^- \backslash B^-_{\mathbb{A}_{\mathbb{Q}}}} \mathcal{L}(f,f')(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g)\psi(-\mathrm{tr}(\xi x))dx$$

with the standard additive character ψ on $\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}$. Here we normalize the measure dx so that the volume of $B^-\backslash B^-_{\mathbb{A}_{\mathbb{Q}}}$ is one. For $\xi \in B^- \setminus \{0\}$ we let $E_\xi := \mathbb{Q}(\xi)$, which is isomorphic to an imaginary quadratic field, and X_ξ be the set of unitary characters on $\mathbb{A}_{\mathbb{Q}}^\times E_\xi^\times \backslash \mathbb{A}_{E_\xi}^\times$. The Fourier expansion is then refined as follows:

$$\mathcal{L}(f,f')(g) = \sum_{\xi \in B^- \setminus\{0\}} \sum_{\chi \in X_\xi} \mathcal{L}(f,f')_\xi^\chi(g),$$

with

$$\mathcal{L}(f,f')_\xi^\chi(g) := \mathrm{vol}(\mathbb{R}_+^\times \mathbb{A}_{\mathbb{Q}}^\times \backslash \mathbb{A}_{E_\xi}^\times)^{-1} \int_{\mathbb{R}_+^\times E_\xi^\times \backslash \mathbb{A}_{E_\xi}^\times} \mathcal{L}(f,f')_\xi(s1_2 \cdot g)\chi(s)^{-1} ds.$$

1.3

To review our explicit formula for $\mathcal{L}(f,f')_\xi^\chi$, we let $(f,f') \in S_\kappa(D) \times \mathcal{A}_\kappa$ and assume the following two:

(1) The two forms f and f' are Hecke eigenforms and have the same eigenvalue for the "Atkin-Lehner involution". More precisely, for each $p|D$, let ϵ_p (resp. ϵ'_p) be the eigenvalue for the involutive action of $\begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}$ (resp. a prime element $\varpi_{B,p} \in B_p$) on f (resp. f'). Then

$$\epsilon_p = \epsilon'_p.$$

Otherwise $\mathcal{L}(f,f') \equiv 0$ (cf. [M-N-1, Remark 5.2 (ii)]).

(2) We assume that $\xi \in B^- \setminus \{0\}$ is primitive. Namely, letting $\mathfrak{a}_p := \begin{cases} \mathfrak{O}_p & (p \nmid d_B \text{ or } p|D) \\ \mathfrak{P}_p & (p|\frac{d_B}{D}) \end{cases}$ for each finite prime p,

$$\xi \in \mathfrak{a}_p \setminus p\mathfrak{a}_p.$$

For the second assumption we note that, in general, a Fourier coefficient F_ξ of an automorphic form F on $G_{\mathbb{A}_{\mathbb{Q}}}$ satisfies

$$F_\xi(g) = F_\xi(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} g) = F_{t\xi}(g) \quad (t \in \mathbb{Q}^\times).$$

We then see that the problem determining F_ξ is reduced to the case where ξ is primitive.

1.4

For each $\xi \in B^- \setminus \{0\}$, d_ξ denotes the discriminant of $E := E_\xi$. We put

$$a := \begin{cases} 2\sqrt{-n(\xi)}\sqrt{d_\xi} & (d_\xi \text{ is odd}) \\ \sqrt{-n(\xi)}\sqrt{d_\xi} & (d_\xi \text{ is even}) \end{cases}, \quad b := \xi^2 - \frac{a^2}{4}.$$

With these a and b we define $\iota_\xi : E^\times \hookrightarrow GL_2(\mathbb{Q})$ by

$$\iota_\xi(x + y\xi) = x \cdot 1_2 + y \cdot \begin{pmatrix} \frac{a}{2} & b \\ 1 & -\frac{a}{2} \end{pmatrix} \quad (x, y \in \mathbb{Q}).$$

Put $\theta := r^{-1}(\xi - \frac{a}{2})$ with $r = \frac{2\sqrt{n(\xi)}}{\sqrt{d_\xi}} \in \mathbb{Q}^\times$. Then $\{1, \theta\}$ forms a $\mathbb{Z}$-basis of the integer ring $\mathcal{O}_E$ of E. We can rewrite ι_ξ as

$$\iota_\xi(x + y\theta) = \begin{pmatrix} x & -rN_{E/\mathbb{Q}}(\theta)y \\ r^{-1}y & x + \mathrm{Tr}_{E/\mathbb{Q}}(\theta)y \end{pmatrix} \quad (x, y \in \mathbb{Q}).$$

The completion E_∞ of E at ∞ is identified with $\mathbb{C}$ by

$$\delta_\xi : E_\infty \ni x + y\xi \mapsto x + y\sqrt{-n(\xi)} \in \mathbb{C} \ (x, y \in \mathbb{R}).$$

For a Hecke character $\chi = \prod_{v \leq \infty} \chi_v$ of $\mathbb{R}_+^\times E^\times \backslash \mathbb{A}_E^\times$, we let $w_\infty(\chi) \in \mathbb{Z}$ be such that

$$\chi_\infty(u) = (\delta_\xi(u)/|\delta_\xi(u)|)^{w_\infty(\chi)} \ (u \in E_\infty).$$

Furthermore, for each prime $v = p < \infty$, we let $p^{i_p(\chi)}$ be the conductor of χ at p and

$$\mu_p := \frac{\mathrm{ord}_p(2\xi)^2 - \mathrm{ord}_p(d_\xi)}{2},$$

which coincides with $\mathrm{ord}_p(r)$. We now state the following (cf. [M-N-2, Theorem 5.1.1]):

Proposition 1.1. *$\mathcal{L}(f, f')_\xi^\chi \equiv 0$ unless $i_p(\chi) = 0$ for any $p | d_B$ and $w_\infty(\chi) = -\kappa$.*

1.5

In what follows, we assume that χ satisfies the assumption in the proposition above. We need further notations to recall our formula for $\mathcal{L}(f, f')_\xi^\chi$.

We define $\gamma_0 = (\gamma_{0,p})_{p \leq \infty} \in H_{\mathbb{A}_\mathbb{Q}}$ and $\gamma_0' = (\gamma_{0,p}')_{p < \infty} \in H_{\mathbb{A}_f}'$ as follows:

$$\gamma_{0,p} := \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & p^{-\mu_p + i_p(\chi)} \end{pmatrix} & (p \not| D), \\ 1_2 & (p | D \text{ and } p \text{ is inert in } E), \\ \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & (p | D \text{ and } p \text{ ramifies in } E), \\ n_H(a/2) d_H(N(\xi)^{1/4}) & (p = \infty), \end{cases}$$

$$\gamma_{0,p}' := \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & p^{-\mu_p + i_p(\chi)} \end{pmatrix} & (p \not| d_B), \\ \varpi_{B,p}^{-1} & (p | d_B). \end{cases}$$

Here recall that $\varpi_{B,p}$ denotes a prime element of B_p (cf. §1.3).

In addition, we introduce the following local constants:

$$C_p(f, \xi, \chi) := \begin{cases} p^{2\mu_p - i_p(\chi)}(1 - \delta(i_p(\chi) > 0) e_p(E) p^{-1}) & (p \not| d_B), \\ 1 & (p | \frac{d_B}{D}), \\ 2\epsilon_p & (p | D \text{ and } p \text{ is inert in } E), \\ (p+1)^{-1} & (p | D \text{ and } p \text{ ramifies in } E), \end{cases}$$

where

$$e_p(E) = \begin{cases} -1 & (p \text{ is inert in } E), \\ 0 & (p \text{ ramifies in } E), \\ 1 & (p \text{ splits in } E). \end{cases}$$

For $(f, f') \in S_\kappa(D) \times \mathcal{A}_\kappa$ we introduce their toral integrals with respect to χ (cf. [Wa]).

$$P_\chi(f;h) := \int_{\mathbb{R}_+^\times E_\xi^\times \backslash \mathbb{A}_{E_\xi}} f(\iota_\xi(s)h)\chi(s)^{-1}ds, \; P_\chi(f';h') := \int_{\mathbb{R}_+^\times E_\xi^\times \backslash \mathbb{A}_{E_\xi}^\times} f(sh')\chi(s)^{-1}ds,$$

where $(h, h') \in GL_2(\mathbb{A}_\mathbb{Q}) \times B_{\mathbb{A}_\mathbb{Q}}^\times$. As in [M-N-2, §2.4] we normalize the measure ds of $\mathbb{A}_E^\times$ so that

$$\mathrm{vol}(\mathcal{O}_{E,p}^\times) = 1 \text{ for any } p < \infty, \quad \mathrm{vol}(E_\infty^1) = 1.$$

We denote by $\mathbf{h}(E)$ and $w(E)$ the class number of E and the number of the roots of unity in E respectively. Then we are able to state our formula for $\mathcal{L}(f, f')_\xi^\chi$ (cf. [M-N-2, Theorem 5.2.1]).

Theorem 1.2. *Let (f, f') be Hecke eigenforms and $\xi \in B^- \setminus \{0\}$ be primitive. Suppose that χ satisfies the assumption in Proposition 1.1 and that (1) and (2) in §1.3 hold. We then have the following formula:*

$$\begin{aligned} &\mathcal{L}(f, f')_\xi^\chi(g_{0,f} d_G(\sqrt{\eta_\infty})) \\ &= 2^{\kappa-1} n(\xi)^{\kappa/4} \frac{w(E)}{\mathbf{h}(E)} \cdot \left(\prod_{p<\infty} C_p(f, \xi, \chi)\right) \eta_\infty^{\kappa/2+1} \exp(-4\pi\sqrt{n(\xi)}\eta_\infty)\, \overline{P_\chi(f;\gamma_0)} P_\chi(f';\gamma_0'). \end{aligned}$$

Here $\eta_\infty \in \mathbb{R}_+^\times$ and $g_{0,f} = (g_{0,p})_{p<\infty} \in G_{\mathbb{A}_f}$ is given by

$$g_{0,p} := \begin{cases} \mathrm{diag}(p^{i_p(\chi)-\mu_p}, p^{2(i_p(\chi)-\mu_p)}, 1, p^{i_p(\chi)-\mu_p}) & (p \not| d_B), \\ 1_2 & (p | d_B) \end{cases}$$

Remark 1.3. According to Sugano [Su, Theorem 2-1], the Fourier coefficient $\mathcal{L}(f, f')_\xi^\chi$ is determined by the evaluation at $g_{0,f} \begin{pmatrix} \sqrt{\eta_\infty} & 0 \\ 0 & \sqrt{\eta_\infty}^{-1} \end{pmatrix}$.

2 Relation with central L-values.

2.1

Let (f, f') be Hecke eigenforms and assume that f is a primitive form (for the definition, see [Mi, §4.6]). Let $\pi(f)$ (resp. $\pi(f')$) be the irreducible automorphic representation generated by f (resp. f'), and let $\pi(\mathrm{JL}(f'))$ be the irreducible automorphic representation generated by the Jacquet-Langlands lift $\mathrm{JL}(f')$ of f'. It is known that $\pi(f)$ and $\pi(\mathrm{JL}(f'))$ (resp. $\pi(f')$) decompose into restricted tensor products over $v \leq \infty$ of irreducible admissible representations of $GL_2(\mathbb{Q}_v)$ (resp. $B_v^\times$). By π_v, π_v' and π_v'' we denote the v-component of $\pi(f)$,

$\pi(\mathrm{JL}(f'))$ and $\pi(f')$ respectively. According to such decomposition of $\pi(f)$ and $\pi(f')$, f and f' admit decompositions into pure tensor products

$$\rho(f) = \prod_{v\leq\infty} f_v, \ \rho'(f') = \prod_{v\leq\infty} f'_v,$$

where we fix an isomorphism ρ (resp. ρ') between $\pi(f)$ and $\otimes'_{v\leq\infty}\pi_v$ (resp. $\pi(f')$ and $\otimes'_{v\leq\infty}\pi''_v$). We denote by Π (resp. Π') the quadratic base change of $\pi(f)$ (resp. $\pi(\mathrm{JL}(f'))$) to $GL_2(\mathbb{A}_E)$. These Π and Π' also decompose into the restricted tensor products $\otimes'_{v\leq\infty}\Pi_v$ and $\otimes'_{v\leq\infty}\Pi'_v$ respectively, where each Π_v or Π'_v is a local base change lift of π_v or π'_v at every place v respectively.

2.2 Review on the adjoint L-functions and the L-functions of base change lifts for GL_2.

Let $L(\pi,s)$ be the standard L-function for an automorphic representation π of $GL_2(\mathbb{A}_{\mathbb{Q}})$ in the sense of Jacquet-Langlands [J-L]. We denote by $L(\Pi,\chi^{-1},s)$ (resp. $L(\Pi',\chi^{-1},s)$) the L-function of Π (resp. Π') with χ^{-1}-twist, and let $L(\pi(f),\mathrm{Ad},s)$ (resp. $L(\pi(\mathrm{JL}(f')),\mathrm{Ad},s)$) be the adjoint L-function of $\pi(f)$ (resp. $\pi(\mathrm{JL}(f'))$).

We describe the local factors of $L(\Pi,\chi^{-1},s)$ and $L(\Pi',\chi^{-1},s)$ (resp. $L(\pi(f),\mathrm{Ad},s)$ and $L(\pi(\mathrm{JL}(f')),\mathrm{Ad},s)$), following Jacquet [J] (resp. Gelbart-Jacquet [G-J]). We note that π_p (resp. $\pi'_p \simeq \pi''_p$) is a unitary unramified principal series representation for each finite prime p (resp. $p \not| d_B$). This is due to the Ramanujian conjecture for holomorphic cusp forms on GL_2. In addition, we remark that, for $p|d_B$, π''_p is written as

$$B_p^\times \ni b \mapsto \delta_p \cdot n(b) \in \{\pm 1\},$$

with a character δ_p of $\mathbb{Q}_p^\times$ of order two. For $p|d_B$, π'_p is thus the special representation of $GL_2(\mathbb{Q}_p)$ given by the irreducible subquotient of the induced representation $\mathrm{Ind}(\delta_p\cdot|*|_p^{\frac{1}{2}},\delta_p\cdot|*|_p^{-\frac{1}{2}})$ (cf. [J-L, Theorem 4.2 (iii)]), where $|*|_p$ is the p-adic absolute value. We furthermore note that, when p is inert or ramified in E and χ_p is unramified, χ_p can be written as

$$\chi_p = \omega_p \cdot n_{E_p/\mathbb{Q}_p}$$

with a character ω_p of $\mathbb{Q}_p^\times$ of order two and the norm $n_{E_p/\mathbb{Q}_p}$ of E_p. In fact, $\omega_p \equiv 1$ when p is inert. In addition, at the Archimedean place, π_∞ and π'_∞ are the discrete series representations with weight κ and $\kappa+2$ respectively (see [Sh, §6]).

We let π_{ur} be a unitary unramified principal series representation of $GL_2(\mathbb{Q}_p)$ with Satake parameter (α_p,α_p^{-1}) and the trivial central character, and let π_{sp} be the special representation of $GL_2(\mathbb{Q}_p)$ corresponding to π''_p. We denote by Π_{ur} (resp. Π_{sp}) the base change lift of π_{ur} (resp. π_{sp}) to $GL_2(E_p)$.

We first deal with the local L-functions of π_{ur} and Π_{ur}.

Lemma 2.1. *(1)*

$$L_p(\pi_{\mathrm{ur}},s) = (1-\alpha_p p^{-s})^{-1}(1-\alpha_p^{-1}p^{-1})^{-1}.$$

(2)

$$L_p(\pi_{\mathrm{ur}}, \mathrm{Ad}, s) = (1-p^{-s})^{-1}(1-\alpha_p^2 p^{-s})^{-1}(1-\alpha_p^{-2}p^{-s})^{-1}.$$

(3)

$$L_p(\Pi_{\mathrm{ur}}, \chi_p^{-1}, s) =$$
$$\begin{cases} \prod_{i=1,2}(1-\alpha_p\chi_p(\varpi_{p,i})^{-1}p^{-s})^{-1}(1-\alpha_p^{-1}\chi_p(\varpi_{p,i})^{-1}p^{-s})^{-1} & (p{:}split,\ i_p(\chi)=0), \\ (1-\alpha_p^2 p^{-2s})^{-1}(1-\alpha_p^{-2}p^{-2s})^{-1} & (p{:}inert,\ i_p(\chi)=0), \\ (1-\alpha_p\chi_p(\varpi_p)^{-1}p^{-s})^{-1}(1-\alpha_p^{-1}\chi_p(\varpi_p)^{-1}p^{-s})^{-1} & (p{:}ramified,\ i_p(\chi)=0), \\ 1 & (i_p(\chi)>0), \end{cases}$$

where $\varpi_{p,i} \in E_p$ with $i=1,2$ (resp. $\varpi_p \in E_p$) denote two distinct prime elements dividing p (a prime element dividing p) when p is split (resp. p is ramified).

We next deal with the case of π_{sp} and Π_{sp}.

Lemma 2.2. *We have*

$$L_p(\pi_{\mathrm{sp}}, s) = (1-\delta_p(p)p^{-(s+\frac{1}{2})})^{-1}, \tag{2.1}$$
$$L_p(\pi_{\mathrm{sp}}, \mathrm{Ad}, s) = (1-p^{-(s+1)})^{-1}, \tag{2.2}$$
$$L_p(\Pi_{\mathrm{sp}}, \chi_p^{-1}, s) = \begin{cases} (1-p^{-(2s+1)})^{-1} & (p{:}inert), \\ (1-\delta_p(p)\omega_p(p)p^{-(s+\frac{1}{2})})^{-1} & (p{:}ramified). \end{cases} \tag{2.3}$$

For a positive integer $k \geq 2$ we let π_k be the discrete series representation with weight k and Π_k denote its base change. We give their Archimedean local L-functions as follows.

Lemma 2.3. *We have*

$$L_\infty(\pi_k, s) = \Gamma_{\mathbb{C}}(s+\frac{k-1}{2}), \tag{2.4}$$
$$L_\infty(\pi_k, \mathrm{Ad}, s) = \Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{C}}(s+k-1), \tag{2.5}$$
$$L_\infty(\Pi_k, \chi_l, s) = \begin{cases} \Gamma_{\mathbb{C}}(s+\frac{k-1}{2}+|l|)\Gamma_{\mathbb{C}}(s-\frac{k-1}{2}+|l|) & (|l| \geq \frac{\kappa-1}{2}) \\ \Gamma_{\mathbb{C}}(s+\frac{k-1}{2}+|l|)\Gamma_{\mathbb{C}}(s+\frac{k-1}{2}-|l|) & (|l| \leq \frac{\kappa-1}{2}) \end{cases}, \tag{2.6}$$

where $l \in \frac{1}{2}\mathbb{Z}$.

2.3 Relation between $P_\chi(f;\gamma_0)$ and $L(\Pi, \chi^{-1}, \frac{1}{2})$.

By η we denote the quadratic character attached to the quadratic extension $E/\mathbb{Q}$. We let $L(\eta, s)$ be the L-function defined by η and $L_v(\eta_v, s)$ the local factor of $L(\eta, s)$ at a place v. Let $\mathcal{T}_p^{\pm} f = \lambda_p^{\pm} f$ for $p|D$ (for the definition of $\mathcal{T}_p^{\pm}$, see [Mu, 2.4]) and let

$$S_1 := \{p < \infty \mid p|D,\ p \text{ is inert in } E\},$$
$$S_2^{\pm} := \{p < \infty \mid p|D,\ p \text{ ramifies in } E,\ \chi_p(\varpi_p)\lambda_p^{+} = \pm 1\},$$

where ϖ_p denotes a prime element of E_p (cf. Lemma 2.1). Note that $S_1 \cup S_2^+ \cup S_2^-$ coincides with $\delta(D) := \{p < \infty \mid p|D\}$. We furthermore put $A(\chi) := \prod_{p<\infty} p^{i_p(\chi)}$.

Let us introduce the Petersson norm

$$\langle f, f \rangle := \int_{Z(\mathbb{A}_\mathbb{Q})GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_\mathbb{Q})} |f(g)|^2 dg$$

of f, where Z denotes the center of GL_2. Keeping the assumption on χ in Proposition 1.1 we can quote [Mu, Theorem 1.1] with a modification.

Proposition 2.4. *We have*

$$\frac{|P_\chi(f;\gamma_0)|^2}{\langle f, f\rangle} = C(f,\chi)L(\Pi, \chi^{-1}, \frac{1}{2}),$$

with

$$C(f,\chi) := \begin{cases} \frac{2^{|\delta(D)|}|d_\xi| \prod_{p|A(\chi)} L_p(\eta_p,1)^2}{4D^{\frac{3}{2}}A(\chi)L(\pi(f),\mathrm{Ad},1)} & (S_1 = S_2^+ = \emptyset), \\ 0 & (\textit{otherwise}). \end{cases}$$

2.4 Relation between $P_\chi(f';\gamma_0')$ and $L(\Pi',\chi^{-1},\frac{1}{2})$.

Next we provide an explicit relation between $||P_\chi(f';\gamma_0')||^2$ and the central L-value $L(\Pi', \chi^{-1}, \frac{1}{2})$, which we prove in §3. To write down the relation we need several notations. We denote by r_p the ramification index of p for the quadratic extension $E/\mathbb{Q}$, i.e. $r_p := \begin{cases} 1 & (p\text{:non-ramified}) \\ 2 & (p\text{:ramified}) \end{cases}$.

We set

$$\langle f', f' \rangle := \int_{Z'(\mathbb{A}_\mathbb{Q})B^\times\backslash B^\times_{\mathbb{A}_\mathbb{Q}}} (f'(b), f'(b))_\kappa db,$$

where Z' denotes the center of $B^\times$ and $(*,*)_\kappa$ stands for an inner product of V_κ. By $\langle *, * \rangle_\infty$ we denote an inner product of π''_∞. As this inner product we can take

$$\langle h_\infty, h'_\infty \rangle_\infty := \int_{\mathbb{H}^1} (h_\infty(u), h'_\infty(u))_\kappa du \quad (h_\infty,\ h'_\infty \in \pi''_\infty \simeq V_\kappa),$$

where du denotes the invarient measure of $\mathbb{H}^1$ normalized so that $\mathrm{vol}(\mathbb{H}^1) = 1$. Let $v_{\kappa,\xi}$ be a highest weight vector of V_κ with respect to $\sigma_\kappa(\mathbb{R}(\xi)^\times)$-action, and let $v^*_{\kappa,\xi} \in V_\kappa$ be the dual of $v_{\kappa,\xi}$ with respect to $\langle *, * \rangle_\infty$. We set $f'_{\infty,\kappa}(b_\infty) := \langle f'_\infty(b_\infty), v^*_{\kappa,\xi}\rangle_\infty v_{\kappa,\xi}$ for $b_\infty \in B^\times_\infty$. The notation $\zeta(s) = \prod_{v\le\infty} \zeta_v(s)$ stands for the Riemann zeta function with a local factor $\zeta_v(s)$ at each place v. With the assumption on χ in Proposition 1.1 we then have the following:

Proposition 2.5.

$$\frac{||P_\chi(f';\gamma_0')||^2}{\langle f', f' \rangle} = \begin{cases} C(f',\chi)L(\Pi',\chi^{-1},\frac{1}{2}) & (\pi''_p|_{E_p^\times} = \chi_p \textit{ when } p \textit{ divides } d_B \textit{ and is ramified in } E), \\ 0 & (\textit{otherwise}), \end{cases}$$

where

$$C(f',\chi) := \frac{\sqrt{|d_\xi|}(\kappa+1)}{2^5\pi A(\chi)} \prod_{p|A(\chi)} L_p(\eta_p,1)^2 \cdot \prod_{p|d_B} r_p(1-p^{-1}) \cdot \frac{\langle f'_{\infty,\kappa}, f'_{\infty,\kappa}\rangle_\infty}{\langle f'_\infty, f'_\infty\rangle_\infty} \cdot \frac{\zeta(2)}{L(\pi(\mathrm{JL}(f')),\mathrm{Ad},1)}.$$

2.5 Main result (first form)

Theorem 1.2, Proposition 2.4 and Proposition 2.5 imply the theorem as follows:

Theorem 2.6. *Under the assumption in Theorem 1.2 we have*

$$\frac{||\mathcal{L}(f,f')^{\chi}_{\xi}(g_0)||^2}{\langle f,f\rangle\langle f',f'\rangle} = C(f,f',\xi,\chi)L(\Pi,\chi^{-1},\frac{1}{2})L(\Pi',\chi^{-1},\frac{1}{2}),$$

where, if $\pi''_p|_{E_p^\times} = \chi_p$ *for* $p|d_B$ *ramified in* E *and* $S_1 = S_2^+ = \emptyset$, $C(f,f',\xi,\chi) =$

$$\frac{2^{2\kappa+|\delta(D)|-9}(\kappa+1)n(\xi)^{\frac{\kappa}{2}}|d_\xi|^{\frac{3}{2}}w(E)^2}{\pi \mathbf{h}(E)^2A(\chi)^2D^{\frac{3}{2}}}\prod_{p|A(\chi)}L_p(\eta_p,1)^4\prod_{p|d_B}r_p(1-p^{-1})\prod_{p<\infty}C_p(f,\xi,\chi)^2$$
$$\times\, e^{-8\pi\sqrt{n(\xi)}}\cdot\frac{\langle f'_{\infty,\kappa},f'_{\infty,\kappa}\rangle_\infty}{\langle f'_\infty,f'_\infty\rangle_\infty}\cdot\frac{\zeta(2)}{L(\pi(f),\mathrm{Ad},1)L(\pi(\mathrm{JL}(f')),\mathrm{Ad},1)},$$

and $C(f,f',\xi,\chi)=0$ *otherwise.*

2.6 Main result (second form)

We introduce the degree eight L-function attached to $\mathcal{L}(f,f')$ and χ, and relate its central value to the square norm $||\mathcal{L}(f,f')^{\chi}_{\xi}(g_0)||^2$ when $D=1$.

We now recall that $\mathcal{L}(f,f')$ belongs to $\mathcal{S}_\kappa$ (cf. §1.1). Before introducing the degree eight L-function, we define the spinor L-function for a Hecke eigenform $F\in\mathcal{S}_\kappa$. In [M-N-1, §5.1] we introduced three Hecke operator $\mathcal{T}_p^i$ with $0\le i\le 2$ for $p\not|d_B$. Let Λ_p^i be the Hecke eigenvalue of $\mathcal{T}_p^i$ for F with $0\le i\le 2$. For $p\not|d_B$ we put

$$Q_{F,p}(t) := 1-p^{-\frac{3}{2}}\Lambda_p^1t+p^{-2}(\Lambda_p^2+p^2+1)t^2-p^{-\frac{3}{2}}\Lambda_p^1t^3+t^4.$$

For this we note that $Q_{F,p}(p^{-s})^{-1}$ coincides with the local spinor L-function for an unramified principal series of the group of $\mathbb{Q}_p$-rational points for the split symplectic $\mathbb{Q}$-group $GSp(2)$ of degree two with similitudes. Here recall that $GSp(2)$ is defined by

$$GSp(2)_{\mathbb{Q}} := \left\{g\in GL(4)_{\mathbb{Q}}\;\middle|\; {}^tg\begin{pmatrix}0_2 & 1_2\\ -1_2 & 0_2\end{pmatrix}g=\nu(g)\begin{pmatrix}0_2 & 1_2\\ -1_2 & 0_2\end{pmatrix},\ \nu(g)\in\mathbb{Q}^\times\right\}.$$

On the other hand, in [M-N-1, §5.2], we introduced two Hecke operators $\mathcal{T}_p^i$ with $0\le i\le 1$ for $p|d_B$. Let Λ'^i_p be the Hecke eigenvalue of $\mathcal{T}_p^i$ for F with $0\le i\le 1$. For $p|d_B$ we put

$$Q_{F,p}(t) := (1-p^{-\frac{3}{2}}(\Lambda'^1_p-(p^{A_p}-1)\Lambda'^0_p)t+t^2)(1-\Lambda'^0_pp^{-\frac{1}{2}}t),$$

where $A_p := \begin{cases}1 & (p\not|D)\\ 2 & (p|D)\end{cases}$. This is due to Sugano [Su, (3.4)].

We define the spinor L-function $L(F,\mathrm{spin},s)$ of F by

$$L(F,\mathrm{spin},s) := \prod_{v\le\infty}L_v(F,\mathrm{spin},s),$$

where

$$L_v(F,\mathrm{spin},s) := \begin{cases} Q_{F,p}(p^{-s})^{-1} & (v = p < \infty), \\ \Gamma_{\mathbf{C}}(s+\frac{\kappa-1}{2})\Gamma_{\mathbf{C}}(s+\frac{\kappa+1}{2}) & (v = \infty). \end{cases}$$

This is a modification of the definition in [M-N-1, §5.3]. We can then reformulate [M-N-1, Corollary 5.3] when $D = 1$.

Proposition 2.7. *The spinor L-function for $\mathcal{L}(f,f')$ decomposes into*

$$L(\mathcal{L}(f,f'),\mathrm{spin},s) = L(\pi(f),s)L(\pi(\mathrm{JL}(f')),s).$$

Of course, when $D = 1$, we see that $L(\mathcal{L}(f,f'),\mathrm{spin},s)$ has the meromorphic continuation and satisfies the functional equation between s and $1-s$ since so do $L(\pi(f),s)$ and $L(\pi(\mathrm{JL}(f')),s)$.

We now introduce the L-function

$$L(F,\chi^{-1},s) := \prod_{v\leq\infty} L_v(F,\chi^{-1},s)$$

of degree eight for a Hecke eigenform $F \in \mathcal{S}_\kappa$ and χ. Here the local factors $L_v(F,\chi^{-1},s)$ are given as

$$L_v(F,\chi^{-1},s) := \begin{cases} Q_{F,p}(\alpha_p^\chi p^{-s})^{-1}Q_{F,p}(\beta_p^\chi p^{-s})^{-1} & (\chi \text{ is unramified at } v = p < \infty) \\ 1 & (\chi \text{ is ramified at } v = p < \infty), \\ \Gamma_{\mathbf{C}}(s+\kappa-\frac{1}{2})\Gamma_{\mathbf{C}}(s+\frac{1}{2})\Gamma_{\mathbf{C}}(s+\kappa+\frac{1}{2})\Gamma_{\mathbf{C}}(s+\frac{1}{2}) & (v = \infty), \end{cases}$$

where

$$(\alpha_v^\chi,\beta_v^\chi) := \begin{cases} (\chi_p(\varpi_{p,1})^{-1},\chi_p(\varpi_{p,2})^{-1}) & (v = p\text{: split}), \\ (\chi_p(p)^{-1},-\chi_p(p)^{-1}) = (1,-1) & (v = p\text{: inert}), \\ (\chi_p(\varpi_p)^{-1},0) & (v = p\text{: ramified}) \end{cases}$$

with prime elements $\varpi_{p,1}$, $\varpi_{p,2}$ and ϖ_p introduced in Lemma 2.1.

Proposition 2.8. *Let $D = 1$. We have*

$$L(\mathcal{L}(f,f'),\chi^{-1},s) = L(\Pi,\chi^{-1},s)L(\Pi',\chi^{-1},s).$$

This has meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$L(\mathcal{L}(f,f'),\chi^{-1},s) = \epsilon(\Pi,\chi^{-1})\epsilon(\Pi',\chi^{-1})L(\mathcal{L}(f,f'),\chi^{-1},1-s),$$

where $\epsilon(\Pi,\chi^{-1})$ (resp. $\epsilon(\Pi,\chi^{-1})$) denotes the ϵ-factor of $L(\Pi,\chi^{-1},s)$ (resp. $L(\Pi',\chi^{-1},s)$).

In view of Proposition 2.4, Proposition 2.5 and Proposition 2.8, the central value $L(\mathcal{L}(f,f'),\chi^{-1},\frac{1}{2})$ is meaningful when $D = 1$. We are now able to reformulate Theorem 2.6 as follows:

Theorem 2.9. *Let the assumption be as in Theorem 1.2 and let $D = 1$. We have*

$$\frac{||\mathcal{L}(f,f')_\xi^\chi(g_0)||^2}{\langle f,f\rangle\langle f',f'\rangle} = C(f,f',\xi,\chi)L(\mathcal{L}(f,f'),\chi^{-1},\frac{1}{2}).$$

2.7 Main result (third form)

Note that the group $G = GSp(1,1)$ is an inner form of the symplectic $\mathbb{Q}$-group $GSp(2)$ of degree two with similitudes. As is well-known, the Langlands principle of functoriality [Lg-1] suggests that the L-function for a Hecke eigenform $F \in S_\kappa$ should be an L-function for some automorphic form (or automorphic representation) of $GSp(2)_{\mathbb{A}_\mathbb{Q}}$.

T. Okazaki [O] has recently constructed cusp forms on $GSp(2)_{\mathbb{A}_\mathbb{Q}}$, which contain forms whose spinor L-functions coincide with those of $\mathcal{L}(f,f')$'s. More precisely his construction uses a theta lifting from $GL(2)_{\mathbb{A}_\mathbb{Q}} \times GL(2)_{\mathbb{A}_\mathbb{Q}}$ to $GSp(2)_{\mathbb{A}_\mathbb{Q}}$ (or from $GO(2,2)_{\mathbb{A}_\mathbb{Q}}$ to $GSp(2)_{\mathbb{A}_\mathbb{Q}}$) and follows the formulation of Harris-Kudla [H-K]. He specifies an appropriate Schwarts-Bruhat function on $M_2(\mathbb{A}_\mathbb{Q})^{\oplus 2}$ to construct the theta kernel for the lifting. We denote this lifting by $\mathcal{L}_{GSp(2)}$. His result is then stated as follows:

Theorem 2.10 (Okazaki). *For two non-zero primitive cusp forms $(f_1, f_2) \in S_{\kappa_1}(N_1) \times S_{\kappa_2}(N_2)$, $\mathcal{L}_{GSp(2)}(f_1, f_2)$ is a non-zero generic cusp form on $GSp(2)_{\mathbb{A}_\mathbb{Q}}$ with the following properties:*

1. *it is a paramodular form of level N_1N_2, namely, at a prime $p|N := N_1N_2$, it is right invariant by a paramodular group*

$$\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p & N^{-1}\mathbb{Z}_p & \mathbb{Z}_p \\ N\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ N\mathbb{Z}_p & N\mathbb{Z}_p & \mathbb{Z}_p & N\mathbb{Z}_p \\ N\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \cap GSp(2)_{\mathbb{Q}_p},$$

2. *at the Archimedean place, the highest weight of $\mathcal{L}_{GSp(2)}(f_1, f_2)$ is*

$$(\frac{\kappa_1+\kappa_2}{2}, -\frac{|\kappa_1-\kappa_2|}{2}).$$

The global spinor L-function of $\mathcal{L}_{GSp(2)}(f_1, f_2)$ (in the sense of Novodvorsky [No]) decomposes into

$$L(\pi(f_1), s)L(\pi(f_2), s).$$

Also for $\mathcal{L}_{GSp(2)}(f_1, f_2)$ and a Hecke character χ we can similarly define the degree eight L-function $L(\mathcal{L}_{GSp(2)}(f_1, f_2), \chi^{-1}, s)$. For a Hecke eigenform $f' \in \mathcal{A}_\kappa$ we have put $\mathrm{JL}(f')$ to be the Jacquet-Langlands lift of f' (cf. §2.1) and note that $\mathrm{JL}(f')$ is primtive (cf. [E-1], [E-2], [Sh, §6]). From the theorem just mentioned and Proposition 2.8 we deduce the following:

Proposition 2.11. *Let $(f, f') \in S_\kappa(1) \times \mathcal{A}_\kappa$ be Hecke eigenforms with $\mathcal{L}(f,f') \not\equiv 0$. Then we have*

$$L(\mathcal{L}(f,f'), \mathrm{spin}, s) = L(\mathcal{L}_{GSp(2)}(f, \mathrm{JL}(f')), \mathrm{spin}, s),$$
$$L(\mathcal{L}(f,f'), \chi^{-1}, s) = L(\mathcal{L}_{GSp(2)}(f, \mathrm{JL}(f')), \chi^{-1}, s).$$

We thus know that Theorem 2.9 is restated as follows:

Theorem 2.12. *Let the assumption be as in Theorem 1.2 and let $D = 1$. We have*

$$\frac{||\mathcal{L}(f,f')^{\chi}_{\xi}(g_0)||^2}{\langle f,f\rangle\langle f',f'\rangle} = C(f,f',\xi,\chi)L(\mathcal{L}_{GSp(2)}(f,\mathrm{JL}(f')),\chi^{-1},\frac{1}{2}).$$

Remark 2.13. (1) This theorem is compatible with the conjecture of Furusawa-Martin [F-M] and Furusawa-Shalika [F-S], which are inspired by Böcherer [B].
(2) The example of the functorial correspondence above is essentially predicted by T. Ibukiyama in his study of automorphic forms on the compact inner form of $GSp(2)$ (cf. [I]). We can expect that automorphic forms in $\mathcal{S}_\kappa$ (cf. §1.1) with $D = 1$ should functorially correspond to paramodular forms on $GSp(2)_{\mathbb{A}_\mathbb{Q}}$ of level d_B.
(3) At the Archimedean place $\mathcal{L}(f,f')$ generates the quaternionic discrete series with minimal K_∞-type τ_κ, while $\mathcal{L}_{GSp(2)}(f,\mathrm{JL}(f'))$ generates the large discrete series with the Blattner parameter $(\kappa+1,-1)$. The latter is due to Przebinda [P, Chap.III, §3]. These two discrete series representations have the same infinitesimal character. That is, the functorial correspondence of these two global theta lifts is compatible with the Archimedean local functorial correspondence established by Langlands [Lg-2].

2.8 Strictly positive central L-values

As an application of our main results we show the existence of strictly positive central values for the L-functions in our concern. In this subsection we fix the quaternion algebra B and a maximal order $\mathfrak{O}$ of B as

$$B = \mathbb{Q} + \mathbb{Q}\cdot i + \mathbb{Q}\cdot j + \mathbb{Q}\cdot k \quad (i^2 = j^2 = -1,\ ij = -ji = k),$$
$$\mathfrak{O} = \mathbb{Z}\cdot\frac{1+i+j+k}{2} + \mathbb{Z}\cdot i + \mathbb{Z}\cdot j + \mathbb{Z}\cdot k.$$

We note that $d_B = 2$ for this B. In [M-N-2, §14] we have shown the following:

Proposition 2.14. *Suppose that $\xi = i/2$ and χ is unramified at every finite prime, i.e. $i_p(\chi) = 0$ for any finite prime p. Let $D \in \{1,2\}$ and $\kappa \geq 12$ (resp. $\kappa \geq 8$) be divisible by 4 (resp. by 8) when $D = 1$ (resp. $D = 2$). Then there exist Hecke eigenforms $(f,f') \in \mathcal{S}_\kappa(D)\times\mathcal{A}_\kappa$ such that $P_\chi(f,\gamma_0)P_\chi(f',\gamma_0') \neq 0$ (which implies $\mathcal{L}(f,f')^{\chi}_{\xi} \not\equiv 0$).*

We now assume that $D = 1$ and χ is unramified at every finite prime. From Proposition 2.4, Proposition 2.5 and this proposition we deduce the following:

Theorem 2.15. *Under the assumption in Proposition 2.14 there exist Hecke eigenforms $(f,f') \in \mathcal{S}_\kappa(1)\times\mathcal{A}_\kappa$ such that*

$$L(\Pi,\chi^{-1},\frac{1}{2}) > 0, \quad L(\Pi',\chi^{-1},\frac{1}{2}) > 0.$$

The point of our proof for this theorem is to show the positivity of the constants $C(f,\chi)$ and $C(f',\chi)$ for (f,f') in Proposition 2.14. As an immediate consequence of this theorem and Proposition 2.11 we have obtained the theorem as follows:

Theorem 2.16. *Keeping the assumption in Proposition 2.14, there are Hecke eigenforms* $(f, f') \in S_\kappa(1) \times \mathcal{A}_\kappa$ *such that*

$$L(\mathcal{L}(f, f'), \chi^{-1}, \frac{1}{2}) = L(\mathcal{L}_{GSp(2)}(f, \mathrm{JL}(f')), \chi^{-1}, \frac{1}{2}) > 0.$$

Remark 2.17. We remark that there are several results on the non-negativity of the central-values of the L-functions in Theorem 2.15 and Theorem 2.16 (cf. [Gu], [J-C], [Lp] etc.).

3 Outline of the proof of Proposition 2.5

In order to verify Theorem 2.6 and Theorem 2.9 it remains to show Proposition 2.5. In this section we choose the measure of $\mathbb{A}_\mathbb{Q}^\times$ so that

$$\mathrm{vol}(\mathbb{Z}_p^\times) = 1 \text{ for any } p < \infty.$$

For the proof of the proposition we need two formulas. The first formula is due to Waldspurger [Wa, Proposition 7]. In order to review it we need several remarks. We first remark that every π''_v is unitary and thus equipped with a unitary inner product. When $v = p \not| d_B$ we embed $E_p^\times$ into $B_p^\times = GL_2(\mathbb{Q}_p)$ by ι_ξ (cf. §1.4). Recall that, for the quadratic character η attached to the quadratic extension $E/\mathbb{Q}$, we have let $L(\eta, s)$ be the L-function defined by η and $L_v(\eta_v, s)$ the local factor of $L(\eta, s)$ at a place v (cf. §2.3).

Proposition 3.1 (Waldspurger). *For* $b = (b_v)_{v \le \infty} \in B_{\mathbb{A}_\mathbb{Q}}^\times$,

$$\frac{||P_\chi(f'; b)||^2}{\langle f', f' \rangle} = \frac{\sqrt{|d_\xi|}}{8\pi} \frac{\zeta(2) L(\Pi', \chi^{-1}, \frac{1}{2})}{L(\pi(\mathrm{JL}(f')), \mathrm{Ad}, 1)} \prod_{v \le \infty} \alpha_v(f'_v, \chi_v, b_v),$$

where

$$\alpha_v(f'_v, \chi_v, b_v) := \frac{L_v(\eta_v, 1) L_v(\pi'_v, \mathrm{Ad}, 1)}{\zeta_v(2) L_v(\Pi'_v, \chi_v^{-1}, \frac{1}{2})} \int_{\mathbb{Q}_v^\times \backslash E_v^\times} \frac{\langle \pi''_v(tb_v) f'_v, \pi''_v(b_v) f'_v \rangle_v}{\langle f'_v, f'_v \rangle_v} \chi_v(t)^{-1} dt$$

with an inner product $\langle *, * \rangle_v$ *of* π''_v *for each place* v.

Here we note that the normalization of our measure dt differs from that of [Wa] by $\frac{\sqrt{|d_\xi|}}{4\pi}$-multiple, and furthermore note that the toral integral in the sense of [Wa] is replaced by ours under the normalization of the measure of $\mathbb{A}_\mathbb{Q}^\times$ mentionted above.

The second formula is the well-known explicit formula for zonal spherical functions by Macdonald [Ma, Chap.V, §3, (3.4)]. We put

$$\phi_v(g) := \frac{\langle \pi''_v(g) f'_v, f'_v \rangle_v}{\langle f'_v, f'_v \rangle_v} \quad (g \in B_v^\times)$$

for each place v. We note that π''_p at $p \not| d_B$, which is isomorphic to π'_p, is a unitary unramified principal series representation of $GL_2(\mathbb{Q}_p)$ with a spherical vector f'_p. The Satake parameter of π''_p is of the form $(\alpha_p, \alpha_p^{-1}) \in (\mathbb{C}^\times)^2$ with $|\alpha_p| = 1$. We see that $\phi_p(g)$ is a zonal spherical function on $GL_2(\mathbb{Q}_p)$ for $v = p \not| d_B$.

Proposition 3.2 (Macdonald). *Let p be a finite prime not dividing d_B. For $a_m := \begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix}$ with $m \geq 0$ we have*

$$\phi_p(a_m) = \frac{p^{-\frac{m}{2}}}{1+p^{-1}}(\alpha_p^m \frac{1-p^{-1}\alpha_p^{-2}}{1-\alpha_p^{-2}} + \alpha_p^{-m}\frac{1-p^{-1}\alpha_p^2}{1-\alpha_p^2}).$$

This formula is useful to evaluate the local integrals involved in $\alpha_v(f'_v, \chi_v, b_v)$ for $v = p$ not dividing d_B. The evaluation of such local integrals at other places is settled by a direct calculation. We furthermore note that $\alpha_v(f'_v, \chi_v, b_v)$'s have factors contributed by ratios of local L-functions, which are computed by Lemma 2.1, Lemma 2.2 and Lemma 2.3. We now state the following formula.

Proposition 3.3. *(1) For $p \not| d_B$,*

$$\alpha_p(f'_p, \chi_p, \gamma'_{0,p}) = \begin{cases} 1 & (i_p(\chi) = 0), \\ p^{-i_p(\chi)} L_p(\eta_p, 1)^2 & (i_p(\chi) > 0). \end{cases}$$

(2) When $p | d_B$ we have

$$\alpha_p(f'_p, \chi_p, \gamma'_{0,p}) = \begin{cases} r_p(1-p^{-1}) & (p \text{ is inert or } p \text{ is ramified and } \omega_p = \delta_P), \\ 0 & (p \text{ is ramified and } \omega_p \neq \delta_p), \end{cases}$$

where r_p is the ramification index of p for the quadratic extension $E/\mathbb{Q}$ (cf. §2.4).
(3) When $v = \infty$ we have

$$\alpha_\infty(f'_\infty, \chi_\infty^{-1}, \gamma'_{0,\infty}) = \frac{\kappa+1}{4} \frac{\langle f'_{\infty,\kappa}, f'_{\infty,\kappa}\rangle_\infty}{\langle f'_\infty, f'_\infty\rangle_\infty},$$

where see §2.4 for $f'_{\infty,\kappa}$.

As a result of Proposition 3.1 and Proposition 3.3 we have proved Proposition 2.5.

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