数理解析研究所講究録
第 1715 巻 2010 年 169-182

The Taylor expansion of Jacobi forms of general degree and some application to explicit structures of higher indices

Tomoyoshi Ibukiyama
Department of Mathematics, Graduate School of Science
Osaka University

We denote by H_n the Siegel upper half space of degree n. Jacobi forms $F(\tau,z)$ of degree n are functions of $(\tau,z)\in H_n\times\mathbb{C}^n$ which satisfy the same automorphic properties as those fuctions appearing as coefficients of the Fourier expansion of Siegel modular forms of degree $n+1$ with respect to the $(n+1,n+1)$-component of H_{n+1}. A systematic extensive study was done in Eichler-Zagier's book [2] in the case $n=1$. In this short note, we announce the following two results.
(1) For general degree n, the Taylor coefficients of $F(\tau,z)$ along $z=0$ are described by vector valued Siegel modular forms of various weights.
(2) We apply (1) to give explicit structures of the modules of Jacobi forms of degree $n=2$ w.r.t. $\Gamma_2=Sp(2,\mathbb{Z})$ of index one and two over the ring of Siegel modular forms of even weights.
The assertion (1) is a generalization of Eichler-Zagier, where a mapping from Jacobi forms of degree one to a product of modular forms of various weights is explicitly given. In (2), the results for the index one case was already given before in [3] by using correspondence with Siegel modular forms of half-integral weight in [4], but we give an alternative simpler proof here.

More details of the results and proofs in this article will appear elsewhere.

The author would like to thank Samuel Grushevsky for a discussion in June in 2009 at Osaka, which definitely convinced the author that the vanishing order of Jacobi forms for higher degree is much more complicated than one variable case.

1 Jacobi forms and Siegel modular forms

We review several definitions here. We denote by $Sp(n,\mathbb{R})$ the symplectic group of rank n defined by

$$Sp(n,\mathbb{R}) = \{g \in M_{2n}(\mathbb{R});\ gJ_n\ {}^tg = J_n\}$$

where $J_n = \left(\begin{smallmatrix} 0_n & -1_n \\ 1_n & 0_n \end{smallmatrix}\right)$ and 1_n is the unit matrix of size n. We denote by Γ_n the Siegel modular group of level one defined by $\Gamma_n = Sp(n,\mathbb{R}) \cap M_{2n}(\mathbb{Z})$. For any finite dimensional rational representation (ρ, V) of $GL_n(\mathbb{C})$, any V-valued function $F(\tau)$ on H_n, and any element $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$, we write

$$(F|_\rho[g])(\tau) = \rho(CZ+D)^{-1}F(g\tau)$$

A holomorphic function $F(\tau)$ on H_n is called a Siegel modular form of weight ρ w.r.t. Γ_n if $F|_\rho[\gamma] = F$ for all $\gamma \in \Gamma_n$ (and is holomorphic at $i\infty$ if $n=1$.) We denote by $A_\rho(\Gamma_n)$ the vector space of such functions. In this article, we mainly treat the case when the weight is $\rho_{k,\nu} = \det^k Sym_\nu$, the tensor product of $\det^k$ and the symmetric tensor representation Sym_ν of degree ν. When $\rho = \rho_{k,\nu}$ we write $A_\rho(\Gamma_n) = A_{k,\nu}(\Gamma_n)$ and if $\nu = 0$ besides, we write $A_\rho(\Gamma_n) = A_k(\Gamma_n)$. Elements of $A_k(\Gamma_n)$ is called of weight k.

The representation $\rho_{k,\nu}$ is realized as follows. The representation space V_ν of $\rho_{k,\nu}$ is the vector space of homogeneous polynomials $P(u) = P(u_1,\ldots,u_n)$ of degree ν of n variables and the action of $g \in GL_n(\mathbb{C})$ on V_ν is given by $P \to \det(g)^k P(ug)$. For $\alpha = (\alpha_1,\ldots,\alpha_n) \in (\mathbb{Z}_{\geq 0})^n$ and a variable vector $u = (u_1,\ldots,u_n)$, we write $u^\alpha = \prod_{i=1}^n u_i^{\alpha_i}$. We also write $|\alpha| = \sum_{i=1}^n \alpha_i$. Then a holomorphic V_ν-valued function F is identified with

$$F = \sum_{|\alpha|=\nu} f_\alpha(\tau)u^\alpha.$$

So to emphasize that it is a polynomial of u, we sometimes write $F = F(\tau, u)$. The automorphy of $F \in A_{k,\nu}(\Gamma_n)$ means

$$F(g\tau, u) = \det(c\tau + d)^k F(\tau, ug).$$

Or if we write u as a column vector, this relation is written also as

$$F(g\tau,\ {}^tg^{-1}u) = \det(c\tau+d)^k F(\tau,u).$$

Example: When $n = \nu = 2$, $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2$, $C\tau + D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, and $F(\tau,u) = f_{20}(\tau)u_1^2 + f_{11}(\tau)u_1u_2 + f_{02}(\tau)u_2^2 \in A_{k,2}(\Gamma_2)$, we have

$$\begin{pmatrix} f_{20}(g\tau) \\ f_{11}(g\tau) \\ f_{02}(g\tau) \end{pmatrix} = \det(c\tau+d)^k \begin{pmatrix} \alpha^2 & \alpha\beta & \beta^2 \\ 2\alpha\gamma & \alpha\delta+\beta\gamma & 2\beta\delta \\ \gamma^2 & \gamma\delta & \delta^2 \end{pmatrix} \begin{pmatrix} f_{20}(\tau) \\ f_{11}(\tau) \\ f_{02}(\tau) \end{pmatrix}.$$

Next we review the definition of Jacobi forms. We define the Jacobi modular group of degree n by

$$\begin{aligned}\Gamma_n^J &= \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1_n & 0 & 0 & \mu \\ {}^t\lambda & 1 & {}^t\mu & \kappa \\ 0 & 0 & 1_n & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_n, \lambda, \mu \in \mathbb{Z}^n, \kappa \in \mathbb{Z} \right\} \\ &\cong \Gamma_n \cdot (\mathbb{Z}^n \times \mathbb{Z}^n) \cdot \mathbb{Z}.\end{aligned}$$

We write element of H_{n+1} by $\begin{pmatrix} \tau & z \\ {}^t z & \omega \end{pmatrix}$ where $(\tau, z) \in H_n \times \mathbb{C}^n$.

For any integer m and a complex number x, we write $e^m(x) = exp(2\pi i m x)$. For any $\gamma \in \Gamma_n^J$ and a holomorphic function $F(\tau, z)$ on $H_n \times \mathbb{C}^n$, we have $(F(\tau, z)e^m(\omega))|_k[\gamma] = \widetilde{F}(\tau, z)e^m(\omega)$ for some unique holomorphic function $\widetilde{F}$ on $H_n \times \mathbb{C}^n$. We write $\widetilde{F} = F|_{k,m}[\gamma]$. When $n \geq 2$, we say that a holomorphic function F on $H_n \times \mathbb{C}^n$ is a Jacobi form of weight k of index m w.r.t. Γ_n^J if $f|_{k,m}[\gamma] = f$ for any $\gamma \in \Gamma_n^J$. When $n = 1$, we need some conditions of the Fourier expansion at cusps besides (see below), but this is unnecessary when $n \geq 2$ by Koecher principle proved by Ziegler in [14]. By automorphy, any Jacobi form $F(\tau, z)$ has the following Fourier expansion.

$$F(\tau, z) = \sum_{N,r} a(N, r) e(Tr(N\tau) + {}^t r z)$$

where N runs over positive semi-definite half integral symmetric matrices and r over $\mathbb{Z}^n$. We have $a(N, r) = 0$ unless $4Nm - r\,{}^t r \geq 0$ (positive semi-definie) by Koecher principle for $n \geq 2$ or the definition for $n = 1$. Here note that r is a column vector, so $r\,{}^t r$ is an $n \times n$ matrix. We say that F is a Jacobi cusp form when $a(N, r) = 0$ unless $4Nm - r\,{}^t r > 0$ (positive definite). We denote by $J_{k,m}(\Gamma_n^J)$ the space of Jacobi forms defined above and $J_{k,m}^{cusp}(\Gamma_n^J)$ the space of Jacobi cusp forms. We note that if $m > 0$, then $J_{0,m}(\Gamma_n^J) = 0$.

2 Taylor expansion and Theta expansion

Since a Jacobi form $F(\tau, z)$ is a holomorphic function, we have the Taylor expansion along $z = 0$. We write this expansion as

$$F(\tau, z) = \sum_{\nu=0}^{\infty} \left(\sum_{|\alpha|=\nu} f_\alpha(\tau) z^\alpha \right)$$

where $\alpha \in (\mathbb{Z}_{\geq 0})^n$. We also write $f_\nu(\tau, z) = \sum_{|\alpha|=\nu} f_\alpha(\tau) z^\alpha$. The coefficients f_α are holomorphic functions on H_n. They are closely related to Siegel modular forms of degree n as we shall see later. When $n = 1$, Eichler-Zagier proved the following claims. (cf. [2])

Claim 1. For each integer $l \geq 0$, we can construct a modular form $\xi_{k+2l}(\tau) \in M_{k+2l}(\Gamma_1)$ from Taylor coefficients $(f_0(\tau), f_2(\tau), \ldots, f_{2l}(\tau))$ of a Jacobi form in $J_{k,m}(\Gamma_1^J)$. This is explicitly given by using differential operators on $f_\nu(\tau)$ w.r.t. variables τ.

Claim 2 The linear mapping
$J_{k,m}(\Gamma_1^J) \to M_k(\Gamma_1) \times M_{k+2}(\Gamma_1) \times \cdots \times M_{k+2m}(\Gamma_1)$
induced by the above construction is injective. In other words, the Jacobi form F is determined by the Taylor coefficients up to z^{2m}.

Claim 3 This induces a surjective isomorphism from $J_{k,1}(\Gamma_1^J)$ to $M_k(\Gamma_1) \oplus S_{k+2}(\Gamma_1)$ for $k > 0$.

Now we generalize this for higher n. For the sake of simplicity, we assume now that nk is even. Then we have $f_\nu(\tau, z) = 0$ for any odd ν. We denote by u a variable column vector of length n. We denote by $Hol_{2\nu}[u]$ the vector space of polynomials in u_1, u_2, $\ldots$, u_n of degree 2ν with holomorphic coefficients. We define a differential operator $\mathcal{D}$ of $Hol_{2\nu}[u]$ to $Hol_{2\nu+2}[u]$ by

$$\mathcal{D} = {}^t u \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial \tau_{ij}} \right) u = \sum_{i \leq j} u_i u_j \frac{\partial}{\partial \tau_{ij}}$$

where δ_{ij} are Kronecker's delta. For Taylor coefficients of $F(\tau, z)$ up to degree 2ν: $(f_0(\tau, z), f_2(\tau, z), \ldots, f_{2\nu}(\tau, z))$, which are polynomials in z, we define $\xi_{2\nu}(\tau, u) \in Hol_{2\nu}[u]$ by

$$\begin{aligned} \xi_{k,2l}(\tau, u) &= \sum_{\mu=0}^{l} \frac{(k+2l-\mu-2)!}{\mu!(k+2l-2)!} (-2\pi i m)^\mu (\mathcal{D}^\mu f_{2l-2\mu})(\tau, u) \\ &= f_{2\nu}(\tau, u) + \text{ constant times derivations of } f_{2l}(\tau, u) \text{ with } l < \nu. \end{aligned}$$

For example, we have

$$\begin{aligned} \xi_0(\tau, u) &= \chi_0(\tau), \\ \xi_2(\tau, u) &= \sum_{|\alpha|=2} f_\alpha(\tau) u^\alpha - \frac{2\pi i m}{k} \sum_{1 \leq i \leq j \leq n} \frac{\partial f_0(\tau)}{\partial \tau_{ij}} u_i u_j. \end{aligned}$$

To make it readable, we give a concrete shape of $\xi_4(\tau, u)$ only in the case

$n = 2$. In this case we have

$$\begin{aligned}
\xi_4(\tau, u) &= (f_{40}(\tau)u_1^4 + f_{31}(\tau)u_1^3u_2 + f_{22}(\tau)u_1^2u_2^2 + f_{13}(\tau)u_1u_2^3 + f_{04}(\tau)u_2^4) \\
&\quad -\frac{2\pi im}{k+2}\left(\frac{\partial f_{20}(\tau)}{\partial \tau_1}u_1^4 + \left(\frac{\partial f_{20}(\tau)}{\partial z_0} + \frac{\partial f_{11}(\tau)}{\partial \tau_1}\right)u_1^3u_2\right. \\
&\quad + \left(\frac{\partial f_{20}(\tau)}{\partial \tau_2} + \frac{\partial f_{11}(\tau)}{\partial z_0} + \frac{\partial f_{02}(\tau)}{\partial \tau_1}\right)u_1^2u_2^2 \\
&\quad \left. + \left(\frac{\partial f_{11}(\tau)}{\partial \tau_2} + \frac{\partial f_{02}(\tau)}{\partial z_0}\right)u_1u_2^3 + \frac{\partial f_{02}(\tau)}{\partial \tau_2}u_2^4\right) \\
&\quad + \frac{(2\pi im)^2}{2(k+2)(k+1)}\left(\frac{\partial^2 f_0(\tau)}{\partial \tau_1^2}u_1^4 + 2\frac{\partial^2 f_0(\tau)}{\partial \tau_1 \partial z_0}u_1^3u_2\right. \\
&\quad \left. + \left(\frac{\partial^2 f_0(\tau)}{\partial z_0^2} + 2\frac{\partial^2 f_0(\tau)}{\partial \tau_1 \partial \tau_2}\right)u_1^2u_2^2 + 2\frac{\partial^2 f_0(\tau)}{\partial z_1 \partial \tau_2}u_1u_2^3 + \frac{\partial^2 f_0(\tau)}{\partial \tau_2^2}u_2^4\right),
\end{aligned}$$

where we write

$$F(\tau, z) = f_0(\tau) + f_{20}(\tau)z_1^2 + f_{11}(\tau)z_1z_2 + f_{02}(\tau)z_2^2 + f_{40}(\tau)z_1^4 + \cdots$$

Theorem 2.1 *We have $\xi_{k,2l}(\tau, u) \in A_{k,2l}(\Gamma_n)$. Conversely, $f_0(\tau, u)$ to $f_{2\nu}(\tau, u)$ are determined by $\xi_{k,0}(\tau, u)$, ..., $\xi_{k,2\nu}(\tau, u)$.*

So this induces a linear mapping from $J_{k,m}(\Gamma_n^J)$ to $A_k(\Gamma_n) \times A_{k,2}(\Gamma_n) \times \cdots \times A_{k,2l}(\Gamma_n)$ for any $l \in \mathbb{Z}_{\geq 0}$.

This is a kind of generalization of the case $n = 1$ since when $n = 1$ we have $\det^k Sym_{2l} = \det^{k+2l}$. When $n = 1$ the induced mapping from $J_{k,m}(\Gamma_1)$ to $A_k(\Gamma_1) \times A_{k+2}(\Gamma_2) \times \cdots \times A_{k,2m}(\Gamma_1)$ is injective. This is not true for general n. In fact, there exist non-zero Jacobi forms whose Taylor coefficients vanish up to degree $2m$, as we see later. It does not seem to be known how many vanishings of Taylor coefficients of $F(\tau, z)$ assure $F(\tau, z) = 0$ in general, and this seems an interesting question. (There are several algebro-geometric results for each fixed τ but they do not answer well to our stand point on modular forms.)

We omit the details of the proof of the above theorem, but there are two ways to do this. One is to show this directly by calculation, which is possible and not too complicated. The other is to apply a genaral theory of differential operators on Siegel modular forms which preserve automorphy well under restriction from H_{n+1} to $H_n \times H_1$. (cf. [5] for a general theory.)

Now we explain another expansion of $F(\tau, z)$ which we call "theta expansion". First of all, for any $m \in \mathbb{Z}_{>0}$, if $F \in J_{k,m}(\Gamma_n^J)$, then we have

$$F(\tau, z + \tau\lambda + \mu) = e(-m({}^t\lambda\tau\lambda + 2\,{}^t\lambda z))F(\tau, z)$$

for any λ, $\mu \in \mathbb{Z}^n$, where we put $e(x) = e^{2\pi i x}$ for any $x \in \mathbb{C}$. For any $\nu \in \mathbb{Z}^n$, we put

$$\vartheta_{\nu,m}(\tau, z) = \sum_{p\in\mathbb{Z}^n} e\left({}^t\left(p+\frac{\nu}{2m}\right)(m\tau)\left(p+\frac{\nu}{2m}\right) + {}^t\left(p+\frac{\nu}{2m}\right)(2mz)\right).$$

This depends only on ν mod $2m$, so there are $(2m)^n$ functions. Then by the well-known theory of theta functions, we have

$$F(\tau, z) = \sum_{\nu\in(\mathbb{Z}/2m)^n} c_\nu(\tau)\vartheta_{\nu,m}(\tau, z)$$

for some holomorphic functions $c_\nu(\tau)$ on H_n. But if F is a Jacobi form, then it satisfies automorphy also for Γ_n, so we can say a little more. By the action of $-1_{2n} \in \Gamma_n$, we have $F(\tau, -z) = (-1)^{nk}F(\tau, z)$, so for example if nk is even, then $F(\tau, z)$ is an even function of z. But we also have $\vartheta_{\nu,m}(\tau, -z) = \vartheta_{-\nu,m}(\tau, z)$, so this means that $c_\nu(\tau) = c_{-\nu}(\tau)$. If the index $m = 1$, this does not give any new condition, since $-\nu \equiv \nu \bmod 2$ and theta functions $\vartheta_{\nu,1}(\tau, z)$ are all even functions of z. But when $m > 1$, then the above relation gives a real restriction. We return to this point later for explicit examples.

3 Explicit structures

We define the ring of Siegel modular forms by

$$A(\Gamma_2) = \oplus_{k=0}^{\infty} A_k(\Gamma_2) \qquad \text{and} \qquad A_{even}(\Gamma_2) = \oplus_{k=0}^{\infty} A_{2k}(\Gamma_2).$$

For any fix natural number m, we write $J_m(\Gamma_2^J) = \oplus_{k>0}^{\infty} J_{k,m}(\Gamma_2^J)$ and $J_{m,even}(\Gamma_2^J) = \oplus_{k>0} J_{2k,m}(\Gamma_2^J)$. These modules are obviously an $A(\Gamma_2)$ module and also an $A_{even}(\Gamma_2)$-module. We would like to study the structure of these module only over $A_{even}(\Gamma_2)$ since it becomes inessentially complicated if we regard it as a module over $A(\Gamma_2)$.

First we give a result for $n = 2$ and $m = 1$. When k is odd, we have $A_{k,j}(\Gamma_2) = S_{k,j}(\Gamma_2)$ for any $j \geq 0$. For odd k, we put

$$S_{k,2}^0(\Gamma_2) = \left\{ f(Z, u) \in A_{k,2}(\Gamma_2); f\left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}, u\right) = 0 \right\}.$$

We can define $S_k^0(\Gamma_2)$ in the same way, but this is redundant since any element in $S_k(\Gamma_2) = A_k(\Gamma_2)$ vanishes identically on the diagonals for any odd k.

Theorem 3.1 *We assume that $n = 2$.*
(1) The mapping

$$J_{k,1}(\Gamma_2^J) \to (\xi_{k,0}(\tau), \xi_{k,2}(\tau, u)) \in A_k(\Gamma_2) \times A_{k,2}(\Gamma_2)$$

is injective.
(2) If k is even with $k \geq 2$, this is also surjective.
(3) If k is odd, then the image of the mapping in (1) is $S_k(\Gamma_2) \times S_{k,2}^0(\Gamma_2)$.
(4) $J_{even,1}(\Gamma_2^J)$ is a free $A_{even}(\Gamma_2)$ module spanned by Jacobi forms of weight 4, 6, 10, 12, 21, 27, 29, 35.

The content of this theorem is essentially contained in [3]. The proof there used sutructures of the "plus" space (a kind of space of new forms) of Siegel modular forms of half-integral weight of level 4 with or without character, since $J_{k,1}(\Gamma_2^J)$ is isomorphic to this space (cf. [4], [3]). We roughly sketch a more direct proof here.

For any $F(\tau, z) \in J_{k,1}(\Gamma_2^J)$, we write

$$F(\tau, z) = \chi_0(\tau) + (2\pi i)^2 \left(\frac{1}{2}\chi_{20}(\tau)z_1^2 + \chi_{11}(\tau)z_1z_2 + \frac{1}{2}\chi_{02}(\tau)z_2^2\right) + \cdots$$

where $z = {}^t(z_1, z_2)$. We also use the theta expansion. Here for $n = 2$ and $\nu \in (\mathbb{Z}/2\mathbb{Z})^2$, we put $\vartheta_\nu(\tau, z) = \vartheta_{\nu,1}(\tau, z)$ and $\vartheta_\nu(\tau) = \vartheta_\nu(\tau, 0)$. Then we have

$$F(\tau, z) = c_{00}(\tau)\vartheta_{00}(\tau, z) + c_{01}(\tau)\vartheta_{01}(\tau, z) + c_{10}(\tau)\vartheta_{10}(\tau, z) + c_{11}(\tau)\vartheta_{11}(\tau, z)$$

for some holomorphic functions $c_\nu(\tau)$. Here $c_\nu(\tau)$ are uniquely determined by F. We write $\partial_i = \frac{1}{2\pi i}\frac{\partial}{\partial z_i}$ for $i = 1$ and 2. Then we have a simultaneous equation

$$A(\tau)\begin{pmatrix} c_{00}(\tau) \\ c_{01}(\tau) \\ c_{10}(\tau) \\ c_{11}(\tau) \end{pmatrix} = \begin{pmatrix} \chi_{00}(\tau) \\ \chi_{20}(\tau) \\ \chi_{11}(\tau) \\ \chi_{02}(\tau) \end{pmatrix}$$

where we put

$$A(\tau) = \begin{pmatrix} \vartheta_{00}(\tau) & \vartheta_{01}(\tau) & \vartheta_{10}(\tau) & \vartheta_{11}(\tau) \\ \partial_1^2\vartheta_{00}(\tau, z)|_{z=0} & \partial_1^2\vartheta_{01}(\tau, z)|_{z=0} & \partial_1^2\vartheta_{10}(\tau, z)|_{z=0} & \partial_1^2\vartheta_{11}(\tau, z)|_{z=0} \\ \partial_1\partial_2\vartheta_{00}(\tau, z)|_{z=0} & \partial_1\partial_2\vartheta_{01}(\tau, z)|_{z=0} & \partial_1\partial_2\vartheta_{10}(\tau, z)|_{z=0} & \partial_1\partial_2\vartheta_{11}(\tau, z)|_{z=0} \\ \partial_2^2\vartheta_{00}(\tau, z)|_{z=0} & \partial_2^2\vartheta_{01}(\tau, z)|_{z=0} & \partial_2^2\vartheta_{10}(\tau, z)|_{z=0} & \partial_2^2\vartheta_{11}(\tau, z)|_{z=0} \end{pmatrix}$$

Since theta functions satisfy heat equations, we can replace $\partial_i\partial_j\vartheta_\nu(\tau, z)|_{z=0}$ by $\frac{\partial\vartheta_\nu(\tau)}{\partial\tau_1}$, $\frac{\partial\vartheta_\nu(\tau)}{\partial z_0}$ or $\frac{\partial\vartheta_\nu(\tau)}{\partial\tau_2}$ up to constants, where we write $\tau = \begin{pmatrix} \tau_1 & z_0 \\ z_0 & \tau_2 \end{pmatrix}$

and we can show that $\det(A(\tau)) = \chi_5(\tau)$, where $\chi_5(\tau)$ is the unique cusp form of weight 5 (up to constants) with respect to the subgroup Γ_e of Γ_2 of index two containing $\Gamma(2)$, which is unique. Here it is well known that $\chi_5(\tau)$ vanishes only on the Γ_2-orbit of the diagonals of H_2 and the vanishing order is one. Anyway, $\det(A(\tau))$ does not vanish identically, so the mapping of $J_{k,1}(\Gamma_2^J)$ to $A_k(\Gamma_2) \times A_{k,2}(\Gamma_2)$ is injective. When k is even, by comparing the dimensions, we can see that the mapping is surjective also. This is proved more directly as follows without dimension formula. Denote by $\widetilde{A}(\tau)$ the cofactor matrix of $A(\tau)$. Then we see easily that the first, second and the fourth row are zero on the diagonals. When k is even, by the automorphy of $(\chi_{20}(\tau), 2\chi_{11}(\tau), \chi_{02}(\tau))$ up to derivations of $\chi_{00}(\tau)$ with respect to the transformation $(\tau_1, z_0, \tau_2) \to (\tau_1, -z_0, \tau_2)$ means that $\chi_{11}(\tau)$ vanishes on the diagonals. So $A(\tau)^{-1}\chi(\tau)$ is holomorphic on the diagonals when k is even, where we put $\chi(\tau) = {}^t(\chi_{00}(\tau), \chi_{01}(\tau), \chi_{10}(\tau), \chi_{11}(\tau))$. By automorphy w.r.t. Γ_2, this means that $c(\tau)$ is holomorphic on H_2 too. By the uniqueness of $c_\nu(\tau)$, we see that the corresponding theta expansion gives a Jacobi form. When k is odd, the map to $A_k(\Gamma_2) \times A_{k,2}(\Gamma_2)$ is not surjective since $\chi_{11}(\tau)$ might not vanish on the diagonals. Imposing this condition, we have the results for odd k directly or by comparison of dimensions. More details will appear elsewhere.

By the way, we give generating functions of related dimensions. The first one is due to Igusa and the rests are due to Tsushima (cf. [9], [12]). We have

$$
\begin{aligned}
\sum_{k=0}^{\infty} \dim A_k(\Gamma_2) &= \frac{1+t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})} \\
\sum_{k=0}^{\infty} \dim A_{k,2}(\Gamma_2) t^k &= \frac{t^{10}+t^{14}+2t^{16}+t^{18}-t^{20}-t^{26}-t^{28}+t^{32}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})} \\
&\quad + \frac{t^{21}+t^{23}+t^{27}+t^{29}-t^{33}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})} \\
\sum_{k=1}^{\infty} \dim J_{k,1}(\Gamma_2^J) t^k &= \frac{(t^4+t^6+t^{10}+t^{12})+(t^{21}+t^{27}+t^{29}+t^{35})}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}
\end{aligned}
$$

We also have

$$
\sum_{k=0,k:odd}^{\infty} \dim S_{k,2}^0(\Gamma_2) t^k = \frac{t^{21}+t^{27}+t^{29}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}
$$

This is obtained by an explicit description of $\oplus_{k=0,k:odd}^{\infty} A_{k,2}(\Gamma_2)$ (cf. [6]).

Since $J_{0,1}(\Gamma_2^J) = 0$, when we compare dimensions between Jacobi forms and Siegel modular forms, we should take the sum only over $k > 0$. We have

$$\sum_{k>0,k:even}^{\infty} (\dim A_k(\Gamma_2) + \dim A_{k,2}(\Gamma_2))t^k = \frac{t^4 + t^6 + t^{10} + t^{12}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$$

$$\sum_{k=1,k:odd}^{\infty} (\dim S_k(\Gamma_2) + \dim S^0_{k,2}(\Gamma_2))t^k = \frac{t^{21} + t^{27} + t^{29} + t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$$

When $m = 2$, the situation is much more complicated. We assume here that the weight is even. The dimension formula for $\dim J^{cusp}_{k,2}(\Gamma^J_2)$ is known by Tsushima, but the formula for non-cusp forms was not known before. We put $J^{cusp}_{even}(\Gamma^J_2) = \oplus^{\infty}_{k>0;k:even} J^{cusp}_{k,2}(\Gamma^J_2)$ and $J_{even}(\Gamma^J_2) = \oplus^{\infty}_{k>0;k:even} J_{k,2}(\Gamma^J_2)$. We can give the formula for $\dim J_{k,2}(\Gamma_2)$ when k is even by considering the structure of $J_{even}(\Gamma^J_2)$ as an $A_{even}(\Gamma_2)$-module. The argument is complicated.

Theorem 3.2 *The module $J_{even,2}(\Gamma^J_2)$ is a free $A_{even}(\Gamma_2)$ module and spanned by* 10 *Jacobi forms of weight* 4, 6, 8, 8, 10, 12, 12, 14, 16.

So as a corollary of this theorem, the dimension of $J_{k,2}(\Gamma^J_2)$ for even k is given by

$$\sum_{k>0;k:even}^{\infty} \dim J_{k,2}(\Gamma^J_2) = \frac{t^4 + t^6 + 2t^8 + 2t^{10} + 2t^{12} + t^{14} + t^{16}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.$$

This dimension formula seems new.

Now we sketch the proof of this theorem. When $n = m = 2$, we put

$$\begin{aligned}
t_1(\tau,z) &= \vartheta_{00,2}(\tau,z)\\
t_2(\tau,z) &= \vartheta_{02,2}(\tau,z)\\
t_3(\tau,z) &= \vartheta_{20,2}(\tau,z)\\
t_4(\tau,z) &= \vartheta_{22,2}(\tau,z)\\
t_5(\tau,z) &= \vartheta_{01,2}(\tau,z) + \vartheta_{03,2}(\tau,z)\\
t_6(\tau,z) &= \vartheta_{21,2}(\tau,z) + \vartheta_{23,2}(\tau,z)\\
t_7(\tau,z) &= \vartheta_{10,2}(\tau,z) + \vartheta_{30,2}(\tau,z)\\
t_8(\tau,z) &= \vartheta_{12,2}(\tau,z) + \vartheta_{32,2}(\tau,z)\\
t_9(\tau,z) &= \vartheta_{11,2}(\tau,z) + \vartheta_{33,2}(\tau,z) + \vartheta_{13,2}(\tau,z) + \vartheta_{31,2}(\tau,z)\\
t_{10}(\tau,z) &= \vartheta_{11,2}(\tau,z) + \vartheta_{33,2}(\tau,z) - \vartheta_{13,2}(\tau,z) - \vartheta_{31,2}(\tau,z)
\end{aligned}$$

Then for all i with $1 \le i \le 10$, we have $t_i(\tau,-z) = t_i(\tau,z)$ and $F(\tau,z) \in J_{k,2}(\Gamma^J_2)$ is a linear combination of these 10 theta functions over functions on

H_2. Besides, we have

$$t_i\left(\begin{pmatrix} \tau & -z_0 \\ -z_0 & \tau_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ -z_2 \end{pmatrix}\right) = \epsilon_i t_i(\tau, z)$$

where $\epsilon_i = 1$ for $1 \le i \le 9$ and -1 for $i = 10$.

We define a holomorphic function $F_{18}(\tau, z)$ on $H_2 \times \mathbb{C}^2$ by the following determinant of 10×10 matrix.

$$F_{18}(\tau, z) = \begin{vmatrix} t_1(\tau, z) & t_2(\tau, z) & \cdots\cdots & t_{10}(\tau, z) \\ t_1(\tau, 0) & t_2(\tau, 0) & \cdots\cdots & t_{10}(\tau, 0) \\ \partial_1^2 t_1(\tau, z)\big|_{z=0} & \cdots & \cdots & \partial_1^2 t_{10}(\tau, z)\big|_{z=0} \\ \partial_1\partial_2 t_1(\tau, z)\big|_{z=0} & \cdots & \cdots & \partial_1\partial_2 t_{10}(\tau, z)\big|_{z=0} \\ \partial_2^2 t_1(\tau, z)\big|_{z=0} & \cdots & \cdots & \partial_2^2 t_{10}(\tau, z)\big|_{z=0} \\ \partial_1^4 t_1(\tau, z)\big|_{z=0} & \cdots & \cdots & \partial_1^4 t_{10}(\tau, z)\big|_{z=0} \\ \partial_1^3\partial_2 t_1(\tau, z)\big|_{z=0} & \cdots & \cdots & \partial_1^3\partial_2 t_{10}(\tau, z)\big|_{z=0} \\ \partial_1^2\partial_2^2 t_1(\tau, z)\big|_{z=0} & \cdots & \cdots & \partial_1^2\partial_2^2 t_{10}(\tau, z)\big|_{z=0} \\ \partial_1\partial_2^3 t_1(\tau, z)\big|_{z=0} & \cdots & \cdots & \partial_1\partial_2^3 t(\tau, z)\big|_{z=0} \\ \partial_2^4 t_1(\tau, z)\big|_{z=0} & \cdots & \cdots & \partial_2^4 t_{10}(\tau, z)\big|_{z=0} \end{vmatrix}$$

It is almost trivial by definition that $F_{18}(\tau, z)$ satisfies the property

$$\left.\frac{\partial^4 F(\tau, z)}{\partial z_1^i \partial z_2^j}\right|_{z=0} = 0$$

for all $i + j \le 4$. We denote by $J_{k,2}^{(4)}(\Gamma_2^J)$ the space of Jacobi forms in $J_{k,2}(\Gamma_2^J)$ which satisfy this property.

Theorem 3.3 *(1) $F_{18}(\tau, z)$ is not identically zero and belongs to $J_{18,2}(\Gamma_2^J)$.*
(2) $F_{18}(\tau, z)$ is divisible by $\chi_{10}(\tau) = \chi_5(\tau)^2 \in S_{10}(\Gamma_2)$.
(3) If we put $F_8(\tau, z) = F_{18}(\tau, z)/\chi_{10}(\tau)$, then $F_8(\tau, z) \in J_{8,2}^{cusp}(\Gamma_2^J)$.
(4) When k is even, we have $J_{k,2}^{(4)}(\Gamma_2^J) = F_8(\tau, z)A_{k-8}(\Gamma_2)$. All such Jacobi forms are Jacobi cusp forms. In particular, we have $J_{8,2}^{(4)}(\Gamma_2^J) = \mathbb{C}F_8(\tau, z)$ and $J_{k,2}^{(4)}(\Gamma_2^J) = 0$ for $k < 8$.

We do not know if (4) is true also for odd k. The difficult point of this theorem is as follows. By the usual linear algebra, we can say that any element of $J_{k,2}^{(4)}(\Gamma_2^J)$ is equal to $f(\tau)F_8(\tau, z)$ for some function $f(\tau)$. We can say that $f(\tau)$ is a meromorphic function but this does not automatically mean that $f(\tau)$ is a *holomorphic* modular form of weight $k-8$. For example, the zeros of $f(\tau)$ might cancel with zeros of $F_8(\tau, z)$. To avoid such difficulty, we use here an explicit structure theorem of $\oplus_{k=0,k:even}^{\infty} A_{k,6}(\Gamma_2)$ in [7]. We have a mapping from $J_{k,2}^{(4)}(\Gamma_2^J)$ to $A_{k,6}(\Gamma_2)$ and the image of F_8 to $A_{8,6}(\Gamma_2)$ does not vanish. Besides this is one of the vectors which form a free basis of $\oplus_{k=0,k:even}^{\infty} A_{k,6}(\Gamma_2)$ over $A^{even}(\Gamma_2)$. So by comparing the image of $f(\tau)F_8(\tau,z) \in J_{k,2}^{(4)}(\Gamma_2^J)$ in $A_{k,6}(\Gamma_2)$ with the expression as linear combination of a free basis, we can say that $f(\tau)$ is also holomorphic. Since we know by dimension formula that $\dim J_{8,2}^{cusp}(\Gamma_2^J) = 1$, F_8 is a cusp form. As for general Jacobi cusp forms, we know the dimensions of $J_{k,2}^{cusp}(\Gamma_2^J)$ by Tsushima, which is given by

$$\sum_{k=0}^{\infty} \dim J_{2k,2}^{cusp}(\Gamma_2^J) = \frac{t^8 + 2t^{10} + 2t^{12} + 2t^{14} + 3t^{16} + 2t^{18} + t^{20} - t^{26} - t^{28} - t^{30}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.$$

On the other hand, we have

$$\begin{aligned}
\sum_{k=0,k:even}^{\infty} \dim S_k(\Gamma_2)t^k &= \frac{t^{10}+t^{12}-t^{22}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})} \\
\sum_{k=0,k:even}^{\infty} \dim S_{k,2}(\Gamma_2)t^k &= \frac{t^{14}+2t^{16}+t^{18}+t^{22}-t^{26}-t^{28}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})} \\
\sum_{k=0,k:even} \dim S_{k,4}(\Gamma_2)t^k &= \frac{t^{10}+t^{12}+t^{14}+t^{16}+t^{18}+t^{20}-t^{30}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})} \\
\sum_{k=0,k:even}^{\infty} \dim A_{k-8}(\Gamma_2)t^k &= \frac{t^8}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}
\end{aligned}$$

These dimension formulas are due to [9], [13], [12]. When k is even and $k>0$, by these we see

$$\dim J_{k,2}^{cusp}(\Gamma_2^J) = \dim S_k(\Gamma_2) + \dim S_{k,2}(\Gamma_2) + \dim S_{k,4}(\Gamma_2) + \dim A_{k-8}(\Gamma_2)$$

There is an injective map from $J_{k,2}^{cusp}(\Gamma_2^J)/J_{k,2}^{(4)}(\Gamma_2^J)$ to $S_k(\Gamma_2) \times S_{k,2}(\Gamma_2) \times S_{k,4}(\Gamma_2)$, and hence by dimensional coincidence, we have

Theorem 3.4 *When k is even with $k>0$, the natural mapping from $J_{k,2}^{cusp}(\Gamma_2^J)$ to $S_k(\Gamma_2) \times S_{k,2}(\Gamma_2) \times S_{k,4}(\Gamma_2)$ is surjective.*

On the other hand, we have

$$\sum_{k=0,k:even}^{\infty} \dim A_{k,4}(\Gamma_2) = \frac{t^8+t^{10}+t^{12}+t^{14}+t^{16}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$$

and

$$\sum_{k>0,k:even} (\dim A_k(\Gamma_2) + \dim A_{k,2}(\Gamma_2) + \dim A_{k,4}(\Gamma_2) + A_{k-8}(\Gamma_2))t^k$$
$$= \frac{t^4 + t^6 + 2t^8 + 2t^{10} + 2t^{12} + t^{14} + t^{16}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$$

Now the proof of Theorem 3.2 follows from the claim that the natural map from $J_{k,2}(\Gamma_2^J)$ to $A_k(\Gamma_2) \times A_{k,2}(\Gamma_2) \times A_{k,4}(\Gamma_2)$ is surjective for even $k > 0$ with kernel $F_8(\tau, z)A_{k-8}(\Gamma_2)$. To prove this with the aid of Theorem 3.4, we still need a construction of several other Jacobi forms of weight 4, 6, and 8. This can be done by using Eisenstein series, theta functions, and a square of a Jacobi form of index one. Also in the proof of this theorem, the structure theorem of $\oplus_{k=0,k:even}^{\infty} A_{k,2}(\Gamma_2)$ in [11] is used in a very natural context. The details will appear elsewhere.

4 Image of the Witt operator

After my talk in the conference, B. Heim asked me if the Witt operator W on $J_{k,1}(\Gamma_2^J)$ is surjective or not. I could answer this affirmatively there after a little consideration and I would like to add this here.

For any $F(\tau, z) \in J_{k,1}(\Gamma_2^J)$, we define a holomorphic function on $H_1^2 \times \mathbb{C}^2$ by

$$(WF)(\tau_1, z_1, \tau_2, z_2) = F\left(\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right).$$

We see that by the automorphy of F w.r.t. the elements

$$\begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix} \in \Gamma_2$$

where $a_i d_i - c_i d_i = 1$ for $i = 1, 2$ and $\mathbb{Z}^4 \cdot \mathbb{Z}$, we see that WF is a Jacobi form of variable (τ_1, z_1) or (τ_2, z_2) for each fixed (τ_2, z_2) or (τ_1, z_1). Besides, by the action of

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

on F, we see that WF is invariant by exchange of (τ_1, z_1) and (τ_2, z_2) when k is even. Hence we see that WF is in the symmetric tensors $Sym^2(J_{k,1}(\Gamma_2^J))$

of degree two, i.e.

$$WF = \sum_{i,j}(f_i(\tau_1, z_1)g_j(\tau_2, z_2) + g_j(\tau_1, z_1)f_i(\tau_2, z_2))$$

for some f_i, $g_j \in J_{k,1}(\Gamma_1^J)$. When k is odd, W is just the zero map since $J_{k,1}(\Gamma_1^J) = \{0\}$ and W is trivially surjective, Also for even k, we can show the surjectivity.

Theorem 4.1 *The Witt operator on $J_{k,1}(\Gamma_2^J)$ is surjective to $Sym^2(J_{k,1}(\Gamma_1^J))$.*

The proof can be obtained by using the explicit structure theorems. We omit the proof here. We note that even if the restricition of the Taylor coefficients of F to the diagonals vanish up to degree two (i.e. even if the restricion to the diagonals of the coefficients at 1, z_1^2, z_1z_2, z_2^2 vanish), WF might not vanish, since the Taylor expansion of WF might contain non-vanishing coefficient at $z_1^2z_2^2$. There exists such form in $Sym^2(J_{k,1}(\Gamma_1^J))$ of course, since $J_{k,1}(\Gamma_1^J) \cong A_k(\Gamma_1) \times S_{k+2}(\Gamma_1)$ and essentially $S_{k+2}(\Gamma_1)$ part controls the coefficients at z_i^2.

It would be very interesting to ask the same question for the higher degree cases. For example, it seems plausible that the surjectivity holds also for the case $n = 3$ when the index $m = 1$.

References

[1] S. Böcherer, Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen, Math. Zeit. **183**(1983), 21-46, (II) ibid **189** (1985), 81–110.

[2] M. Eichler and D. Zagier, *The theory of Jacobi forms.* Progress in Mathematics, 55. Birkhäuser Boston, Inc., Boston, MA, (1985), v+148pp.

[3] S. Hayashida and T. Ibukiyama, Siegel modular forms of half integral weight and a lifting conjecture, J. Math. Kyoto Univ. Vol. 45 No. 3 (2005), 489–530.

[4] T. Ibukiyama, On Jacobi forms and Siegel modular forms of half integral weights. Comment. Math. Univ. St. Paul. 41 (1992), no. 2, 109–124.

[5] T. Ibukiyama, On differential operators on automorphic forms and invariant pluri-harmonic polynomials, Comment. Math. Univ. St. Pauli 48 (1999), 103–118.

[6] T. Ibukiyama, Vector valued Siegel modular forms of symmetric tensor representation of degree two, preprint (2000).

[7] T. Ibukiyama, Vector valued Siegel modular forms of Sym(4) and Sym(6), preprint (2001).

[8] T. Ibukiyama and R. Kyomura, A generalization of vector valued Jacobi forms, to appear in Osaka J. Math.

[9] J. Igusa, On Siegel modular forms of genus two, Amer. J. Math. **84**(1962),175–200, (II), ibid. **86** (1964), 392–412.

[10] J. Igusa, On the graded ring of theta-constants, Amer. J. Math. **86**(1964), 219-246, (II) ibid. **88** (1966), 221-236.

[11] T. Satoh, On certain vector valued Siegel modular forms of degree two. Math. Ann. 274 (1986), no. 2, 335–352.

[12] R. Tsushima, The spaces of Siegel cusp forms of degree two and the representation of Sp(2, F_p). Proc. Japan Acad. Ser. A Math. Sci. 60 (1984), no. 6, 209–211.

[13] R. Tsushima, On the Dimension Formula for the Spaces of Jacobi Forms of Degree Two, in RIMS Kokyuroku, Study of automorphic forms and L functions No. 1103 (1999), 96–110.

[14] C. Ziegler, Jacobi forms of higher degree. Abh. Math. Sem. Univ. Hamburg 59 (1989), 191–224.

Tomoyoshi Ibukiyama
Department of Mathematics,
Graduate School of Science
Osaka University
Toyonaka, Osaka 560-0043 Japan
ibukiyam@math.sci.osaka-u.ac.jp