数理解析研究所講究録
第 1717 巻 2010 年 1-7

Coefficient estimates of functions in the class concerning with spirallike functions

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Abstract

For analytic functions $f(z)$ normalized by $f(0)=0$ and $f'(0)=1$ in the open unit disk $\mathbb{U}$, a new subclass $\mathcal{S}_{\alpha}$ of $f(z)$ concerning with spirallike functions in $\mathbb{U}$ is introduced. The object of the present paper is to discuss an extremal function for the class $\mathcal{S}_{\alpha}$ and coefficient estimates of functions $f(z)$ belonging to the class $\mathcal{S}_{\alpha}$.

1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

(1.1) $$f(z)=z+\sum_{n=2}^{\infty}a_{n}z^{n}$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in \mathbb{C};\ |z|<1\}$.
If $f(z)\in \mathcal{A}$ satisfies the following inequality

(1.2) $$\mathrm{Re}\left(\frac{1}{\alpha}\frac{zf'(z)}{f(z)}\right)>1 \qquad (z\in \mathbb{U})$$

for some complex number α ($|\alpha-\frac{1}{2}|<\frac{1}{2}$), then we say that $f(z)\in \mathcal{S}_{\alpha}$. If $\alpha=|\alpha|e^{i\varphi}$, then the condition (1.2) is equivalent to

$$\mathrm{Re}\left(e^{-i\varphi}\frac{zf'(z)}{f(z)}\right)>|\alpha| \qquad (z\in \mathbb{U}).$$

Therefore, we note that a function $f(z)\in \mathcal{S}_{\alpha}$ is spirallike in $\mathbb{U}$ which implies that $f(z)$ is univalent in $\mathbb{U}$.

Further, if $0<\alpha<1$, then $f(z)\in \mathcal{S}_{\alpha}$ is starlike of order α (cf. Robertson[3]).
Let $\mathcal{P}$ denote the class of functions $p(z)$ of the form

(1.3) $$p(z)=1+\sum_{k=1}^{\infty}c_{k}z^{k}$$

2000 *Mathematics Subject Classification*: Primary 30C45.
Key Word and Phrases: Analytic, univalent, spirallike, extremal function.

which are analytic in $\mathbb{U}$ and satisfy

$$\operatorname{Re} p(z) > 0 \qquad (z \in \mathbb{U}).$$

Then we say that $p(z) \in \mathcal{P}$ is the Carathéodory function (cf. Caratéodory [1] or Duren [2]).

Remark 1.1 Let us consider a function $f(z) \in \mathcal{A}$ which satisfies

$$\left| \frac{f(z)}{zf'(z)} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha} \qquad (z \in \mathbb{U}) \tag{1.4}$$

for $|\alpha - \frac{1}{2}| < \frac{1}{2}$. If we write that $F(z) = \frac{zf'(z)}{f(z)}$, then the inequality (1.4) can be written by

$$\left| \frac{2\alpha - F(z)}{F(z)} \right| < 1 \qquad (z \in \mathbb{U}).$$

This implies that

$$\alpha \overline{F(z)} + \bar{\alpha} F(z) > 2|\alpha|^2 \qquad (z \in \mathbb{U}).$$

It follows that

$$\left(\frac{F(z)}{\alpha}\right) + \overline{\left(\frac{F(z)}{\alpha}\right)} > 2 \qquad (z \in \mathbb{U}).$$

Therefore, the inequality (1.4) is equivalent to

$$\operatorname{Re}\left(\frac{1}{\alpha} \frac{zf'(z)}{f(z)}\right) > 1 \qquad (z \in \mathbb{U}).$$

2 Coefficient estimates

In this section, we discuss the coefficient estimates of a_n for $f(z) \in \mathcal{S}_\alpha$. To establish our results, we need the following lemma due to Carathéodory [1].

Lemma 2.1 *If a function $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in \mathcal{P}$ satisfies the following inequality*

$$\operatorname{Re} p(z) > 0 \qquad (z \in \mathbb{U}),$$

then

$$|c_k| \leqq 2 \qquad (k = 1, 2, 3, \cdots)$$

with equality for

$$p(z) = \frac{1+z}{1-z}.$$

Now, we introduce the following theorem.

Theorem 2.2 *Extremal function for the class $\mathcal{S}_\alpha$ is $f(z)$ defined by*

$$f(z) = \frac{z}{(1-z)^{2\alpha(\operatorname{Re}(\frac{1}{\alpha})-1)}}. \tag{2.1}$$

Proof. From the definition of the class $\mathcal{S}_\alpha$, we have that

$$\mathrm{Re}\left(\frac{1}{\alpha}\frac{zf'(z)}{f(z)}-1\right)>0.$$

Moreover, it is clear that

$$\mathrm{Re}\left(\frac{1}{\alpha}\right)>1 \qquad \left(|\alpha-\frac{1}{2}|<\frac{1}{2}\right).$$

Then, if the function $F(z)$ is defined by

$$F(z)=\frac{\frac{1}{\alpha}\frac{zf'(z)}{f(z)}-1-i\mathrm{Im}\left(\frac{1}{\alpha}\right)}{\mathrm{Re}\left(\frac{1}{\alpha}\right)-1},$$

we see that

$$\mathrm{Re}F(z)>0 \;\; \textit{and} \;\; F(0)=1,$$

so that, $F(z)\in\mathcal{P}$.
Therefore, Lemma 2.1 shows us that

$$F(z)=\frac{\frac{1}{\alpha}\frac{zf'(z)}{f(z)}-1-i\mathrm{Im}\left(\frac{1}{\alpha}\right)}{\mathrm{Re}\left(\frac{1}{\alpha}\right)-1}=\frac{1+z}{1-z}.$$

It follows that,

$$\frac{f'(z)}{f(z)}-\frac{1}{z}=2\alpha\left(\mathrm{Re}\left(\frac{1}{\alpha}\right)-1\right)\frac{1}{1-z}.$$

Integrating both sides from 0 to z on t, we have that

$$\int_0^z\left(\frac{f'(t)}{f(t)}-\frac{1}{t}\right)dt=2\alpha\left(\mathrm{Re}\left(\frac{1}{\alpha}\right)-1\right)\int_0^z\frac{1}{1-t}dt,$$

which implies that

$$\log\frac{f(z)}{z}=\log\frac{1}{(1-z)^{2\alpha(\mathrm{Re}(\frac{1}{\alpha})-1)}}.$$

Therefore, we obtain that

$$f(z)=\frac{z}{(1-z)^{2\alpha(\mathrm{Re}(\frac{1}{\alpha})-1)}}.$$

This is the extremal function of the class $\mathcal{S}_\alpha$. □

Next, we discuss the coefficient estimates of $f(z)$ belonging to the class $\mathcal{S}_\alpha$.

Theorem 2.3 *If a function $f(z)\in\mathcal{S}_\alpha$, then*

$$|a_n|\leqq\frac{1}{(n-1)!}\prod_{k=1}^{n-1}(2(\cos(\arg(\alpha))-|\alpha|)+(k-1)) \qquad (n=2,3,4\cdots).$$

Equality holds true for $f(z)$ given by (2.1).

Proof. By using same method with Theorem 2.2 , if we set $F(z)$ that

$$F(z)=\frac{\frac{1}{\alpha}\frac{zf'(z)}{f(z)}-1-i\mathrm{Im}\left(\frac{1}{\alpha}\right)}{\mathrm{Re}\left(\frac{1}{\alpha}\right)-1}, \tag{2.2}$$

then it is clear that $F(z)\in\mathcal{P}$.
Letting

$$F(z)=1+c_1z+c_2z^2+\cdots,$$

Lemma 2.1 gives us that

$$|c_m|\leqq 2 \quad (m=1,2,3\cdots).$$

Now, from (2.2),

$$\left(\mathrm{Re}(\frac{1}{\alpha})-1\right)F(z)=\frac{1}{\alpha}\frac{zf'(z)}{f(z)}-1-i\mathrm{Im}(\frac{1}{\alpha}).$$

Let $\mathrm{Re}(\frac{1}{\alpha})-1=s$ and $1+i\mathrm{Im}(\frac{1}{\alpha})=A$.
This implies that

$$(\alpha sF(z)+\alpha A)f(z)=zf'(z).$$

Then, the coefficients of z^n in both sides lead to

$$na_n=(\alpha s+\alpha A)a_n+\alpha s(a_{n-1}c_1+a_{n-2}c_2+\cdots+a_{n-r}c_r+\cdots+a_2c_{n-2}+c_{n-1}).$$

Therefore, we see that

$$a_n=\frac{\alpha s}{n-\alpha s-\alpha A}(a_{n-1}c_1+a_{n-2}c_2+\cdots+a_{n-r}c_r+\cdots+a_2c_{n-2}+c_{n-1}).$$

This shows that

$$\begin{aligned}|a_n|&=\frac{|\alpha(\mathrm{Re}(\frac{1}{\alpha})-1)|}{|n-\alpha(\mathrm{Re}(\frac{1}{\alpha})-1)-\alpha(1+i\mathrm{Im}(\frac{1}{\alpha}))|}|a_{n-1}c_1+a_{n-2}c_2+\cdots+a_{n-r}c_r+\cdots+a_2c_{n-2}+c_{n-1}|\\
&=\frac{\cos(\arg(\alpha))-|\alpha|}{n-1}|a_{n-1}c_1+a_{n-2}c_2+\cdots+a_{n-r}c_r+\cdots+a_2c_{n-2}+c_{n-1}|\\
&\leqq\frac{\cos(\arg(\alpha))-|\alpha|}{n-1}(|a_{n-1}||c_1|+|a_{n-2}||c_2|+\cdots+|a_{n-r}||c_r|+\cdots+|a_2||c_{n-2}|+|c_{n-1}|)\\
&\leqq\frac{\cos(\arg(\alpha))-|\alpha|}{n-1}(2|a_{n-1}|+2|a_{n-2}|+\cdots+2|a_2|+2)\\
&\leqq\frac{2(\cos(\arg(\alpha))-|\alpha|)}{n-1}\sum_{k=1}^{n-1}|a_k| \qquad (|a_1|=1).\end{aligned}$$

To prove that

$$|a_n|\leqq\frac{1}{(n-1)!}\prod_{k=1}^{n-1}(2(\cos(\arg(\alpha))-|\alpha|)+(k-1)),$$

we need to show that

$$|a_n| \leqq \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{n-1} \sum_{k=1}^{n-1} |a_k| \leqq \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (2(\cos(\arg(\alpha)) - |\alpha|) + (k-1)). \tag{2.3}$$

We use the mathematical induction for the proof.
When $n = 2$, this assertion is true.
We assume that the proposition is true for $n = 2, 3, 4, \cdots, m-1$.
For $n = m$, we obtain that

$$|a_m| \leqq \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-1} \sum_{k=1}^{m-1} |a_k|$$

$$= \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-1} \left(\sum_{k=1}^{m-2} |a_k| + |a_{m-1}| \right)$$

$$= \frac{m-2}{m-1} \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-2} \sum_{k=1}^{m-2} |a_k| + \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-1} |a_{m-1}|$$

$$\leqq \frac{m-2}{(m-1)!} \prod_{k=1}^{m-2} (2(\cos(\arg(\alpha)) - |\alpha|) + k - 1)$$

$$+ \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-1} \frac{1}{(m-2)!} \prod_{k=1}^{m-2} (2(\cos(\arg(\alpha)) - |\alpha|) + k - 1)$$

$$= \frac{1}{(m-1)!} \prod_{k=1}^{m-2} (2(\cos(\arg(\alpha)) - |\alpha|) + k - 1) \big(m - 2 + 2(\cos(\arg(\alpha)) - |\alpha|)\big)$$

$$= \frac{1}{(m-1)!} \prod_{k=1}^{m-1} (2(\cos(\arg(\alpha)) - |\alpha|) + k - 1).$$

Thus the inequality (2.3) is true for $n = m$. By the mathematical induction, we prove that

$$|a_n| \leqq \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (2(\cos(\arg(\alpha)) - |\alpha|) + (k-1)) \qquad (n = 2, 3, 4 \cdots).$$

For the equality, we consider the extremal function $f(z)$ given by Theorem 2.2. Since

$$f(z) = \frac{z}{(1-z)^{2\alpha(\mathrm{Re}(\frac{1}{\alpha})-1)}},$$

if we let

$$2\alpha(\mathrm{Re}(\frac{1}{\alpha}) - 1) = j,$$

then $f(z)$ becomes that

$$f(z) = z(1-z)^{-j} = z\left(\sum_{n=0}^{\infty} \binom{-j}{n} (-z)^n \right) = z + \sum_{n=2}^{\infty} \frac{j(j+1)\cdots(j+n-2)}{(n-1)!} z^n.$$

From the above, we obtained

$$a_n = \frac{1}{(n-1)!}\prod_{k=1}^{n-1}\left(2\alpha(\mathrm{Re}(\frac{1}{\alpha}) - 1) + k - 1\right).$$

For $n = 2$,

$$|a_2| = 2|\alpha||\mathrm{Re}(\frac{1}{\alpha}) - 1| = 2(\cos(\arg(\alpha)) - |\alpha|).$$

Furthermore, for $n \geqq 3$, we have that

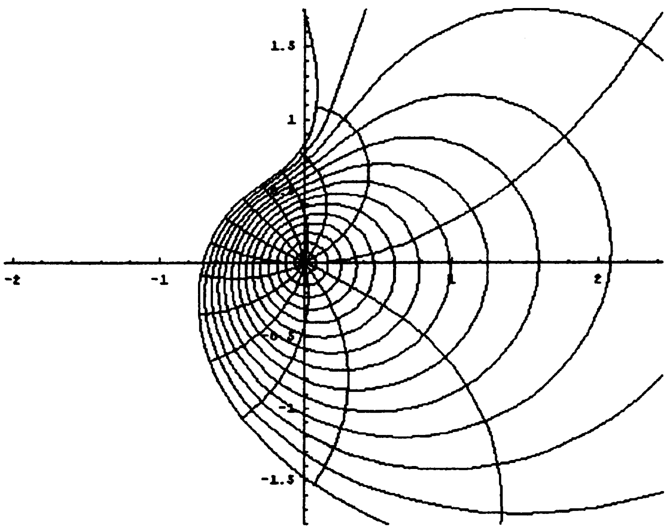
$$\begin{aligned}|a_n| &= \left|\frac{1}{(n-1)!}\prod_{k=1}^{n-1}(2\alpha(\mathrm{Re}(\frac{1}{\alpha}) - 1) + k - 1)\right| \\ &= \frac{1}{(n-1)!}\prod_{k=1}^{n-1}|2\alpha(\mathrm{Re}(\frac{1}{\alpha}) - 1) + k - 1| \\ &\leqq \frac{1}{(n-1)!}\prod_{k=1}^{n-1}(2(\cos(\arg(\alpha)) - |\alpha|) + k - 1).\end{aligned}$$

Equality holds true for some real α $(0 < \alpha < 1)$.
This completes the proof of Theorem 2.3 . □

Example 2.4 Let $\alpha = \frac{1}{2} + \frac{1}{4}i$ in (2.1). Then we have that

$$f(z) = \frac{z}{(1-z)^{\frac{6+3i}{10}}}.$$

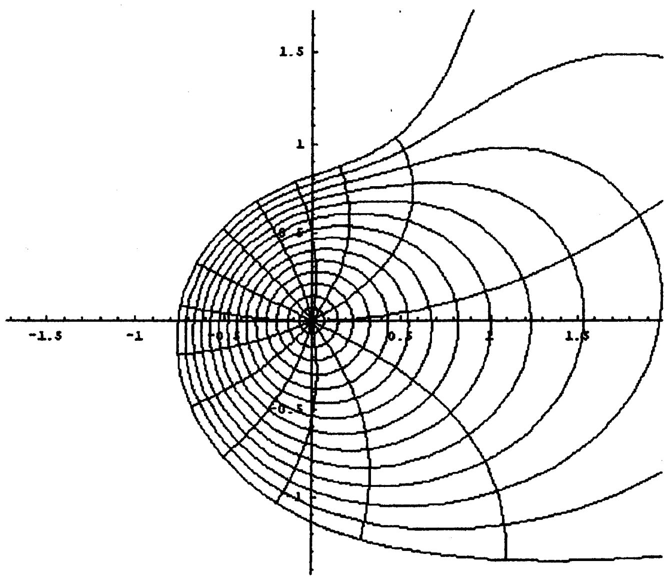
This function $f(z)$ maps the unit disk $\mathbb{U}$ onto the following domain.



Example 2.5 If we take $\alpha = \frac{2}{3} + \frac{1}{4}i$ in (2.1), then we have that

$$f(z) = \frac{z}{(1-z)^{\frac{184+69i}{438}}}.$$

This function $f(z)$ maps the unit disk $\mathbb{U}$ onto the following domain.



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