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Critical points parameters for triply connected Bell domains

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1 Introduction

The fundamental problem in the geometric function theory is to find a family of canonical domains. Recently, S. Bell proposed a new family of domains which admit canonically a *simple* proper holomorphic map to the unit disc U. Actually, they are enough.

Theorem 1 ([1]). *Every non-degenerate d-ply connected planar domain W with $d>1$ is mapped biholomorphically (or, conformally) onto a domain $W_{\mathbf{a},\mathbf{b}}$, defined by*

$$W_{\mathbf{a},\mathbf{b}}=\left\{z\in\mathbb{C}:\left|z+\sum_{k=1}^{d-1}\frac{a_k}{z-b_k}\right|<1\right\}$$

with suitable complex vectors

$$\mathbf{a}=(a_1,a_2,\cdots,a_{d-1}),\quad \mathbf{b}=(b_1,b_2,\cdots,b_{d-1}).$$

This theorem can be considered as a natural generalization of the classical Riemann mapping theorem for simply connected planar domains.

We call such a domain $W_{\mathbf{a},\mathbf{b}}$ as in Theorem 1.1 a *Bell representation* of W. The function $f_{\mathbf{a},\mathbf{b}}$ defined by

$$f_{\mathbf{a},\mathbf{b}}(z)=z+\sum_{k=1}^{d-1}\frac{a_k}{z-b_k}$$

is a proper holomorphic map from $W_{\mathbf{a},\mathbf{b}}$ onto U. Set B_d be the set of all vectors $(\mathbf{a},\mathbf{b})$ in $\mathbb{C}^{2d-2}$ such that $W_{\mathbf{a},\mathbf{b}}$ is a Bell representation of d-ply connected planar domains, and we call B_d the *coefficient body of degree d.* (Cf. [2].)

Now, from a well-known fact on the theory of moduli, we can conclude that d-ply connected non-degenerate planar domains have real $3d-6$ moduli (or Teichmüller) parameters if $d \geq 3$. First we state this fact more precisely.

Definition 1. Let $d \geq 2$. We call a d-ply connected non-degenerate planar domain W equipped with an order of boundery components of W a *boundary-marked* planar domain of type d.

Two marked planar domains W_1 and W_2 of type d are *conformally equivalent* if there is a conformal mapping $f : W_1 \to W_2$ which preserves the boundary-markings.

Let D_d be the set of all equivalence classes of boundary-marked planar domains of type d. We call D_d the *deformation space* of a boundary-marked planar domain of type d.

Then the following fact is classical.

Proposition 2. *If $d \geq 3$, then D_d can be considered as a domain in $\mathbb{R}^{3d-6}$.*

Proof. By Koebe's theorem ([3]), every d-ply connected non-degenerate planar domain can be mapped conformally onto a Koebe circle domain.

On the other hand, it is easy to see that boundary-marked Koebe circle domains have real $3d-6$ real global parameters up to Möbius tranformations.

□

In the case of triply connected planar domains, there always exists a canonical symmetry for every such one. Moreover, it is believed that the intersection of the coefficient body B_3 with each one of the following families gives an explicit model of D_3. We will discuss about it.

Definition 2. Set

$$B^+ = \{(a,b,d) \in \mathbb{R}^3 \mid a > 0, b > 0, d > 0\},$$

and

$$B^- = \{(a,b,d) \in \mathbb{R}^3 \mid a > 0, b < 0, d < 0\}.$$

We assume that $B^\pm$ are naturally embedded in $\mathbb{C}^3$. Also in the sequel, we write as

$$W_{a,b,d} = \{z \in \mathbb{C} \mid |f_{a,b,d}(z)| < 1\},$$

where

$$f_{a,b,d}(z) = z + \frac{b}{z-a} + \frac{d}{z+a}.$$

2 Main results

First, we clarify the correspondence of (a, b, d) with the set of critical points and the phase transition of the covering structures of $f_{a,b,d}$ for the case of B^+.

First note the following

Lemma 3. *For every $f = f_{a,b,d}$ with $(a, b, d) \in B^+$, either*

1) f has for real critical points $\{r, p, s, t\}$, or

2) f has two real critical points $\{r, t\}$ and two others $\{p+si, p-si\}$. Here we may assume that

$$1) \quad r < p \leq s < t, \ \text{or } 2) \quad r < t, \ s > 0,$$

respectively.

For every $f = f_{a,b,d}$ with $(a, b, d) \in B^-$, f has two pair of complex conjugates $\{r + it, r - it\}$ and $\{p + si, p - si\}$. Here we assume that

$$r \leq p, \ t > 0, \ s > 0.$$

In the case of B^+, the phase transition occurs at the locus $\mathrm{Discr}(F) = 0$, *where* $\mathrm{Discr}(F)$ *is the constant times*

$$bda^2 \left((4a^2 - b - d)^3 - 108bda^2\right)$$

$$F(z) = (z - a)^2(z + a)^2 - b(z + a)^2 - d(z - a)^2.$$

Here, we include the figures which show the typical manner of the phase transition.

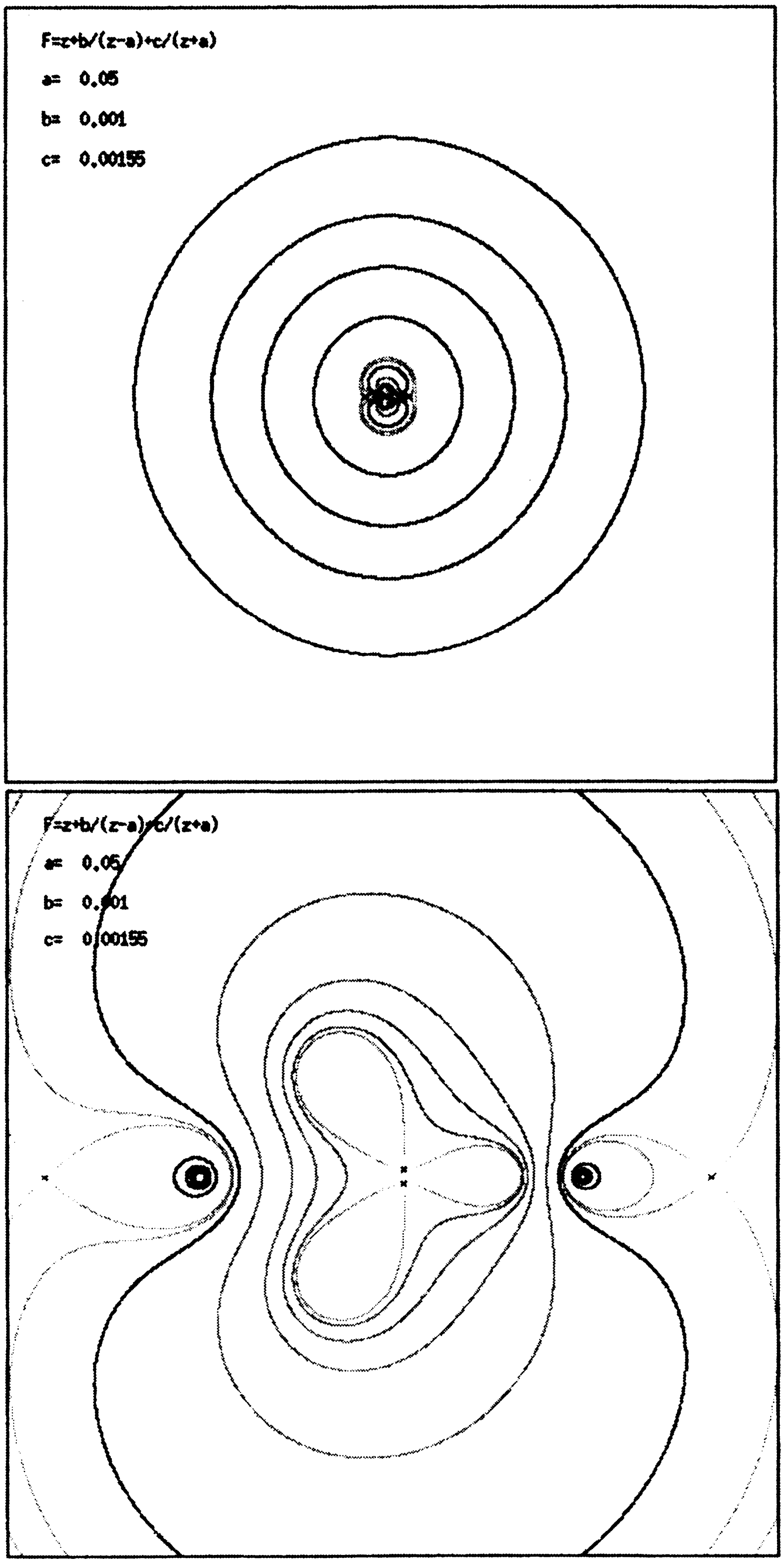


Figure 1: $a = 0.05,\ b = 0.001,\ c = 0.00155$

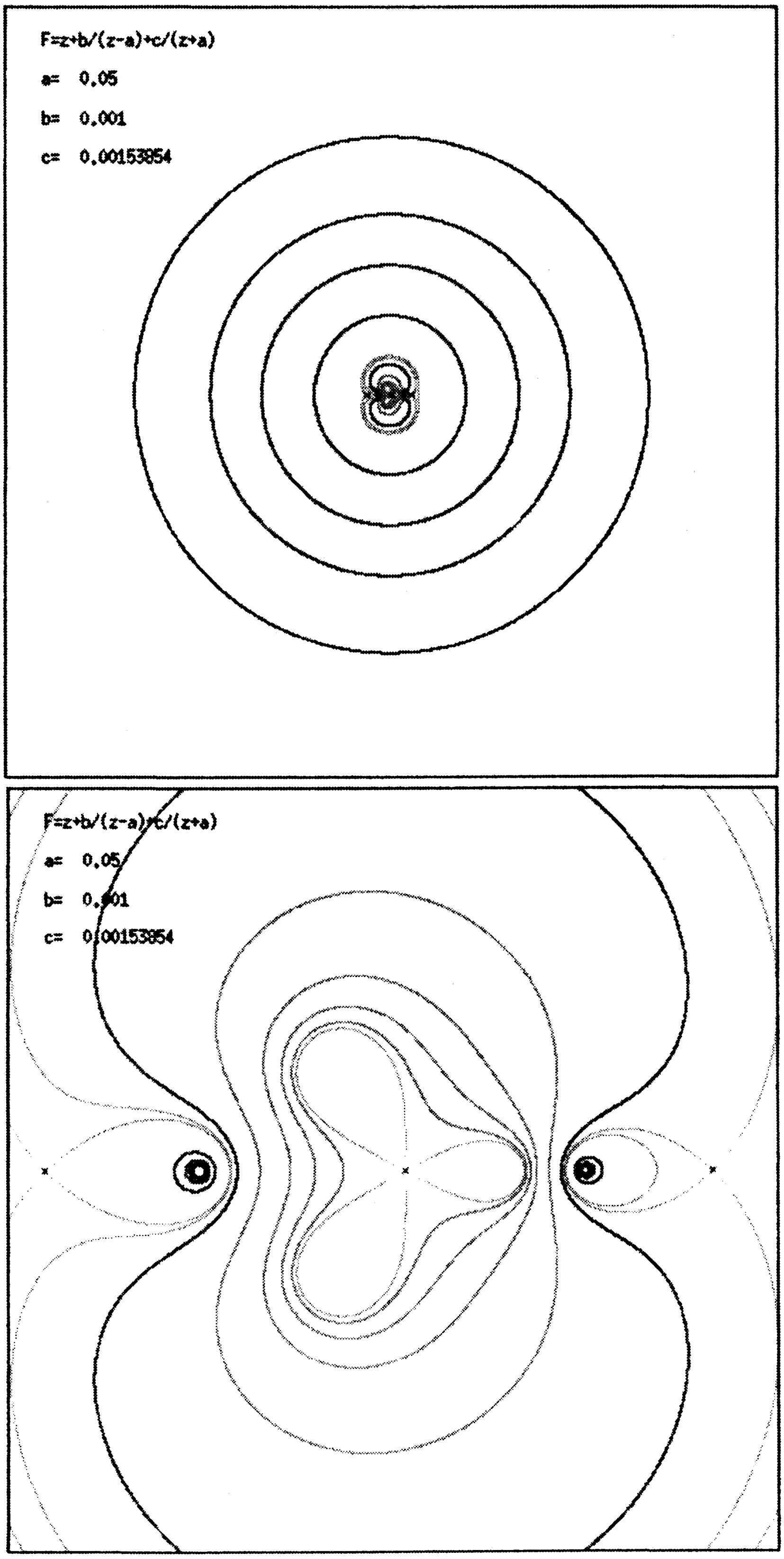


Figure 2: $a = 0.05,\ b = 0.001,\ c = 0.0015385375692573147 9$

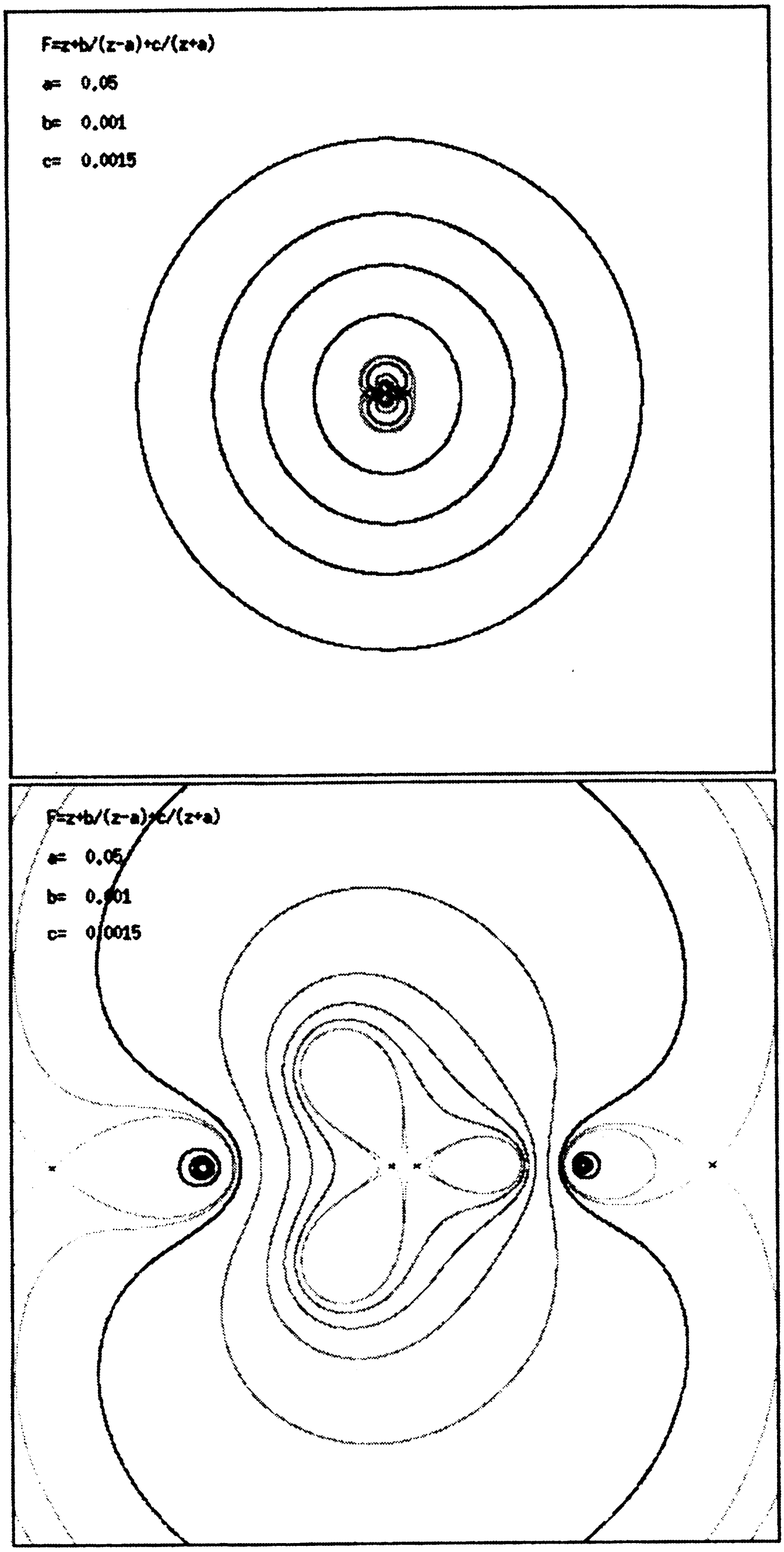


Figure 3: $a = 0.05,\ b = 0.001,\ c = 0.0015$

Next, recall that $F(z)$ is represented also as

$$F(z) = z^4 + \sigma_1 z^3 + \sigma_2 z^2 + \sigma_3 z + \sigma_4.$$

Clearly, $\sigma_1 = 0$ and the vectors $(\sigma_2, \sigma_3, \sigma_4)$ correspond to the sets $\{r, s, t\}$ bijectively, which is called the relations between solutions and coefficients. Also a direct computation gives

Lemma 4. *The Jacobian*

$$\frac{\partial(\sigma_2, \sigma_3, \sigma_4)}{\partial(\ a,\ b,\ d)}$$

is

$$-8a^2(4a^2 - b - d).$$

Now, the main theorems are the following

Theorem 5. *In the case of B^-, the set of three real parameters*

$$(r, s, t)$$

gives the set of global coordinates of B^-. In other words, the map Π^- of B^- to $(r, s, t) \in \mathbb{R}^3$ is a homeomorphism onto the image.

Proof. First, the map

$$\phi : (a, b, d) \mapsto (\sigma_2, \sigma_3, \sigma_4)$$

is locally homeomorphic by Lemma 4 and the assumptions that $b < 0$ and $d < 0$. Also ϕ is injective. Indeed, a^2 is a positive solution of

$$3x^2 + \sigma_2 x - \sigma_4 = 0.$$

And since $\sigma_4 > 0$, it has exactly one positve solution.

Next, we can show by a direct computation that the Jacobian

$$\begin{aligned} \frac{\partial(\sigma_2, \sigma_3, \sigma_4)}{\partial(\ r,\ s,\ t)} &= 4st\left(2(t^2 - s^2)^2 + 16r^2(2r^2 + s^2 + t^2)\right) \\ &= 8st\left(4r^2 + (s-t)^2\right)\left(4r^2 + (s+t)^2\right), \end{aligned}$$

whcih is non-negative, and equals 0 if and only if $r = 0, s = t$. But these conditions imply that $a = b = d = 0$, and hence can not occur. Thus we conclude that

$$\psi : (r, s, t) \mapsto (\sigma_2, \sigma_3, \sigma_4)$$

is also locally homeomorphism and clearly ψ^{-1} is injective.

Thus we can show that the map Π^- of B^- to $(r, s, t) \in \mathbb{R}^3$ is injective and locally homeomorphic, and hence is a homeomorphism onto the image. □

Theorem 6. *In the case B^+, the map $\Pi^+ : (a,b,d) \mapsto (r,s,t)$ is locally homeomorphic except for the degenerate locus*

$$E_1 = \{(a,b,d) \mid 4a^2 - b - d = 0\},$$

The bifurcation locus is

$$E_2 = \{(a,b,d) \mid Discr(F) = (4a^2 - b - d)^3 - 108bda^2 = 0\}.$$

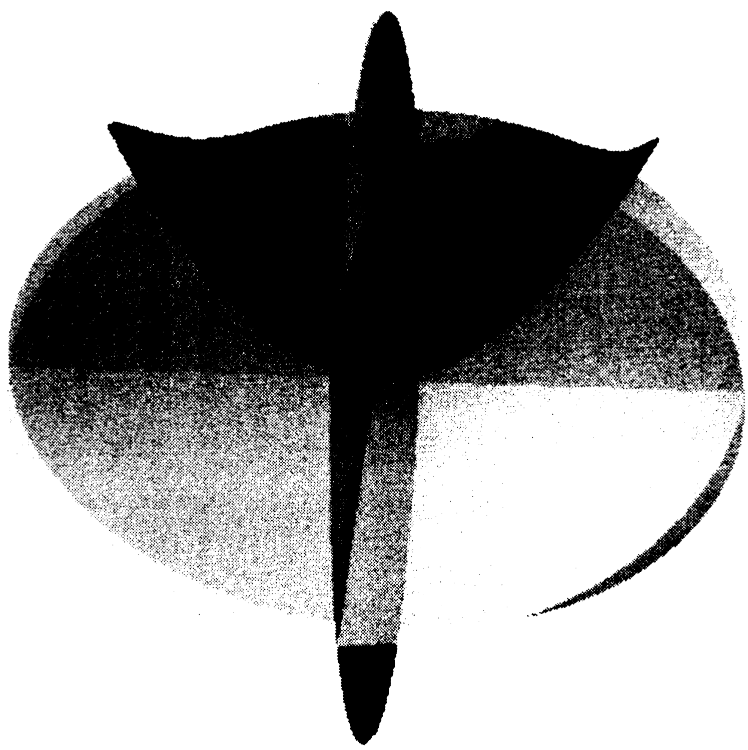
Proof. The first assertion follows from Lemma 4. And the second assertion is already stated before Lemma 4. □

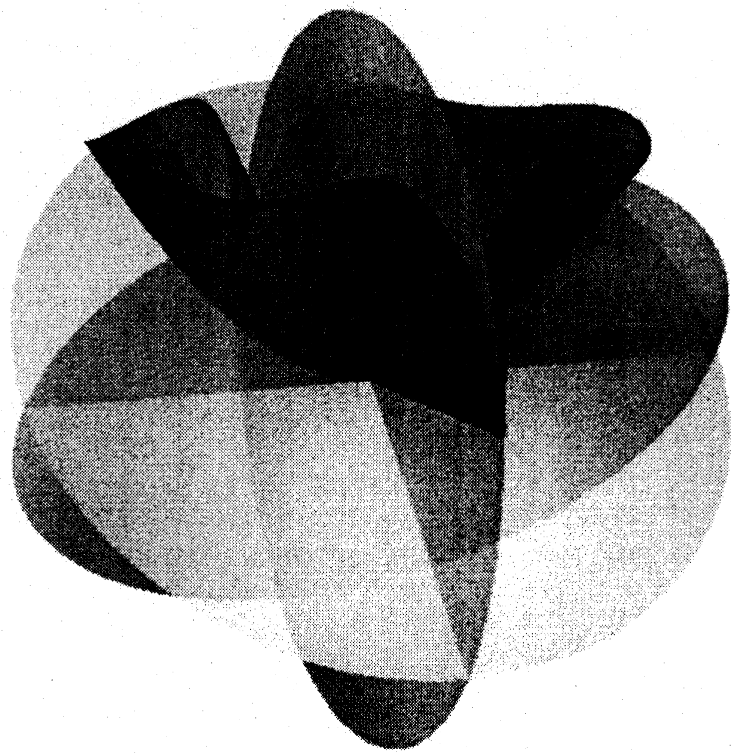
Remark 1. On the subset of B^+ where $s^2 - b - d > 0$, Π^+ is injective.

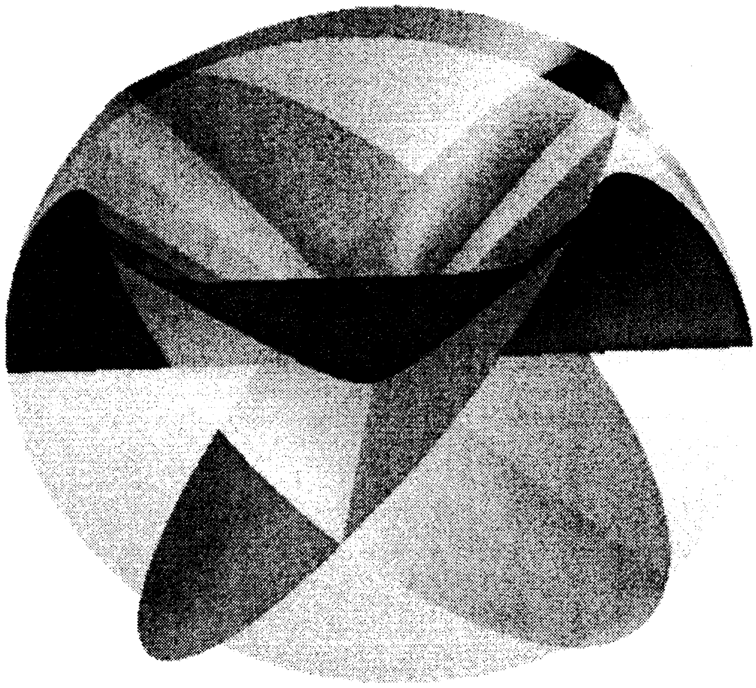
Finally we include the figures of

$$(4a^2 - b - d)^3 - 108bda^2 = 0,$$

which are symmetric with respect to $\{a = 0\}$ aqnd $\{b = c\}$. The planes in the figures are a-, b-, c-planes.







References

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[3] P. Koebe, Abhandlungen zur Theorie der Konformen Abbildung; iV, Math. Z. **7** (1920), 235–301.