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Topological representation of the branched covering structure induced from a real rational function

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1 Introduction

Let $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be a rational function of degree n. We say that f is a *generic* (complex) rational function if the preimage $f^{-1}(z)$ of any point $z\in\widehat{\mathbb{C}}$ consists of either n or $n-1$ points; the points of the latter type are called *simple ramification points.* The set of all simple ramification points of f is denoted $\Sigma(f)$ and consists of $2n-2$ points.

Two rational functions $f_{i}:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ are called *covering-equivalent* if there exists a Möbius transformation $\phi:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ such that $f_{1}=f_{2}\circ\phi$. Let $CH_{0,n}$ be the set of all equivalence classes of complex generic rational functions of degree n. The correspondence $f\mapsto\Sigma(f)$ generates a covering $C\Phi_{n}$: $CH_{0,n}\to CQ_{0,n}$, where $CQ_{0,n}$ is the configuration space consisting of all n-tuples of unordered distinct points on $\widehat{\mathbb{C}}$. We assume that $CH_{0,n}$ is provided with the weakest topology for which the map $C\Phi_{n}$ is continuous.

Remark 1. The degree h_{n} of the covering $C\Phi_{n}$, and its analogs for arbitrary meromorphic functions, are called the *Hurwitz numbers.* These numbers arise in many situations in mathematical physics. The Hurwitz numbers h_{n} can be calculated.

$$h_{n}=\frac{n^{n-3}(2n-2)!}{n!};$$

in fact, this result was apparently known already to Hurwitz himself. Also see [2] and [3].

Next, let τ be an anti-holomorphic involution. A *real rational function* is a complex rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\overline{f(\tau p)} = f(p)$ for any $p \in \widehat{\mathbb{C}}$. A real rational function (τ, f) is said to be *generic* if f is generic. Clearly, $\overline{\Sigma(f)} = \Sigma(f)$ for any real rational function f.

Definition 1. Two real rational functions (τ_i, f_i) $(i = 1, 2)$ are called *equivalent* if there exists a Möbius transformation ϕ such that

$$f_1 = f_2 \circ \phi, \quad \phi \circ \tau_1 = \tau_2 \circ \phi.$$

Let $RH_{0,n}$ denote the space of all equivalence classes of generic real rational functions of degree n.

The topology of $CH_{0,n}$ generates a topology on $RH_{0,n}$; in this topology $RH_{0,n}$ is not connected.

The points of $RH_{0,n}$ are called the equivalence classes of *generic real rational functions*. Since there might be no confusions, we write (τ, f) simply as f and $f \in RH_{0,n}$ means that the equivalence class of f belongs to $RH_{0,n}$. Now the main result of Natanzon, Shapiro, and Vainshtein is as follows.

Theorem 1 ([2]). *The set of all connected components of the space $RH_{0,n}$ is in a 1-1-correspondence with the set of the equivalence classes of all gardens of weight n.*

Here, the object, called a *garden*, consists of a weighted labeled directed planar chord diagram and of a set of weighted rooted trees each of which corresponds to a face of the diagram.

Definition 2. By a *planar chord diagram* (of *order* $2l$) we mean a circle drawn on the plane together with $2l$ points on this circle partitioned into l pairs in such a way that, for any two pairs, the chords joining the points from the same pair do not intersect. The above $2l$ points are called the *vertices* of the chord diagram; the chords joining the vertices from the same pair and the arcs of the circle joining adjacent vertices are called the *edges*. The notion of its *faces* is defined in a usual way (except for the outer face of the graph, which is not a face of the diagram).

We say that a planar chord diagram is *directed* if its edges are directed in such a way that the boundary of each face becomes a directed cycle. Fixing the standard orientation of the plane, we call a face of a directed planar chord diagram *positive* if the face lies to the left when we traverse its boundary according to the chosen direction, and *negative* otherwise.

A planar chord diagram is said to be *weighted* if each edge is equipped with a nonnegative integer (*weight*), and *labeled* if there exists a bijection β (*labeling*) that takes the vertex set of the diagram to the set $\{1, 2, \ldots, 2l\}$. Two labelings β_1 and β_2 are said to be *cyclically equivalent* if $\beta_1(v) - \beta_2(v)$ mod $2l$ is a constant not depending on the choice of a vertex v.

Definition 3. A *rooted tree* is, by definition, a tree with one distinguished vertex called the *root*; all the other vertices of the tree are said to be *inner*. We say that a rooted tree is *weighted* if each its vertex is equipped with a positive integer (*weight*).

Definition 4. A *garden* is a weighted labeled directed planar chord diagram with a weighted rooted tree (possibly consisting just of its root) corresponding to each face of the diagram. The weights of the *inner* vertices of the trees are arbitrary positive integers, and the weight of the root of the tree corresponding to the face j equals t_j defined below. The (*total*) *weight* of the garden equals twice the sum of the weights of all the inner vertices of all trees plus the sum of the weights of all roots.

Here, for any face j of a labeled directed planar chord diagram, we denote by d_j the number of descents in the sequence of vertex labels ordered cyclically along the boundary of the face, and by t_j the sum of d_j and the weights of all the edges along the boundary of the face j.

Two gardens are said to be *equivalent* if there exists a bijection of the vertex sets of the corresponding chord diagrams which preserves chords, their orientation, labels (up to the cyclic equivalence), rooted trees, and weights.

2 The case of degree 2 or 3

Every real rational function is equivalent to another real rational function with $\tau = J$, where J is the complex conjugation. And we can see that the latter is a rational function with real coefficients.

(Indeed, if such a funciton $R(z)$ is $P(z)/Q(z)$ with polynomials $P(z), Q(z)$, we may assume that the leading coefficient of $Q(z)$ is 1. Then from the assumtion, $P(z)/Q(z) = \bar{P}(z)/\bar{Q}(z)$, where $\bar{P}(z), \bar{Q}(z)$ are the polynomials obtained from $P(z), Q(z)$ by replacing the coefficients with the complex conjugates of them. Since $Q(z)$ and $\bar{Q}(z)$ are monic, and have the same zeros, we conclude that $Q(z) = \bar{Q}(z)$. And hence, we can conclude similarly that $P(z) = \bar{P}(z)$.)

Here, we also recall how to get a garden from a real rational function

Definition 5. Take an $f \in RH_{0,n}$, and represent $\Sigma = \Sigma(f)$ as $\Sigma = \Sigma_R \cup \Sigma_I$, where Σ_R is the set of real critical values of f and Σ_I is the set of its non-real critical values. The number of elements in Σ_R is denoted by $2l(\Sigma)$.

Let $S(f)$ be the preimage of the real line $\bar{\mathbb{R}} = \mathbb{R} \cup \infty$ under f. For every element in Σ_R, $S(f)$ contains exactly 4 arcs incident to it. These arcs together with $\bar{\mathbb{R}} \subset S(f)$ define a 2-dimensional cell complex on $\widehat{\mathbb{C}}$. The 2-cells of this complex are called the *faces* of $S(f)$. Here, $S(f)$ may contain simple closed curves called *ovals* as the connected components.

To construct $G(f)$ we start from a planar chord diagram of order $2l(\Sigma)$. The vertices of the diagram correspond to the critical points with real critical values, and the chords correspond to the arcs of $S(f)$ lying inside the circle. Thus, the faces of the diagram correspond to the faces of $S(f)$ lying inside the circle. The orientation of the edges is induced by the orientation of $\bar{\mathbb{R}}$ in the image. To define the labeling of the chord diagram, consider the natural order $<$ on Σ_R (if ∞ belongs to Σ_R, we assume that it is the biggest critical value). The label of a critical point equals the number of the corresponding critical value under this order. To define the weights, consider an arbitrary point $x \in \bar{\mathbb{R}} - \Sigma_R$, and for any given arc (or oval) let $w(x)$ be the number of preimages of x lying on this arc (or oval). The weight of the arc (or oval) is then defined as the minimum of $w(x)$ over all $x \in \bar{\mathbb{R}} - \Sigma_R$.

The root of the tree corresponds to the boundary of the face, and the inner vertices correspond to the ovals contained in the face. The weight of an inner vertex is equal to the weight of the corresponding oval.

Now noting that covering-equivalence should be taken by a real Möbius transformation, we see the following

Theorem 2. *Every real rational function of degree* 2 *is covering-equivalent to an element of the families*

$$\left\{c + \frac{az+b}{z^2+1}\right\}, \quad \left\{z + c + \frac{b}{z}\right\}, \quad \{\pm z^2 + a\},$$

where $a, b, c \in \mathbb{R}$.

Next, gardens of the total weight 2 *are*

1. *a garden of order* 0 *with one root only,*

2. *a garden of order* 2 *with two roots.*

The first garden represents any generic element in the subfamily

$$\left\{ z + c + \frac{b}{z} \;\middle|\; b < 0 \right\},$$

and the second garden represents any generic element in the subfamilies

$$\left\{ c + \frac{az+b}{z^2+1} \right\}, \quad \left\{ z + c + \frac{b}{z} \;\middle|\; b > 0 \right\}, \quad \{\pm z^2 + a\}.$$

Proof. According as ∞ is a non-critical point or a critical one, we can show that the given quadratic rational function is covering-equivalent to an element of the first two families or of the third one, respectively. Here, every element of the first family has poles $\pm i$, and every one of the second family has two poles on $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. (See the remark below.)

Next, it is easy to see from the definition, that there are two kinds of gardens.

Now, for the first family, both of the critical points are real and finite if $a \neq 0$ and $0, \infty$ if $a = 0$. For the second family, it is clear that $b \neq 0$. If $b > 0$, the family $z + c + (b/z)$ has two real critical points. For the third family, critical points are $0, \infty$. Thus every generic functions in these families are represented by the second garden.

While, if $b < 0$ in the second family, a generic element has no real critical points, and is represented by the first garden. □

Remark 2. In the first family, by rotating around $\pm i$, we may assume that ∞ is always a critical point. Hence we can replace the first family by a simpler one:

$$\left\{ c + \frac{b}{z^2+1} \right\}.$$

Lemma 1. *Every real rational functoin of degree* 3 *is covering-equivalent to an element of the families*

$$\left\{ z + c + \frac{az+b}{z^2+d} \right\} \quad (a, b, c, d \in \mathbb{R})$$

$$\left\{ \pm z^2 + bz + c + \frac{a}{z} \right\} \quad (a, b, c \in \mathbb{R})$$

$$\{z^3 + bz + c\} \quad (b, c \in \mathbb{R})$$

Proof. Write the given real rational function as $P(z)/Q(z)$ with suitable polynomials $P(z)$ and $Q(z)$. If the degree of $Q(z)$ is 3, then the equation $Q(z) = 0$ has a real solution x_0. Sending x_0 to ∞, we may assume that the degree of $Q(z)$ is less than 3.

The rest of the proof is similar to that in case of degree 2. □

Remark 3. The third family contains no generic real rational functions.

The next lemma is easy to see from the definition.

Lemma 2. *Gardens of the weight* 3 *are*

1. *a garden of order* 0 *with one root only,*
2. *anotgher garden of order* 0 *with one rooted tree with one inner vertex,*
3. *a garden of order* 2 *with two roots,*
4. *a garden of order* 4 *with three roots.*

Now, if $d < 0$ in the first family in Lemma 1, then by setting $d = -A^2$ with a positive A and using new real parameters B and D, we can rewrite the function as

$$f(z) = z + c + \frac{B}{z-A} + \frac{D}{z+A}.$$

If $B > 0, D > 0$, then there are two cases:

1. $f(z)$ has 4 real critical points, or
2. $f(z)$ has two real critical points and two non-real ones.

Here, the phase transition occurs at the locus defined by

$$\Phi = (4A^2 - B - D)^3 - 108BDA^2 = 0.$$

So we devide the first family into the followings

1. $E_2 = \{(A, B, D) \mid \Phi > 0, BD < 0, \text{ or } \Phi < 0, B > 0, D > 0\}$
2. $E_4 = \{(A, B, D) \mid \Phi < 0, BD < 0, \text{ or } \Phi > 0, B > 0, D > 0\}$
3. $E' = \{(A, B, D) \mid B < 0, D < 0\}$.

Here the locus defined by $\Phi = 0$ is as in Figure 1.

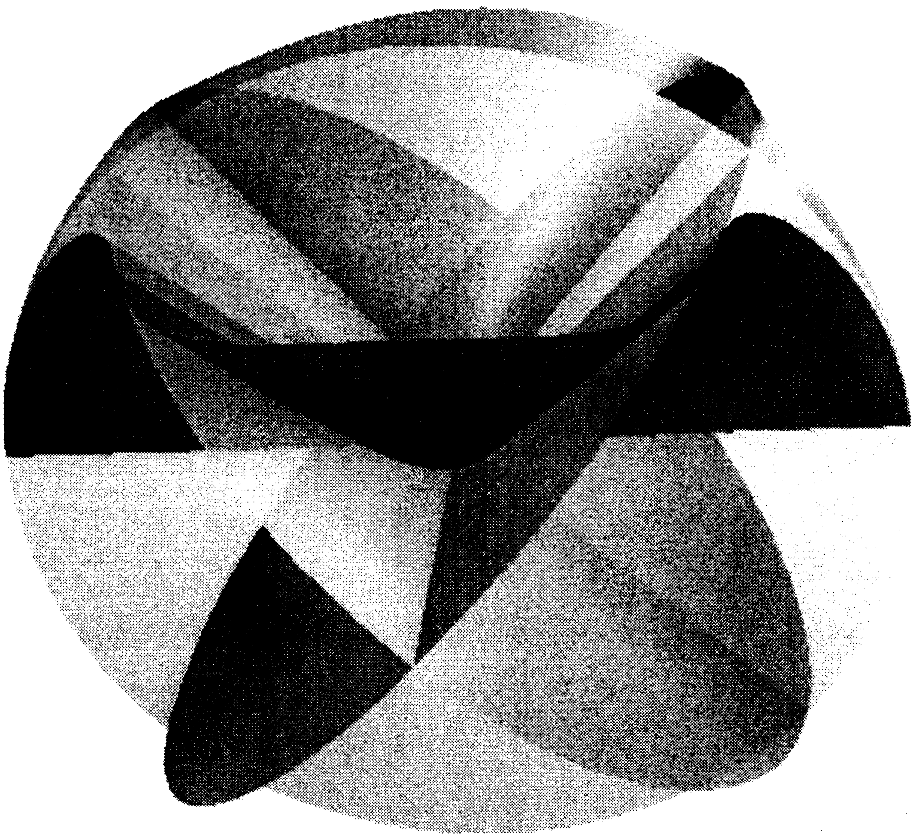

Figure 1: The horizontal plane is $\{B + D = 0\}$; the vertical one is $\{A = 0\}$.

Theorem 3. *1. In the first family, if $d < 0$, then generic elements in the subfamily E_2 or E_4, or E' are represented by the third garden, the 4-th one, or the first one, respectively.*

2. If $d > 0$ in the first family, then we can write elements as

$$f(z) = z + c + \frac{B}{z - iA} + \frac{\bar{B}}{z + iA} \qquad (d = A^2)$$

with real A, c and a complex $B = a + ib$. And if

$$\Phi' = -(4d + 2a)^3 + 108(a^2 + b^2)d > 0,$$

then generic elements are represented by the third garden, and if $\Phi' < 0$, *generic ones are represented by the second one.*

3. *For the second family, set*

$$\Psi = a(a - b^3/27).$$

Then generic elements are represented by the third garden or the 4-th one, respectively, according as $\Psi > 0$ *or* $\Psi < 0$.

Proof. First, we consider the first family, and note that, if $d = 0$, then the function has a critical value ∞, and is equivalent to an element of the second family. Hence we may assume that $d \neq 0$.

The case 1). Assume that $d < 0$, and set

$$F(z) = (z^2 - A^2)^2 - B(z + A)^2 - D(z - A)^2.$$

Elements in the first subfamily E_2 has 2 real critical points, and hence is represented by the third kind of gardens. Elements in E_4 has 4 real critical points.

These were shown in [1] when $B > 0, D > 0$. Actually, $\Phi = 0$ gives the locus where the number of critical points changes by 2. Here for instance, if we put $B = D = A^2/4$, then $F(0) > 0$ and $F(\pm A) < 0$, and hence the equation $F = 0$ has 4 real solutions. On the other hand, $\Phi > 0$, we have the assertion in this case.

And if $BD < 0$, assume for instance $B + D = 0$. Then

$$F(z) = (z^2 - A^2)^2 - 4ABz$$

and hence $F = 0$ has exactly 2 real solutions, since

$$F(-A)F(A) < 0.$$

Here it is clear that $\Phi > 0$.

Finally, we also know (cf. [1]) that, if $B < 0, D < 0$, then the element has no real critical points. Since all poles belongs to $\bar{\mathbb{R}}$, we have the assertion for E'.

The case 2). If $d > 0$ in the first family, then the above Φ' is the same as Φ, and again $\Phi' = 0$ gives the locus where the number of critical points changes by 2. Here for instance, if B is sufficiently near to 0, then $f(z)$ as in Theorem 3.2) is strictly increasing, and hence a diffeomorphism of $\mathbb{R}$ onto itself. Since iA is a pole, such a function has a single inner vertex.

On the other hand, if B is sufficiently near to 0, then Φ' is negative, and we have the assertion 2).

The case 3). Generic elements in the second family always have a critical point at ∞, and we can conclude the assertion by direct calculations. □

References

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