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(α, δ)-neighborhood defining by a new operator for certain analytic functions

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Abstract

For analytic functions $f(z)$ in the open unit disk $\mathbb{U}$, a new operator $D^j f(z)$ for any integer j which is the generalization of Sălăgean differential operator and Alexander integral operator is introduced. The object of the present paper is to discuss some properties for (α, δ)-neighborhood defining by a new operator $D^j f(z)$ and to apply Miller-Mocanu lemma (J. Math. Anal. Appl. **65**(1978)) for (α, δ)-neighborhood.

1 Introduction and definitions

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $f(z) \in \mathcal{A}$, Sălăgean [3] has introduced the following operator $D^j f(z)$ which is called Sălăgean differential operator.

$$D^0 f(z) = f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

$$D^1 f(z) = Df(z) = zf'(z) = z + \sum_{n=2}^{\infty} n a_n z^n$$

and

$$D^j f(z) = D(D^{j-1} f(z)) = z + \sum_{n=2}^{\infty} n^j a_n z^n \qquad (j = 1, 2, 3, \cdots).$$

Also, Alexander [1] has defined the following Alexander integral operator

$$D^{-1} f(z) = \int_0^z \frac{f(\zeta)}{\zeta} d\zeta = z + \sum_{n=2}^{\infty} n^{-1} a_n z^n.$$

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Futher, we introduce

$$D^{-j}f(z)=D^{-1}(D^{-(j-1)}f(z))=z+\sum_{n=2}^{\infty}n^{-j}a_nz^n \qquad (j=1,2,3,\cdots)$$

which is the generalization integral operator of Alexander integral operator. Therefore, combining Sălăgean differential operator and Alexander integral operator, we introduce the operator $D^jf(z)$ by

$$D^jf(z)=z+\sum_{n=2}^{\infty}n^ja_nz^n$$

for any integer j. Applying the above operator, we consider the subclass $(\alpha_1,\alpha_2,\cdots,\alpha_p;\delta)-N_{m+1}^{j+1}(g_1,g_2,\cdots,g_p)$ of $\mathcal{A}$ as follows. A function $f(z)\in\mathcal{A}$ is said to be in the class $(\alpha_1,\alpha_2,\cdots,\alpha_p;\delta)-N_{m+1}^{j+1}(g_1,g_2,\cdots,g_p)$ if it satisfies

$$\left|\frac{D^{j+1}f(z)}{z}-\sum_{k=1}^{p}\alpha_k\frac{D^{m+1}g_k(z)}{z}\right|<\delta \qquad (z\in\mathbb{U})$$

for some $\delta>\sqrt{1-2\sum_{k=1}^{p}|\alpha_k|\cos\beta+\left(\sum_{k=1}^{p}|\alpha_k|\right)^2}$, where $\beta=\arg\alpha_k$ for all k with $-\pi\leqq\beta\leqq\pi$, and for some $g_k(z)\in\mathcal{A}$ $(k=1,2,\cdots,p)$. Let us define $(\alpha_1,\alpha_2,\cdots,\alpha_p;\delta)-N_{m+1}^{j+1}(g_1,g_2,\cdots,g_p)$ by

$$(\alpha,\delta)-N_{m+1}^{j+1}(g)\equiv(\alpha_1,\alpha_2,\cdots,\alpha_p;\delta)-N_{m+1}^{j+1}(g_1,g_2,\cdots,g_p)$$

through this paper.

2 Main theorem

Let us define $g_k(z)\in\mathcal{A}$ $(k=1,2,\cdots,p)$ by

$$g_k(z)=z+\sum_{n=2}^{\infty}b_{n,k}z^n$$

through this paper. Our first result of $f(z)$ for $(\alpha,\delta)-N_{m+1}^{j+1}(g)$ is contained in

Theorem 2.1 *If $f(z)\in\mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty}n\left|n^ja_n-n^m\sum_{k=1}^{p}\alpha_kb_{n,k}\right|\leqq\delta-\sqrt{1-2\sum_{k=1}^{p}|\alpha_k|\cos\beta+\left(\sum_{k=1}^{p}|\alpha_k|\right)^2}$$

for some $\delta>\sqrt{1-2\sum_{k=1}^{p}|\alpha_k|\cos\beta+\left(\sum_{k=1}^{p}|\alpha_k|\right)^2}$, where $\beta=\arg\alpha_k$ for all k with $-\pi\leqq\beta\leqq\pi$, and for some $g_k(z)\in\mathcal{A}$ $(k=1,2,\cdots,p)$, then $f(z)\in(\alpha,\delta)-N_{m+1}^{j+1}(g)$.

Proof. Note that

$$\left|\frac{D^{j+1}f(z)}{z}-\sum_{k=1}^{p}\alpha_k\frac{D^{m+1}g_k(z)}{z}\right| = \left|1+\sum_{n=2}^{\infty}n^{j+1}a_nz^{n-1}-\sum_{k=1}^{p}\alpha_k\left(1+\sum_{n=2}^{\infty}n^{m+1}b_{n,k}z^{n-1}\right)\right|$$
$$=\left|1-\sum_{k=1}^{p}\alpha_k+\sum_{n=2}^{\infty}n\left(n^ja_n-n^m\sum_{k=1}^{p}\alpha_kb_{n,k}\right)z^{n-1}\right|$$
$$\leqq\left|1-\sum_{k=1}^{p}\alpha_k\right|+\sum_{n=2}^{\infty}n\left|n^ja_n-n^m\sum_{k=1}^{p}\alpha_kb_{n,k}\right||z|^{n-1}$$
$$<\sqrt{1-2\sum_{k=1}^{p}|\alpha_k|\cos\beta+\left(\sum_{k=1}^{p}|\alpha_k|\right)^2}+\sum_{n=2}^{\infty}n\left|n^ja_n-n^m\sum_{k=1}^{p}\alpha_kb_{n,k}\right|.$$

If

$$\sum_{n=2}^{\infty}n\left|n^ja_n-n^m\sum_{k=1}^{p}\alpha_kb_{n,k}\right|\leqq\delta-\sqrt{1-2\sum_{k=1}^{p}|\alpha_k|\cos\beta+\left(\sum_{k=1}^{p}|\alpha_k|\right)^2},$$

then we see that

$$\left|\frac{D^{j+1}f(z)}{z}-\sum_{k=1}^{p}\alpha_k\frac{D^{m+1}g_k(z)}{z}\right|<\delta \qquad (z\in\mathbb{U}).$$

This gives us that $f(z)\in(\alpha,\delta)-N_{m+1}^{j+1}(g)$. □

Example 2.2 For given $g_k(z)=z+\sum_{n=2}^{\infty}b_{n,k}z^n\in\mathcal{A}$, we consider $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{A}$ with

$$a_n=\frac{\delta-\sqrt{1-2\sum_{k=1}^{p}|\alpha_k|\cos\beta+\left(\sum_{k=1}^{p}|\alpha_k|\right)^2}}{n^{j+2}(n-1)}e^{i\gamma}+n^{m-j}\sum_{k=1}^{p}\alpha_kb_{n,k} \qquad (n=2,3,4,\cdots).$$

Then, we have that

$$\sum_{n=2}^{\infty}n\left|n^ja_n-n^m\sum_{k=1}^{p}\alpha_kb_{n,k}\right|=\sum_{n=2}^{\infty}n\left|n^j\frac{\delta-\sqrt{1-2\sum_{k=1}^{p}|\alpha_k|\cos\beta+\left(\sum_{k=1}^{p}|\alpha_k|\right)^2}}{n^{j+2}(n-1)}e^{i\gamma}\right|$$
$$=\left(\delta-\sqrt{1-2\sum_{k=1}^{p}|\alpha_k|\cos\beta+\left(\sum_{k=1}^{p}|\alpha_k|\right)^2}\right)\left(\sum_{n=2}^{\infty}\frac{1}{n(n-1)}\right)$$
$$=\left(\delta-\sqrt{1-2\sum_{k=1}^{p}|\alpha_k|\cos\beta+\left(\sum_{k=1}^{p}|\alpha_k|\right)^2}\right)\left\{\sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n}\right)\right\}$$
$$=\delta-\sqrt{1-2\sum_{k=1}^{p}|\alpha_k|\cos\beta+\left(\sum_{k=1}^{p}|\alpha_k|\right)^2}.$$

Therefore, $f(z)\in(\alpha,\delta)-N_{m+1}^{j+1}(g)$.

In view of Theorem 2.1, we have the following corollary.

Corollary 2.3 *Let $f(z) \in \mathcal{A}$ satisfy*

$$\sum_{n=2}^{\infty} n \left| n^j |a_n| - n^m \sum_{k=1}^{p} |\alpha_k| |b_{n,k}| \right| \leqq \delta - \sqrt{1 - 2\sum_{k=1}^{p} |\alpha_k| \cos\beta + \left(\sum_{k=1}^{p} |\alpha_k| \right)^2}$$

for some $\delta > \sqrt{1 - 2\sum_{k=1}^{p} |\alpha_k| \cos\beta + \left(\sum_{k=1}^{p} |\alpha_k| \right)^2}$, *where* $\beta = \arg \alpha_k$ *for all* k *with* $-\pi \leqq \beta \leqq \pi$, *and for some* $g_k(z) \in \mathcal{A}$ $(k = 1, 2, \cdots, p)$ *with* $\arg a_n - \arg b_{n,k} = \beta$ $(n = 2, 3, 4, \cdots)$ *for all* k, *then* $f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g)$.

Proof. By Theorem 2.1, we have that if $f(z) \in \mathcal{A}$ satisfies

$$\sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k} \right| \leqq \delta - \sqrt{1 - 2\sum_{k=1}^{p} |\alpha_k| \cos\beta + \left(\sum_{k=1}^{p} |\alpha_k| \right)^2},$$

then $f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g)$. Since $\arg a_n - \arg b_{n,k} = \beta$, if $\arg a_n = \varphi_n$, then $\arg b_{n,k} = \varphi_n - \beta$. Therefore, we see that

$$\begin{aligned} n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k} &= n^j |a_n| e^{i\varphi_n} - n^m \sum_{k=1}^{p} |\alpha_k| e^{i\beta} |b_{n,k}| e^{i(\varphi_n - \beta)} \\ &= \left(n^j |a_n| - n^m \sum_{k=1}^{p} |\alpha_k| |b_{n,k}| \right) e^{i\varphi_n}, \end{aligned}$$

that is, that

$$\left| n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k} \right| = \left| n^j |a_n| - n^m \sum_{k=1}^{p} |\alpha_k| |b_{n,k}| \right|.$$

This completes the proof of the corollary. □

Next, we discuss the necessary conditions for neighborhoods.

Theorem 2.4 *If $f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g)$ with*

$$\arg \left(n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k} \right) = (n-1)\varphi \quad (\varphi \in \mathbb{R}),$$

for $n = 2, 3, 4, \cdots$, *then,*

$$\sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k} \right| \leqq -1 + \sum_{k=1}^{p} |\alpha_k| \cos\beta + \sqrt{\delta^2 - \left(\sum_{k=1}^{p} |\alpha_k| \sin\beta \right)^2}.$$

Proof. For $f(z) \in (\alpha,\delta) - N_{m+1}^{j+1}(g)$, if we consider a point $z \in \mathbb{U}$ such that $\arg z = -\varphi$, then

$$z^{n-1} = |z|^{n-1} e^{-i(n-1)\varphi},$$

and hence we have

$$\begin{aligned}\left|\frac{D^{j+1}f(z)}{z} - \sum_{k=1}^{p} \alpha_k \frac{D^{m+1}g_k(z)}{z}\right| &= \left|1 - \sum_{k=1}^{p}\alpha_k + \sum_{n=2}^{\infty} n\left(n^j a_n - n^m \sum_{k=1}^{p}\alpha_k b_{n,k}\right) z^{n-1}\right| \\ &= \left|1 - \sum_{k=1}^{p}\alpha_k + \sum_{n=2}^{\infty} n\left|n^j a_n - n^m \sum_{k=1}^{p}\alpha_k b_{n,k}\right| |z|^{n-1}\right| < \delta.\end{aligned}$$

Letting $|z| \to 1^-$ we have

$$\begin{aligned}&\left|1 - \sum_{k=1}^{p}\alpha_k + \sum_{n=2}^{\infty} n\left|n^j a_n - n^m \sum_{k=1}^{p}\alpha_k b_{n,k}\right|\right| \\ &\quad = \left\{\left(1 - \sum_{k=1}^{p}|\alpha_k|\cos\beta + \sum_{n=2}^{\infty} n\left|n^j a_n - n^m \sum_{k=1}^{p}\alpha_k b_{n,k}\right|\right)^2 + \left(\sum_{k=1}^{p}|\alpha_k|\sin\beta\right)^2\right\}^{\frac{1}{2}} \leqq \delta,\end{aligned}$$

which implies that

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} n\left|n^j a_n - n^m \sum_{k=1}^{p}\alpha_k b_{n,k}\right|\right)^2 + 2\left(1 - \sum_{k=1}^{p}|\alpha_k|\cos\beta\right)\left(\sum_{n=2}^{\infty} n\left|n^j a_n - n^m \sum_{k=1}^{p}\alpha_k b_{n,k}\right|\right) \\ &\qquad + 1 + \left(\sum_{k=1}^{p}|\alpha_k|\right)^2 - 2\sum_{k=1}^{p}|\alpha_k|\cos\beta - \delta^2 \leqq 0.\end{aligned}$$

Therefore, it is easy to see that

$$\sum_{n=2}^{\infty} n\left|n^j a_n - n^m \sum_{k=1}^{p}\alpha_k b_{n,k}\right| \leqq -1 + \sum_{k=1}^{p}|\alpha_k|\cos\beta + \sqrt{\delta^2 - \left(\sum_{k=1}^{p}|\alpha_k|\sin\beta\right)^2}.$$

□

3 Applications of Miller-Mocanu lemma

In this section, we will give a certain implication for the class $(\alpha,\delta) - N_{m+1}^{j+1}(g)$. To considering our probrem, we need the following lemma given by Miller and Mocanu [2].

Lemma 3.1 *Let n be a positive integer, and let $F(z)$ be analytic in $\mathbb{U}$ with $F^{(k)}(0) = 0$ $(k = 1, 2, \cdots, n-1)$, $F(0) = a$ and $F(z) \not\equiv a$ for a complex number a. If there exists a point $z_0 \in \mathbb{U}$ such that*

$$\max_{|z| \leqq |z_0|} |F(z)| = |F(z_0)|,$$

then

$$\frac{z_0 F'(z_0)}{F(z_0)} = m,$$

where m is real and

$$m \geqq n\frac{\left|F(z_0)-a\right|^2}{\left|F(z_0)\right|^2-\left|a\right|^2} \geqq n\frac{\left|F(z_0)\right|-\left|a\right|}{\left|F(z_0)\right|+\left|a\right|}.$$

Applying Lemma 3.1, we derive

Theorem 3.2 *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\frac{D^{j+1}f(z)}{z}-\sum_{k=1}^{p}\alpha_k\frac{D^{m+1}g_k(z)}{z}\right| < \frac{2\delta^2}{\delta+\sqrt{1-2\sum_{k=1}^{p}\left|\alpha_k\right|\cos\beta+\left(\sum_{k=1}^{p}\left|\alpha_k\right|\right)^2}} \qquad (z \in \mathbb{U})$$

for some $\delta > \sqrt{1-2\sum_{k=1}^{p}\left|\alpha_k\right|\cos\beta+\left(\sum_{k=1}^{p}\left|\alpha_k\right|\right)^2}$, where $\beta = \arg\alpha_k$ for all k with $-\pi \leqq \beta \leqq \pi$, and for some $g_k(z) \in \mathcal{A}$ $(k = 1, 2, \cdots, p)$, then

$$\left|\frac{D^{j}f(z)}{z}-\sum_{k=1}^{p}\alpha_k\frac{D^{m}g_k(z)}{z}\right| < \delta \qquad (z \in \mathbb{U}),$$

which implies that $f(z) \in (\alpha,\delta) - N_{m+1}^{j+1}(g)$.

Proof. We define the function $F(z)$ by

$$F(z) = \frac{D^{j}f(z)}{z}-\sum_{k=1}^{p}\alpha_k\frac{D^{m}g_k(z)}{z} \qquad (z \in \mathbb{U}).$$

Then,

$$\frac{zF'(z)}{F(z)} = \frac{\dfrac{D^{j+1}f(z)}{z}-\dfrac{D^{j}f(z)}{z}-\sum_{k=1}^{p}\alpha_k\left(\dfrac{D^{m+1}g_k(z)}{z}-\dfrac{D^{m}g_k(z)}{z}\right)}{\dfrac{D^{j}f(z)}{z}-\sum_{k=1}^{p}\alpha_k\dfrac{D^{m}g_k(z)}{z}}$$
$$= \frac{1}{F(z)}\left(\frac{D^{j+1}f(z)}{z}-\sum_{k=1}^{p}\alpha_k\frac{D^{m+1}g_k(z)}{z}\right)-1.$$

Therefore,

$$\left|\frac{D^{j+1}f(z)}{z}-\sum_{k=1}^{p}\alpha_k\frac{D^{m+1}g_k(z)}{z}\right| = \left(1+\frac{zF'(z)}{F(z)}\right)F(z).$$

Then $F(z)$ is analytic in $\mathbb{U}$ with $F(0) = 1-\sum_{k=1}^{p}\alpha_k$ and $\left|F(0)\right| < \delta$. In view of the condition, let us suppose that there is a point $z_0 \in \mathbb{U}$ such that $\max_{|z|\leqq|z_0|}\left|F(z)\right| = \left|F(z_0)\right| = \delta$. Then, by Lemma 3.1, we can write that

$$F(z_0) = \delta e^{i\theta}, \quad \frac{z_0F'(z_0)}{F(z_0)} = m \quad \text{and} \quad m \geqq \frac{\left|\delta e^{i\theta}-\left(1-\sum_{k=1}^{p}\alpha_k\right)\right|^2}{\delta^2-\left|1-\sum_{k=1}^{p}\alpha_k\right|^2}.$$

Therefore, we see that

$$\begin{aligned}\left|\frac{D^{j+1}f(z_0)}{z_0}-\sum_{k=1}^{p}\alpha_k\frac{D^{m+1}g_k(z_0)}{z_0}\right| &= |1+m||F(z_0)|\\ &= \delta(1+m)\\ &\geqq \delta+\delta\frac{\left|\delta e^{i\theta}-\left(1-\sum_{k=1}^{p}\alpha_k\right)\right|^2}{\delta^2-\left|1-\sum_{k=1}^{p}\alpha_k\right|^2}\\ &\geqq \delta+\delta\frac{\delta-\left|1-\sum_{k=1}^{p}\alpha_k\right|}{\delta+\left|1-\sum_{k=1}^{p}\alpha_k\right|}\\ &= \frac{2\delta^2}{\delta+\sqrt{1-2\sum_{k=1}^{p}|\alpha_k|\cos\beta+\left(\sum_{k=1}^{p}|\alpha_k|\right)^2}}.\end{aligned}$$

This contradicts our condition in Theorem 3.2. Thus, there is no point $z_0 \in \mathbb{U}$ such that $|F(z_0)| = \delta$. This means that $|F(z)| < \delta$ for all $z \in \mathbb{U}$. Therefore, we have that

$$\left|\frac{D^j f(z)}{z}-\sum_{k=1}^{p}\alpha_k\frac{D^m g_k(z)}{z}\right| < \delta \qquad (z \in \mathbb{U}).$$

□

Taking $p = 1$ in Theorem 3.2, and letting

$$\alpha_1 = e^{i\alpha} \quad \text{and} \quad g_1(z) = g(z),$$

we find the following corollary.

Corollary 3.3 *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\frac{D^{j+1}f(z)}{z}-e^{i\alpha}\frac{D^{m+1}g(z)}{z}\right| < \frac{2\delta^2}{\delta+\sqrt{2(1-\cos\alpha)}} \qquad (z \in \mathbb{U})$$

for some $-\pi \leqq \alpha \leqq \pi$, $\delta > \sqrt{2(1-\cos\alpha)}$ and for some $g(z) \in \mathcal{A}$, then

$$\left|\frac{D^j f(z)}{z}-e^{i\alpha}\frac{D^m g(z)}{z}\right| < \delta \qquad (z \in \mathbb{U}).$$

In paticular, by putting $\delta = \tilde{\delta} + \sqrt{2(1-\cos\alpha)}$ for some $-\pi \leqq \alpha \leqq \pi$ and $\tilde{\delta} > 0$, we can obtain the assertion as follows.

Corollary 3.4 *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\frac{D^{j+1}f(z)}{z}-e^{i\alpha}\frac{D^{m+1}g(z)}{z}\right| < 2\tilde{\delta}+\frac{4(1-\cos\alpha)}{\tilde{\delta}+2\sqrt{2(1-\cos\alpha)}} \qquad (z \in \mathbb{U}) \tag{3.1}$$

for some $-\pi \leqq \alpha \leqq \pi$, $\tilde{\delta} > 0$ *and for some* $g(z) \in \mathcal{A}$, *then*

$$\left|\frac{D^j f(z)}{z} - e^{i\alpha}\frac{D^m g(z)}{z}\right| < \tilde{\delta} + \sqrt{2(1-\cos\alpha)} \qquad (z \in \mathbb{U}). \tag{3.2}$$

Remark 3.5 Recently, in the paper by Kugita, Kuroki and Owa [4], we obtained the implication that

$$\left|\frac{D^{j+1} f(z)}{z} - e^{i\alpha}\frac{D^{m+1} g(z)}{z}\right| < 2\tilde{\delta} - \sqrt{2(1-\cos\alpha)} \qquad (z \in \mathbb{U}) \tag{3.3}$$

implies the inequality (3.2), where $\tilde{\delta} > \sqrt{2(1-\cos\alpha)}$. Here, a simple check gives us that if $f(z) \in \mathcal{A}$ satisfies the inequality (3.3), then $f(z)$ satisfies the inequality (3.1). Hence, it follows this fact that if $f(z) \in \mathcal{A}$ satisfies the assertion of Corollary 3.4, then the implication which were proven by Kugita, Kuroki and Owa [4] holds.

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