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Solutions to The Homogeneous Associated Laguerre's Equation by Means of N-Fractional Calculus Operator

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Abstract

In this article, solutions to the homogeneous associated Laguerre's equations

$$\varphi_2 \cdot z + \varphi_1 \cdot (-z + \alpha + 1) + \varphi \cdot \beta = 0 \quad (z \neq 0)$$

$$(\varphi_\nu = d^\nu \varphi / dz^\nu \text{ for } \nu > 0 \,;\, \varphi_0 = \varphi = \varphi(z))$$

are discussed by means of N-fractional calculus operator (NFCO- Method).

By our method, some particular solutions to the above equations are given as below for example, in fractional differintegrated forms.

Group I.

(i) $$\varphi = (e^z \cdot z^{-(\alpha+\beta+1)})_{-(1+\beta)} \equiv \varphi_{[1](\alpha,\beta)} \quad \text{(denote)}$$

and

(ii) $$\varphi = (z^{-(\alpha+\beta+1)} \cdot e^z)_{-(1+\beta)} \equiv \varphi_{[2](\alpha,\beta)}$$

And the familiar forms are

$$\varphi_{[1](\alpha,\beta)} = e^z z^{-(\alpha+\beta+1)} {}_2F_0(\alpha+1, \alpha+\beta+1; \tfrac{1}{z})$$

and

$$\varphi_{[2](\alpha,\beta)} = -e^{i\pi\beta} \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta+1)} z^{-\alpha} e^z \, {}_1F_1(\beta+1; 1-\alpha; -z)$$

respectively.

Where ${}_pF_q(\cdots)$ is the generalized Gauss hypergeometric function.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\,\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\,\mathrm{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D (z \in D)$,

$$f_\nu = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{\nu+1}}\,d\zeta \qquad (\nu \notin \mathbf{Z}^-), \tag{1}$$

$$(f)_{-m} = \lim_{\nu \to -m} (f)_\nu \qquad (m \in \mathbf{Z}^+), \tag{2}$$

where $\quad -\pi \le \arg(\zeta - z) \le \pi$ for C_- , $\quad 0 \le \arg(\zeta - z) \le 2\pi$ for C_+ ,

$\zeta \ne z$, $\ z \in \mathbf{C}$, $\ \nu \in \mathbf{R}$, $\ \Gamma$; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

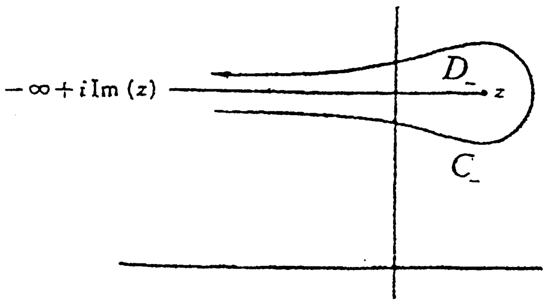

Fig. 1.

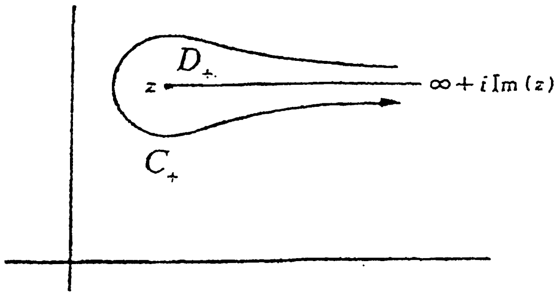

Fig. 2.

Notice that (1) is reduced to Goursat's integral for $\nu = n (\in \mathbf{Z}^+)$ and is reduced to the famous Cauchy's integral for $\nu = 0$. That is, (1) is an extention of Cauchy integral and of Goursat's one, conversely Cauchy and Goursat's ones are spetial cases of (1).

(I I) On the fractional calculus operator N^ν [3]

Theorem A. *Let fractional calculus operator (Nishimoto's Operator) N^ν be*

$$N^{\nu} = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\zeta}{(\zeta - z)^{\nu+1}} \right) \qquad (\nu \notin \mathbb{Z}), \quad [\,Refer\ to\,(1)\,] \qquad (3)$$

with

$$N^{-m} = \lim_{\nu \to -m} N^{\nu} \qquad (m \in \mathbb{Z}^{+}), \qquad (4)$$

and define the binary operation $\circ$ *as*

$$N^{\beta} \circ N^{\alpha} f = N^{\beta} N^{\alpha} f = N^{\beta}(N^{\alpha} f) \qquad (\alpha, \beta \in \boldsymbol{R}), \qquad (5)$$

then the set

$$\left\{ N^{\nu} \right\} = \left\{ N^{\nu} \mid \nu \in \boldsymbol{R} \right\} \qquad (6)$$

is an Abelian product group (having continuous index ν *) which has the inverse transform operator* $(N^{\nu})^{-1} = N^{-\nu}$ *to the fractional calculus operator* N^{ν} *, for the function* f *such that* $f \in F = \left\{ f;\ 0 \neq |f_{\nu}| < \infty,\ \nu \in \boldsymbol{R} \right\}$, *where* $f = f(z)$ *and* $z \in \boldsymbol{C}$.
(*vis.* $-\infty < \nu < \infty$).

(For our convenience, we call $N^{\beta} \circ N^{\alpha}$ as product of N^{β} and N^{α}.)

Theorem B. " F.O.G. $\{N^{\nu}\}$ " *is an* " *Action product group which has continuous index* ν " *for the set of* F . (*F.O.G.* ; Fractional calculus operator group) [3]

Theorem C. *Let*

$$S := \{\pm N^{\nu}\} \cup \{0\} = \{N^{\nu}\} \cup \{-N^{\nu}\} \cup \{0\} \quad (\nu \in \boldsymbol{R}). \qquad (7)$$

Then the set S *is a commutative ring for the function* $f \in F$ *, when the identity*

$$N^{\alpha} + N^{\beta} = N^{\gamma} \qquad (N^{\alpha}, N^{\beta}, N^{\gamma} \in S). \qquad (8)$$

holds. [5]

(III) **Lemma.** We have [1]

(i) $$((z-c)^{b})_{\alpha} = e^{-i\pi\alpha} \frac{\Gamma(\alpha - b)}{\Gamma(-b)} (z-c)^{b-\alpha} \qquad \left(\left| \frac{\Gamma(\alpha - b)}{\Gamma(-b)} \right| < \infty \right),$$

(ii) $$(\log(z-c))_{\alpha} = - e^{-i\pi\alpha} \Gamma(\alpha) (z-c)^{-\alpha} \qquad \left(|\Gamma(\alpha)| < \infty \right),$$

(iii) $$((z-c)^{-\alpha})_{-\alpha} = - e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \qquad \left(|\Gamma(\alpha)| < \infty \right),$$

where $z - c \neq 0$ for (i) and $z - c \neq 0, 1$ for (ii), (iii) ,

(iv) $$(u \cdot v)_{\alpha} := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)} u_{\alpha-k} v_k \qquad \left(\begin{array}{l} u = u(z), \\ v = v(z) \end{array} \right).$$

(Generalized Leibniz rule).

§ 1. Preliminary

(I) The theorem below is reported by the author already (cf. J.F C, Vol. 27, May (2005), 83 - 88.). [31]

Theorem D. *Let*

$$P = P(\alpha,\beta,\gamma) := \frac{\sin \pi\alpha \cdot \sin\pi(\gamma-\alpha-\beta)}{\sin\pi(\alpha+\beta)\cdot\sin\pi(\gamma-\alpha)} \quad (\,|\,P(\alpha,\beta,\gamma)\,| = M < \infty\,) \qquad (1)$$

and

$$Q = Q(\alpha,\beta,\gamma) := P(\beta,\alpha,\gamma), \qquad (\,|\,P(\beta,\alpha,\gamma)\,| = M < \infty) \qquad (2)$$

When $\alpha,\beta,\gamma \notin Z_0^+$, *we have* ;

(i) $$((z-c)^{\alpha}\cdot(z-c)^{\beta})_{\gamma} = e^{-i\pi\gamma}\, P(\alpha,\beta,\gamma)\, \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}(z-c)^{\alpha+\beta-\gamma}, \qquad (3)$$

$$(\,\mathrm{Re}(\alpha+\beta+1) > 0, \quad (1+\alpha-\gamma) \notin Z_0^-\,),$$

(i i) $$((z-c)^{\beta}\cdot(z-c)^{\alpha})_{\gamma} = e^{-i\pi\gamma}\, Q(\alpha,\beta,\gamma)\, \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}(z-c)^{\alpha+\beta-\gamma}, \qquad (4)$$

$$(\,\mathrm{Re}(\alpha+\beta+1) > 0, \quad (1+\beta-\gamma) \notin Z_0^-\,)$$

(i i i) $$((z-c)^{\alpha+\beta})_{\gamma} = e^{-i\pi\gamma}\, \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}(z-c)^{\alpha+\beta-\gamma}, \qquad (5)$$

where

$$z-c \neq 0, \quad \left|\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\right| < \infty .$$

Then the inequalities below are established from this theorem.

Corollary 1. *We have the inequalities*

(i) $$((z-c)^{\alpha}\cdot(z-c)^{\beta})_{\gamma} \neq ((z-c)^{\beta}\cdot(z-c)^{\alpha})_{\gamma}, \qquad (6)$$

and

(i i) $$((z-c)^{\alpha}\cdot(z-c)^{\beta})_{\gamma} \neq ((z-c)^{\alpha+\beta})_{\gamma}, \qquad (7)$$

where

$$\alpha,\beta,\gamma \notin Z_0^+, \quad \alpha \neq \beta, \quad z-c \neq 0.$$

Corollary 2.

(i) *When* $\alpha, \beta, \gamma \notin Z_0^+$, *and*

$$P(\alpha, \beta, \gamma) = Q(\beta, \alpha, \gamma) = 1, \tag{8}$$

we have

$$((z-c)^{\alpha} \cdot (z-c)^{\beta})_{\gamma} = ((z-c)^{\beta} \cdot (z-c)^{\alpha})_{\gamma} = ((z-c)^{\alpha+\beta})_{\gamma}, \tag{9}$$

$$(\mathrm{Re}(\alpha + \beta + 1) > 0, \ (1 + \alpha - \gamma) \notin Z_0^-, (1 + \beta - \gamma) \notin Z_0^-).$$

(ii) *When* $\gamma = m \in Z_0^+$, *we have* ;

$$((z-c)^{\alpha} \cdot (z-c)^{\beta})_m = ((z-c)^{\beta} \cdot (z-c)^{\alpha})_m = ((z-c)^{\alpha+\beta})_m. \tag{10}$$

§2. Solutions to The Homogeneous Associated Laguerre's Equations by N-Fractional Calculus Operator

Theorem 1. *Let* $\varphi = \varphi(z) \in F$ *,then the homogeneous associated Laguerre's equation*

$$[\varphi; z; \alpha, \beta] = \varphi_2 \cdot z + \varphi_1 \cdot (-z + \alpha + 1) + \varphi \cdot \beta = 0 \quad (z \neq 0) \tag{1}$$

$$(\varphi_\nu = d^\nu \varphi / dz^\nu \text{ for } \nu > 0 ; \varphi_0 = \varphi = \varphi(z) \ )$$

has particular solutions of the forms (fractional differintegrated form)

Group I.

(i) $$\varphi = (e^z \cdot z^{-(\alpha+\beta+1)})_{-(1+\beta)} \equiv \varphi_{[1](\alpha,\beta)} \quad \text{(denote)} \tag{2}$$

(ii) $$\varphi = (z^{-(\alpha+\beta+1)} \cdot e^z)_{-(1+\beta)} \equiv \varphi_{[2](\alpha,\beta)} \tag{3}$$

Group II.

(i) $$\varphi = e^z (e^{-z} \cdot z^{\beta})_{\alpha+\beta} \equiv \varphi_{[3](\alpha,\beta)} \tag{4}$$

(ii) $$\varphi = e^z (z^{\beta} \cdot e^{-z})_{\alpha+\beta} \equiv \varphi_{[4](\alpha,\beta)} \tag{5}$$

Group III.

(i) $$\varphi = z^{-\alpha} (e^z \cdot z^{-(\beta+1)})_{-(1+\alpha+\beta)} \equiv \varphi_{[5](\alpha,\beta)} \tag{6}$$

(ii) $$\varphi = z^{-\alpha} (z^{-(\beta+1)} \cdot e^z)_{-(1+\alpha+\beta)} \equiv \varphi_{[6](\alpha,\beta)} \tag{7}$$

and

Group IV.

(i) $$\varphi = z^{-\alpha} e^z (e^{-z} \cdot z^{\alpha+\beta})_{\beta} \equiv \varphi_{[7](\alpha,\beta)} \tag{8}$$

(ii) $$\varphi = z^{-\alpha} e^z (z^{\alpha+\beta} \cdot e^{-z})_{\beta} \equiv \varphi_{[8](\alpha,\beta)} \ . \tag{9}$$

Proof of Group I.

Operate N-fractional calculus (NFC) operator N^{ν} to the both sides of equation (1), we have then

$$(\varphi_2 \cdot z)_\nu + (\varphi_1 \cdot (-z+\alpha+1))_\nu + (\varphi \cdot \beta)_\nu = 0 \quad (\nu \notin Z^-) . \tag{10}$$

Now we have

$$(\varphi_2 \cdot z)_\nu = \sum_{k=0}^{1} \frac{\Gamma(\nu+1)}{k!\,\Gamma(\nu+1-k)} (\varphi_2)_{\nu-k}(z)_k \tag{11}$$

$$= \varphi_{2+\nu} \cdot z + \varphi_{1+\nu} \cdot \nu \quad , \tag{12}$$

$$(\varphi_1 \cdot (-z+\alpha+1))_\nu = \varphi_{1+\nu} \cdot (-z+\alpha+1)) - \varphi_\nu \cdot \nu \tag{13}$$

and

$$(\varphi \cdot \beta)_\nu = \varphi_\nu \cdot \beta \quad , \tag{14}$$

respectively, by Lemmas (i) and (i v).

Therefore, we have

$$\varphi_{2+\nu} \cdot z + \varphi_{1+\nu} \cdot (-z+\alpha+1+\nu) + \varphi_\nu \cdot (\beta - \nu) = 0 \tag{15}$$

from (10), applyimg (12) , (13) and (14).

Choosing ν such that

$$\nu = \beta \tag{16}$$

we obtain

$$\varphi_{2+\beta} \cdot z + \varphi_{1+\beta} \cdot (-z+\alpha+\beta+1) = 0 \quad . \tag{17}$$

Set

$$\varphi_{1+\beta} = \phi = \phi(z) \qquad (\varphi = \phi_{-(1+\beta)}) \, , \tag{18}$$

we have then

$$\phi_1 + \phi \cdot \left(\frac{\alpha+\beta+1}{z} - 1 \right) = 0 \tag{19}$$

from (17). A particular solution to this (variable separable form) equation is given by

$$\phi = e^z z^{-(\alpha+\beta+1)} \quad . \tag{20}$$

Therefore, we obtain

$$\varphi = (e^z \cdot z^{-(\alpha+\beta+1)})_{-(1+\beta)} \equiv \varphi_{[1](\alpha,\beta)} \tag{2}$$

from (20) and (18).

Inversely (20) satisfies equation (19). then (2) satisfies equation (1).

Next, changing the order

e^z and $z^{-(\alpha+\beta+1)}$ in parenthesis $(\quad)_{-(1+\beta)}$

we obtain other solution $\varphi_{[2](\alpha,\beta)}$ which is different from (2) for $-(1+\beta) \notin Z_0^+$, that is,

$$\varphi = (z^{-(\alpha+\beta+1)} \cdot e^z)_{-(1+\beta)} \equiv \varphi_{[2](\alpha,\beta)} \quad . \tag{3}$$

(Refer to Theorem D.)

Proof of Group II.

Set

$$\varphi = e^{\gamma z}\psi \qquad (\psi = \psi(z)) \ , \tag{21}$$

we have then

$$\varphi_1 = e^{\gamma z}(\gamma\psi + \psi_1) \tag{22}$$

and

$$\varphi_2 = e^{\gamma z}(\gamma^2\psi + 2\gamma\psi_1 + \psi_2) \ . \tag{23}$$

We have then

$$\psi_2 \cdot z + \psi_1 \cdot \{z(2\gamma - 1) + \alpha + 1\} + \psi \cdot \{z\gamma(\gamma - 1) + \gamma(\alpha + 1) + \beta\} = 0 \tag{24}$$

from (1), applying (21), (22) and (23).

Here we choose γ such that

$$\gamma(\gamma - 1) = 0 \ ,$$

that is,

$$\gamma = 0, 1 \ . \tag{25}$$

When $\gamma = 0$, (24) is reduced to (1), therefore, we have the same solutions as Group I.

When $\gamma = 1$ we have

$$\psi_2 \cdot z + \psi_1 \cdot \{z + \alpha + 1\} + \psi \cdot (\alpha + \beta + 1) = 0 \tag{26}$$

from (24)

Operate N^ν to the both sides of equation (26), we have then

$$(\psi_2 \cdot z)_\nu + (\psi_1 \cdot (z + \alpha + 1))_\nu + (\psi \cdot (\alpha + \beta + 1))_\nu = 0 \quad (\nu \notin Z^-) \ . \tag{27}$$

hence

$$\psi_{2+\nu} \cdot z + \psi_{1+\nu} \cdot (z + \alpha + 1 + \nu) + \psi_\nu \cdot (\nu + \alpha + \beta + 1) = 0 \ . \tag{28}$$

Choosing ν such that

$$\nu = -(\alpha + \beta + 1) \tag{29}$$

we obtain

$$\psi_{1-(\alpha+\beta)} \cdot z + \psi_{-(\alpha+\beta)} \cdot (z - \beta) = 0 \quad . \tag{30}$$

Set

$$\psi_{-(\alpha+\beta)} = \phi = \phi(z) \qquad (\psi = \phi_{\alpha+\beta}) \ , \tag{31}$$

we have then

$$\phi_1 + \phi \cdot \left(1 - \frac{\beta}{z}\right) = 0 \tag{32}$$

from (30). A particular solution to this (variable separable form) equation is given by

$$\phi = e^{-z} z^{\beta} \ . \tag{33}$$

Hence we obtain

$$\psi = (e^{-z} \cdot z^{\beta})_{\alpha+\beta} \tag{34}$$

from (31) and (33).

Therefore, we obtain

$$\varphi = e^{z}(e^{-z} \cdot z^{\beta})_{\alpha+\beta} \equiv \varphi_{[3](\alpha,\beta)} \tag{4}$$

from (21) and (34), having $\gamma = 1$.

Inversely , (33) satisfies (32), then (4) satisfies equation (1).

Next, changing the order

e^{-z} and z^{β} in parenthesis $(\qquad)_{\alpha+\beta}$ in (4)

we obtain other solution

$$\varphi = e^{z}(z^{\beta} \cdot e^{-z})_{\alpha+\beta} \equiv \varphi_{[4](\alpha,\beta)} \tag{5}$$

which is different from (4) for $(\alpha + \beta) \notin Z_0^+$,

(Refer to Theorem D.)

Proof of Group III.

Set

$$\varphi = z^{\lambda}\psi \qquad (\psi = \psi(z)) \ , \tag{35}$$

we have then

$$\varphi_1 = \lambda z^{\lambda-1}\psi + z^{\lambda}\psi_1 \tag{36}$$

and

$$\varphi_2 = \lambda(\lambda - 1)z^{\lambda-2}\psi + 2\lambda z^{\lambda-1}\psi_1 + z^{\lambda}\psi_2 . \tag{37}$$

respectively.

Hence we obtain

$$\psi_2 \cdot z^{\lambda+1} + \psi_1 \cdot \{-z^{\lambda+1} + z^{\lambda}(2\lambda + \alpha + 1)\}$$
$$+ \psi \cdot \{z^{\lambda}(\beta - \lambda) + z^{\lambda-1}\lambda(\lambda + \alpha)\} = 0 \qquad (38)$$

from (1), applying (35), (36) and (37).

Here we choose λ such that

$$\lambda(\lambda + \alpha) = 0 \ ,$$

that is,

$$\lambda = 0 \, , \, -\alpha \ . \qquad (39)$$

When $\lambda = 0$; (38) is reduced to (1); therefore, we have the same solutions as Group I.

When $\lambda = -\alpha$ we have

$$\psi_2 \cdot z + \psi_1 \cdot \{-z + 1 - \alpha\} + \psi \cdot (\alpha + \beta) = 0 \qquad (40)$$

from (38)

Operate N^{ν} to the both sides of equation (40), we have then

$$\psi_{2+\nu} \cdot z + \psi_{1+\nu} \cdot (-z + 1 - \alpha + \nu) + \psi_{\nu} \cdot (\alpha + \beta - \nu) = 0 \quad (\nu \notin Z^{-}) \ . \qquad (41)$$

Choosing ν such that

$$\nu = \alpha + \beta \qquad (42)$$

we obtain

$$\psi_{2+\alpha+\beta} \cdot z + \psi_{1+\alpha+\beta} \cdot (-z + \beta + 1) = 0 \ . \qquad (43)$$

from (43), applying (42).

Set

$$\psi_{1+\alpha+\beta} = \phi = \phi(z) \qquad (\psi = \phi_{-(1+\alpha+\beta)}) \, , \qquad (44)$$

we have then

$$\phi_1 + \phi \cdot \left(\frac{\beta + 1}{z} - 1 \right) = 0 \qquad (45)$$

from (43). A particular solution to this (variable separable form) equation is given by

$$\phi = e^z z^{-(\beta+1)} \ . \qquad (46)$$

Hence we obtain

$$\psi = (e^z \cdot z^{-(\beta+1)})_{-(1+\alpha+\beta)} \qquad (47)$$

from (44),applying (46).

Therefore, we obtain

$$\varphi = z^{-\alpha}(e^{z} \cdot z^{-(\beta+1)})_{-(1+\alpha+\beta)} \equiv \varphi_{[5](\alpha,\beta)} \tag{6}$$

from (35) and (47), having $\lambda = -\alpha$.

Inversely , (46) satisfies (equation (45), then (47) satisfies equation (43).
Therefore, (6) satisfies equation (1)

Next, changing the order

e^{z} and $z^{-(\beta+1)}$ in parenthesis $(\quad)_{-(1+\alpha+\beta)}$ in (6)

we obtain other solution

$$\varphi = z^{-\alpha}(z^{-(\beta+1)} \cdot e^{z})_{-(1+\alpha+\beta)} \equiv \varphi_{[6](\alpha,\beta)} \quad , \tag{7}$$

which is different from (6) for $-(1+\alpha+\beta) \notin Z_{0}^{+}$,
(Refer to Theorem D.)

Proof of Group IV.

First set

$$\varphi = z^{\lambda}\psi \qquad (\psi = \psi(z)) \ , \tag{35}$$

and substitute (35) into equation (1), we have then (38).

Hence we obtain

$$\psi_2 \cdot z + \psi_1 \cdot \{-z + 1 - \alpha\} + \psi \cdot (\alpha + \beta) = 0 \tag{40}$$

from (38), choosing

$$\lambda = -\alpha \ .$$

Next set

$$\psi = e^{\delta z}\phi \qquad (\phi = \phi(z)) \ , \tag{48}$$

We have then

$$\phi_2 \cdot z + \phi_1 \cdot \{z(2\delta - 1) + 1 - \alpha\}$$
$$+ \phi \cdot \{z(\delta^2 - \delta) + \delta(1 - \alpha) + \alpha + \beta\} = 0 \tag{49}$$

from (40), applying (48).

Choose δ such that

$$\delta^2 - \delta = 0 \ ,$$

that is,

$$\delta = 0, \ 1 \ . \tag{50}$$

When $\delta = 0$, we obtain (40) from (49). Then we have the same solutions as Group III .

When $\delta = 1$ we have

$$\phi_2 \cdot z + \phi_1 \cdot (z + 1 - \alpha) + \phi \cdot (1 + \beta) = 0 \tag{51}$$

from (49) .

Operate N^{ν} to the both sides of equation (51), we have then

$$\phi_{2+\nu}\cdot z+\phi_{1+\nu}\cdot(z+1-\alpha+\nu)+\phi_{\nu}\cdot(\nu+1+\beta)=0 \quad (\nu \notin Z^{-}) . \qquad (52)$$

Choosing ν such that

$$\nu=-1-\beta \qquad (53)$$

we obtain

$$\phi_{1-\beta}\cdot z+\phi_{-\beta}\cdot(z-\alpha-\beta)=0 \quad . \qquad (54)$$

from (52).

Therefore, setting

$$\phi_{-\beta}=u=u(z) \qquad (\phi=u_{\beta}) , \qquad (55)$$

we have

$$u_1+u\cdot\left(1-\frac{\alpha+\beta}{z}\right)=0 \qquad (56)$$

from (54). A particular solution to this equation is given by

$$u=e^{-z}z^{\alpha+\beta} \quad . \qquad (57)$$

Hence we obtain

$$\phi=(e^{-z}\cdot z^{\alpha+\beta})_{\beta} \qquad (58)$$

from (55) and (57).

Therefore, we have

$$\psi=e^{z}(e^{-z}\cdot z^{\alpha+\beta})_{\beta} \qquad (59)$$

from (58) and (48), having $\delta=1$.

We have then

$$\varphi=z^{-\alpha}e^{z}(e^{-z}\cdot z^{\alpha+\beta})_{\beta}\equiv\varphi_{[7](\alpha,\beta)} \qquad (8)$$

from (59) and (35), having $\lambda=-\alpha$.

Inversely , the function shown by (57) satisfies equation (56), then (55) satisfies equation (54), and hence (48)which have (55) satisfies (40).

Therefore, the function given by (8) satisfies equation (1), by (35) wnere $\lambda=-\alpha$.

Next, changing the order

e^{-z} and $z^{\alpha+\beta}$ in parenthesis $(\quad)_{\beta}$ in (8)

we obtain other solution

$$\varphi=z^{-\alpha}e^{z}(z^{\alpha+\beta}\cdot e^{-z})_{\beta}\equiv\varphi_{[8](\alpha,\beta)} \qquad (9)$$

which is different from $\varphi_{[7](\alpha,\beta)}$ for $\beta\notin Z_0^{+}$.

§3. Familiar Forms of The Solutions

In the below, the translated (more familiar) forms of the solutions obtained in §2. are presented.

Corollary 1. *We have*

Group I.

(i) $$\varphi_{[1](\alpha,\beta)} = e^{z} z^{-(\alpha+\beta+1)} {}_2F_0(\beta+1, \alpha+\beta+1; \tfrac{1}{z}) \qquad (1)$$

(ii) $$\varphi_{[2](\alpha,\beta)} = -e^{i\pi\beta} \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta+1)} e^{z} z^{-\alpha} {}_1F_1(\beta+1; 1-\alpha; -z) \qquad (2)$$

Group II.

(i) $$\varphi_{[3](\alpha,\beta)} = e^{-i\pi(\alpha+\beta)} z^{\beta} {}_2F_0(-\alpha-\beta, -\beta; -\tfrac{1}{z}) \qquad (3)$$

(ii) $$\varphi_{[4](\alpha,\beta)} = e^{-i\pi(\alpha+\beta)} \frac{\Gamma(\alpha)}{\Gamma(-\beta)} z^{-\alpha} {}_1F_1(-\alpha-\beta; 1-\alpha; z) \qquad (4)$$

Group III.

(i) $$\varphi_{[5](\alpha,\beta)} = e^{z} z^{-(\alpha+\beta+1)} {}_2F_0(\beta+1, \alpha+\beta+1: \tfrac{1}{z}) \qquad (5)$$

(ii) $$\varphi_{[6](\alpha,\beta)} = -e^{i\pi(\alpha+\beta)} \frac{\Gamma(-\alpha)}{\Gamma(\beta+1)} e^{z} {}_1F_1(\alpha+\beta+1; 1+\alpha; -z) \qquad (6)$$

Group IV.

(i) $$\varphi_{[7](\alpha,\beta)} = e^{-i\pi\beta} z^{\beta} {}_2F_0(-\beta, -\alpha-\beta; -\tfrac{1}{z}) \qquad (7)$$

(ii) $$\varphi_{[8](\alpha,\beta)} = e^{-i\pi\beta} \frac{\Gamma(-\alpha)}{\Gamma(-\alpha-\beta)} {}_1F_1(-\beta; 1+\alpha; z) \qquad (8)$$

where ${}_pF_q(\cdots\cdots)$ *is the generalized Gausss hypergeometric function,* (See §5.)

Proof of Group I.

(i) $$\varphi_{[1](\alpha,\beta)} = (e^{z} \cdot z^{-(\alpha+\beta+1)})_{-(1+\beta)} \qquad (9)$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(-\beta)}{k!\Gamma(-\beta-k)} (e^{z})_{-(1+\beta)-k} (z^{-(\alpha+\beta+1)})_{k} \qquad (10)$$

$$= e^{z} z^{-(\alpha+\beta+1)} \sum_{k=0}^{\infty} \frac{[\beta+1]_k [\alpha+\beta+1]_k}{k!} z^{-k} \tag{11}$$

$$= e^{z} z^{-(\alpha+\beta+1)} {}_2F_0(\beta+1, \alpha+\beta+1; \tfrac{1}{z}) \tag{1}$$

by Lemma (i v), since

$$\Gamma(\lambda-k) = (-1)^{-k} \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{\Gamma(k+1-\lambda)} \qquad (k \in Z_0^+) \ , \tag{12}$$

$$(e^{z})_{\gamma} = e^{z} \quad , \tag{13}$$

$$(z^{\lambda})_k = e^{-i\pi k} \frac{\Gamma(k-\lambda)}{\Gamma(-\lambda)} z^{\lambda-k} \quad , \tag{14}$$

and

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} \quad \text{with} \quad [\lambda]_0 = 1 \ .$$

(Notation of Pochhammer).

(i i) $$\varphi_{[2](\alpha,\beta)} = (z^{-(\alpha+\beta+1)} \cdot e^{z})_{-(1+\beta)} \tag{15}$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(-\beta)}{k!\Gamma(-\beta-k)} (z^{-(\alpha+\beta+1)})_{-(1+\beta)-k} (e^{z})_k \tag{16}$$

$$= e^{i\pi(1+\beta)} z^{-\alpha} e^{z} \sum_{k=0}^{\infty} \frac{[\beta+1]_k \Gamma(\alpha-k)}{k!\Gamma(\alpha+\beta+1)} z^{k} \tag{17}$$

$$= -e^{i\pi\beta} z^{-\alpha} e^{z} \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta+1)} \sum_{k=0}^{\infty} \frac{[\beta+1]_k}{k![1-\alpha]_k} (-z)^{k} \tag{18}$$

$$= -e^{i\pi\beta} \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta+1)} z^{-\alpha} e^{z} {}_1F_1(\beta+1; 1-\alpha; -z) \tag{2}$$

since

$$(z^{-(\alpha+\beta+1)})_{-(1+\beta)-k} = e^{i\pi(1+\beta+k)} \frac{\Gamma(\alpha-k)}{\Gamma(\alpha+\beta+1)} z^{k-\alpha} \quad . \tag{19}$$

Proof of Group II.

(i) $$\varphi_{[3](\alpha,\beta)} = e^{z}(e^{-z}\cdot z^{\beta})_{\alpha+\beta} \tag{20}$$

$$= e^{z}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+\beta+1)}{k!\Gamma(\alpha+\beta+1-k)}(e^{-z})_{\alpha+\beta-k}(z^{\beta})_{k} \tag{21}$$

$$= e^{-i\pi(\alpha+\beta)}z^{\beta}\sum_{k=0}^{\infty}\frac{[-\alpha-\beta]_{k}[-\beta]_{k}}{k!}\left(-\frac{1}{z}\right)^{k} \tag{22}$$

$$= e^{-i\pi(\alpha+\beta)}z^{\beta}{}_{2}F_{0}(-\alpha-\beta\,,-\beta;-\tfrac{1}{z}) \tag{3}$$

since

$$(e^{-z})_{\gamma} = e^{-i\pi\gamma}e^{z} \quad . \tag{23}$$

(ii) $$\varphi_{[4](\alpha,\beta)} = e^{z}(z^{\beta}\cdot e^{-z})_{\alpha+\beta} \tag{24}$$

$$= e^{z}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+\beta+1)}{k!\Gamma(\alpha+\beta+1-k)}(z^{\beta})_{\alpha+\beta-k}(e^{-z})_{k} \tag{25}$$

$$= e^{-i\pi(\alpha+\beta)}z^{-\alpha}\sum_{k=0}^{\infty}\frac{(-1)^{k}[-\alpha-\beta]_{k}\Gamma(\alpha-k)}{k!\,\Gamma(-\beta)}z^{k} \tag{26}$$

$$= e^{-i\pi(\alpha+\beta)}\frac{\Gamma(\alpha)}{\Gamma(-\beta)}z^{-\alpha}\sum_{k=0}^{\infty}\frac{[-\alpha-\beta]_{k}}{k![1-\alpha]_{k}}z^{k} \tag{27}$$

$$= e^{-i\pi\beta}\frac{\Gamma(\alpha)}{\Gamma(-\beta)}z^{-\alpha}\;{}_{1}F_{1}(-\alpha-\beta\,;1-\alpha\,;\,z) \ . \tag{4}$$

Proof of Group III.

(i) $$\varphi_{[5](\alpha,\beta)} = z^{-\alpha}(e^{z}\cdot z^{-(\beta+1)})_{-(1+\alpha+\beta)} \tag{28}$$

$$= z^{-\alpha}\sum_{k=0}^{\infty}\frac{\Gamma(-\alpha-\beta)}{k!\Gamma(-\alpha-\beta-k)}(e^{z})_{-(1+\alpha+\beta)-k}(z^{-(\beta+1)})_{k} \tag{29}$$

$$= z^{-(\alpha+\beta+1)}e^{z}\sum_{k=0}^{\infty}\frac{[1+\alpha+\beta]_{k}[!+\beta]_{k}}{k!}z^{-k} \tag{30}$$

$$= z^{-(\alpha+\beta+1)}e^{z}{}_{2}F_{0}(1+\alpha+\beta,1+\beta\,;\tfrac{1}{z}) \ . \tag{5}$$

(ii) $$\varphi_{[6](\alpha,\beta)}=z^{-\alpha}(z^{-(\beta+1)}\cdot e^{z})_{-(1+\alpha+\beta)} \tag{31}$$

$$=z^{-\alpha}\sum_{k=0}^{\infty}\frac{\Gamma(-\alpha-\beta)}{k!\Gamma(-\alpha-\beta-k)}(z^{-(\beta+1)})_{-(1+\alpha+\beta)-k}(e^{z})_{k} \tag{32}$$

$$=e^{i\pi(1+\alpha+\beta)}\frac{\Gamma(-\alpha)}{\Gamma(\beta+1)}e^{z}\sum_{k=0}^{\infty}\frac{[1+\alpha+\beta]_{k}}{k![1+\alpha]_{k}}(-z)^{k} \tag{33}$$

$$=-e^{i\pi(\alpha+\beta)}\frac{\Gamma(-\alpha)}{\Gamma(\beta+1)}e^{z}\,{}_1F_1(1+\alpha+\beta\,;1+\alpha\,;-z)\ . \tag{6}$$

Proof of Group IV.

(i) $$\varphi_{[7](\alpha,\beta)}=z^{-\alpha}e^{z}(e^{-z}\cdot z^{\alpha+\beta})_{\beta} \tag{34}$$

$$=z^{-\alpha}e^{z}\sum_{k=0}^{\infty}\frac{\Gamma(\beta+1)}{k!\Gamma(\beta+1-k)}(e^{-z})_{\beta-k}(z^{\alpha+\beta})_{k} \tag{35}$$

$$=e^{-i\pi\beta}z^{\beta}\sum_{k=0}^{\infty}\frac{[-\beta]_{k}[-\alpha-\beta]_{k}}{k!}\left(-\frac{1}{z}\right)^{k} \tag{36}$$

$$=e^{-i\pi\beta}z^{\beta}\,{}_2F_0(-\beta\,,-\alpha-\beta;-\tfrac{1}{z})\ . \tag{7}$$

(ii) $$\varphi_{[8](\alpha,\beta)}=z^{-\alpha}e^{z}(z^{\alpha+\beta}\cdot e^{-z})_{\beta} \tag{37}$$

$$=z^{-\alpha}e^{z}\sum_{k=0}^{\infty}\frac{\Gamma(\beta+1)}{k!\Gamma(\beta+1-k)}(z^{\alpha+\beta})_{\beta-k}(e^{-z})_{k} \tag{38}$$

$$=e^{-i\pi\beta}\frac{\Gamma(-\alpha)}{\Gamma(-\alpha-\beta)}\sum_{k=0}^{\infty}\frac{[-\beta]_{k}}{k!\,[1+\alpha]_{k}}z^{k} \tag{39}$$

$$=e^{-i\pi\beta}\frac{\Gamma(\alpha)}{\Gamma(-\alpha-\beta)}\,{}_1F_1(-\beta\,;1+\alpha\,;z)\ . \tag{8}$$

§ 4. Commentary

(I) All solutions shown by (2) ~ (9) in § 2 have a fractional differintegrated form $(\cdots\cdots)_{g(\alpha,\beta)}$, where the index $g(\alpha,\beta)$ is the order of differintegration.

Then notice that only the constants α and β in the equation (1) in § 2 contribute to the order $g(\alpha,\beta)$.

And notice that we have the identities below.

$$(e^{z}\cdot z^{-(\alpha+\beta+1)})_{-(1+\beta)} = z^{-\alpha}(e^{z}\cdot z^{-(\beta+1)})_{-(1+\alpha+\beta)} \qquad (1)$$

from § 3. (1) and § 3. (5), and

$$(e^{-z}\cdot z^{\beta})_{\alpha+\beta} = (-z)^{-\alpha}(e^{-z}\cdot z^{\alpha+\beta})_{\beta} \qquad (2)$$

from § 3. (3) and § 3. (7).

And we have

(i) $\varphi_{[1](\alpha,\beta)} = \varphi_{[2](\alpha,\beta)}$ for $-(1+\beta)\in Z_0^+$.

(i i) $\varphi_{[3](\alpha,\beta)} = \varphi_{[4](\alpha,\beta)}$ for $(\alpha+\beta)\in Z_0^+$.

(i i i) $\varphi_{[5](\alpha,\beta)} = \varphi_{[6](\alpha,\beta)}$ for $-(1+\alpha+\beta)\in Z_0^+$.

and

(iv) $\varphi_{[7](\alpha,\beta)} = \varphi_{[8](\alpha,\beta)}$ for $\beta\in Z_0^+$.

(III) Generalized Associated Laguerre's function of order β and degree α is denoted by $L_{\beta}^{(\alpha)}(z)$ and is defined as

$$L_{\beta}^{(\alpha)}(z) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\,{}_1F_1(-\beta\,;\alpha+1\,;z)\,, \qquad (3)$$

where

$${}_1F_1(-\beta\,;\alpha+1\,;z)$$

is the Kummer's confluent hypergeometric function.

Now we have

$$\varphi_{[8](\alpha,\beta)} = z^{-\alpha}e^{z}(z^{\alpha+\beta}\cdot e^{-z})_{\beta} \qquad (4)$$

$$= e^{-i\pi\beta}\frac{\Gamma(-\alpha)}{\Gamma(-\alpha-\beta)}\,{}_1F_1(-\beta\,;\alpha+1\,;z)\,. \qquad (5)$$

Therefore, we have the presentation below.

$$\varphi_{[8](\alpha,\beta)} = e^{-i\pi\beta}\frac{\Gamma(-\alpha)}{\Gamma(-\alpha-\beta)}\cdot\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}L_{\beta}^{(\alpha)}(z) \qquad (6)$$

and

$$\varphi_{[8](\alpha,n)} = (-1)^{n}\frac{\Gamma(-\alpha)}{\Gamma(-\alpha-n)}\cdot\frac{n!\,\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)}L_{n}^{(\alpha)}(z) \qquad (7)$$

for $\beta = n \in Z_0^+$, using the Laguerre's function.

Where

$$L_n^{(\alpha)}(z) = \frac{e^z z^{-\alpha}}{n!} \cdot \frac{d^n}{dz^n}(z^{\alpha+\beta} e^z) \tag{8}$$

$$= \frac{\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+1)}\, {}_1F_1(-n\,;\alpha+1\,;z)\,. \tag{9}$$

is the polynomial of Laguerre.

(**IV**) Hitherto, to the homogeneous associated Laguerre's equation, mainly the function $L_\beta^{(\alpha)}(z)$ (which is can be derived from our solution $\varphi_{[8](\alpha,\beta)}$) is discussed as its solution.

However, we must notice that there exists many other particular solutions such as

$$\varphi_{[1](\alpha,\beta)},\ \varphi_{[2](\alpha,\beta)},\ \varphi_{[3](\alpha,\beta)},\ \varphi_{[4](\alpha,\beta)},\ \varphi_{[6](\alpha,\beta)}\,,$$

which are different from $L_\beta^{(\alpha)}(z)$, and they are obtained by our NFCO-Method.

(**V**) The solutions obtained by means of NFCO to the nonhomogeneous associated Laguerre's equation shall be reported in a next paper of the author, in a near future.

References

[1] K. Nishimoto ; Fractional Calculus, Vol. 1 (1984), Vol. 2 (1987), Vol. 3 (1989), Vol. 4 (1991), Vol. 5 (1996), Descartes Press, Koriyama, Japan.

[2] K. Nishimoto ; An Essence of Nishimoto's Fractional Calculus (Calculus of the 21st Century) ; Integrals and Differentiations of Arbitrary Order (1991), Descartes Press , Koriyama, Japan.

[3] K. Nishimoto ; On Nishimoto's fractional calculus operator N^ν (On an action group), J. Frac. Calc. Vol. 4, Nov. (1993), 1 - 11.

[4] K. Nishimoto ; Unification of the integrals and derivatives (A serendipity in fractional calculus), J. Frac. Calc. Vol.6, Nov. (1994), 1 - 14.

[5] K. Nishimoto ; Ring and Field produced from The Set of N-Fractional Calculus Operator, J. Frac. Calc. Vol. 24, Nov. (2003), 29 - 36.

[6] K. Nishimoto ; An application of fractional calculus to the nonhomogeneous Gauss equations, J. Coll. Engng. Nihon Univ., B - 28 (1987), 1 - 8.

[7] K. Nishimoto and S. L. Kalla ; Application of Fractional Calculus to Ordinary Differential Equation of Fuchs Type, Rev. Tec. Ing. Univ. Zulia, Vol. 12, No. 1, (1989).

[8] K. Nishimoto ; Application of Fractional Calculus to Gauss Type Partial Differential Equations, J. Coll. Engng. Nihon UNiv., B - 30 (1989), 81 - 87.

[9] Shih - Tong Tu, S. - J. Jaw and Shy - Der Lin ; An application of fractional calculus to Chebychev's equation, Chung Yuan J. Vol. XIX (1990), 1 - 4.

[10] K. Nishimoto, H. M. Srivastava and Shih - Tong Tu ; Application of fractional Calculus in Solving Certain Classes of Fuchsian Differential Equations, J. Coll. Engng. Nihon Univ., B - 32 (1991), 119 - 126.

[11] K. Nishimoto ; A Generalization of Gauss Equation by Fractional Calculus Method, J. Coll. Engng. Nihon Univ., B - 32 (1991), 79 - 87.

[12] Shy - Der Lin, Shih - Tong Tu and K. Nishimoto ; A generalization of Legendre's equation by fractional calculus method, J. Frac. Calc. Vol.1, May (1992),35 - 43.

[13] N. S> Sohi, L. P. Singh and K. Nishimoto ; A generalization of Jacobi's equation by fractional calculus method, J. Frac. Calc. Vol. 1, May (1992), 45 - 51.

[14] K. Nishimoto ; Solutions of Gauss equation in fractional calculus, J. Frac. Calc. Vol. 3, May (1993), 29 - 37.

[15] K. Nishimoto ; Solutions of homogeneous Gauss equations, which have a logarithmic function, in fractional calculus, J. Frac. Calc. Vol. 5, May (1994), 11 - 25.

[16] K. Nishimoto ; Application of N- transformation and N- ftactional calculus method to nonhomogeneous Bessel equations (I), J. Frac. Calc. Vol. 8, Nov. (1995),25 - 30.

[17] K. Nishimoto ; Operator N^{ν} method to nonhomogeneous Gauss and Bessel equations, J. Frac. Calc. Vol. 9, May. (1996), 1 - 15.

[18] K. Nishimoto and Susana S. de Romero ; N- fractional calculus operator N^{ν} method to nonhomogeneous and homogeneous Whitttaker equations (I), J. Frac. Calc. Vol. 9, May. (1996) 17 - 22.

[19] K. Nishimoto and Judith A. de Duran ; N- fractional calculus operator N^{ν} method to nonhomogeneous Fukuhara equations (I), J. Frac. Calc. Vol. 9, May. (1996) ,23 - 31.

[20] K. Nishimoto ; N- fractional calculus operator N^{ν} method to nonhomogeneous Gauss equations , J. Frac. Calc. Vol. 10, Nov. (1996), 33 - 39.

[21] K. Nishimoto ; Kummer's twenty - four functions and N- fractional calculus, Nonlinear Analysis, Theory, Method & Applications, Vol.30, No.2, (1997),1271 - 1282.

[22] Shih-Tong Tu, Ding-Kuo Chyan and Wen-Chieh Luo ; Some solutions to the nonhomogeneous Jacobi equations via fractional calculus operator N^{ν} method, , J. Frac. Calc. Vol. 12, Nov. (1997), 51 - 60.

[23] Shih-Tong Tu, Ding-Kuo Chyan and Erh-Tsung Chin ; Solutions of Gegenbauer and Chebysheff equations via operator N^{ν} method, , J. Frac. Calc. Vol. 12, Nov. (1997), 61 - 69

[24] K. Nishimoto ; N- method to Hermite equations, J. Frac. Calc. Vol. 13, May (1998),21 - 27.

[25] K. Nishimoto ; N- method to Weber equations, J. Frac. Calc. Vol. 14, Nov. (1998), 1 - 8.

[26] K. Nishimoto ; N- method to generalized Laguerre equations, J. Frac. Calc. Vol. 14, Nov. (1998), 9 - 21

[27] Shy-Der Lin, Jaw-Chian Shyu, Katsuyuki Nishimoto and H.M. Srivastava ; Explicit Solutions of Some General Families of Ordinary and Partial Differential Equations Associated with the Bessel Equation by Means of Fractional Calculus, J. Frac. Calc. Vol. 25, May (2004), 33 - 45..

[28] K. Nishimoto ; Solutions to Some Extended Hermite's equations by Means of N-Fractional Calculus, J. Frac. Calc. Vol. 29, May (2006), 45 - 56.

[29] Tsuyako Miyakoda ; Solutions to An Extended Hermite's Equation by Means of N-Fractional Calculus, J. Frac. Calc. Vol. 30, Nov. (2006), 23 - 32.

[30] K. Nishimoto ; Solutions to Some Extended Weber's Equations by Means of N-Fractional Calculus, J. Frac. Calc. Vol. 30, Nov. (2006), 1 - 11.

[31] K. Nishimoto ; N- fractional Calculus of Products of Some Power Functions, J. Frac. Calc. Vol. 27, May (2005), 83 - 88.

[32] K. Nishimoto ; N-Fractional Calculus of Some Composite Functions, J. Frac. Calc. Vol. 29, May (2006),35 -44.

[33] Shy-Der Lin, Yen-Shung Tsai, H.M. Srivastava and Pin-Yu Wang ; Fractional Calculus Derivations of the Explicit Solutions of Confluent Hypergeometric and Associated Laguerre Equations, J. Frac. Calc. , May (2006), 75 - 86.

[34] K. Nishimoto ; N-Fractional Calculus Operator Method to Chebyshev's Equations, J. Frac. Calc. Vol. 33, May (2008), 71 - 90.