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On the solutions of an extended Chebyshev's Equations

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Abstract

In this article, we treat the extended homogeneous Chebyshev's equations

$$\varphi_2 \cdot (z^2 - 2bz - 1) + \varphi_1 \cdot (z - b) - \varphi \cdot \nu^2 = 0,$$

where

$$\varphi_0 = \varphi = \varphi(z), \quad f = f(z), \quad \varphi_\alpha = \frac{d^\alpha \varphi}{dz^\alpha} (for\ \alpha > 0)$$

and $z^2 - 2bz - 1 \neq 0$ in the view of N-fractional calculus and discuss the solutions by means of N- fractional calculus operator. We present the familiar form of the solution like as

$$\varphi(z) = -e^{i\pi\nu} \frac{\sqrt{\pi}}{2^\nu \nu \Gamma(\nu + \frac{1}{2})} (z - b)^{-\nu} {}_2F_1(\frac{\nu}{2}, \frac{\nu}{2} + \frac{1}{2}; \nu + 1; \frac{b^2 + 1}{(z - b)^2})$$

where $|(b^2 + 1)/(z - b)^2| < 1$ and ${}_2F_1(\cdots)$is the Gauss hypergeometric function.

1 Definition of fractional calculus and some properties

We define the N-fractional calculus and N-fractional operator N^α as follows.

For a regular function $f = f(z)$ and a arbitrary number α , N-fractional differintegration of order α is difined as follows,

$$\begin{aligned} N^\alpha f &= f_\alpha = (f)_\alpha = {}_C(f)_\alpha \\ &= \frac{\Gamma(\alpha + 1)}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta - z)^{\alpha+1}} \quad (\alpha \notin Z^-), \end{aligned}$$

$$(f)_{-m} = \lim_{\alpha \to -m} (f)_\alpha \quad (m \in Z^+),$$

where $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

$$-\pi \le arg(\zeta - z) \le \pi \ for\ C_-, \quad 0 \le arg(\zeta - z) \le 2\pi \quad for\ C_+,$$

$$\zeta \ne z, \quad z \in C, \quad \nu \in R, \quad \Gamma;\ Gamma\ function,$$

and C_- is a curve along the cut joining two points z and $-\infty + iIm(z)$, D_- is a domain surrounded by C_- , C_+ is a curve along the cut joining two points z and $\infty + iIm(z)$, D_+ is a domain surrounded by C_+.

When $\alpha > 0$, $(f)_\alpha$ is the fractional derivative of arbitrary order α , and when $\alpha < 0$, it is the integral of order $-\alpha$, if $|(f)_\alpha| < \infty$.

We denote

$$N^\alpha \varphi = \frac{d^\alpha \varphi}{dz^\alpha} = (\varphi)_\alpha.$$

and the binary operation $\circ$ is

$$(N^\beta \circ N^\alpha) f = (N^\beta N^\alpha) f = N^\beta (N^\alpha f) = N^\alpha (N^\beta f) \quad (\alpha, \beta \in R), \qquad (1)$$

then the set

$$\{N^\nu\} = \{N^\nu | \nu \in R\} \qquad (2)$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \ne |f_\nu| \le \infty, \nu \in R\}$, where $f = f(z)$ and $z \in C$. (vis. $-\infty < \nu < \infty$).

As for the properties of the operator, see [2], [3], [5]. We introduce here two necessary lemmas.

Lemma I. We have ([1])

(i)

$$((z-c)^\beta)_\alpha = e^{-i\pi\alpha} \frac{\Gamma(\alpha - \beta)}{\Gamma(-\beta)} (z-c)^{\beta - \alpha} \quad \left(\left|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}\right| < \infty\right)$$

(ii)

$$(log(z-c))_\alpha = -e^{i\pi\alpha} \Gamma(\alpha)(z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty)$$

(iii)

$$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c), \quad (|\Gamma(\alpha)| < \infty)$$

where $z - c \ne 0$ in (i), and $z - c \ne 0, 1$ in (ii) and (iii) ,

(iv)

$$(u \cdot v)_\alpha := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha+1-k)} u_{\alpha-k} v_k \quad (u = u(z), v = v(z))$$

Lemma II We have ([4])

(i)

$$(((z-b)^{\beta}-c)^{\alpha})_{\gamma}=e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma}$$
$$\times\sum_{k=0}^{\infty}\frac{[-\alpha]_k\Gamma(\beta k-\alpha\beta+\gamma)}{k!\Gamma(\beta k-\alpha\beta)}\left(\frac{c}{(z-b)^{\beta}}\right)^k,\quad \left(\left|\frac{\Gamma(\beta k-\alpha\beta+\gamma)}{\Gamma(\beta k-\alpha\beta)}\right|<\infty\right) \quad (3)$$

and

(ii) for $n\in Z_0^+$

$$(((z-b)^{\beta}-c)^{\alpha})_{n}=(-1)^n(z-b)^{\alpha\beta-n}$$
$$\times\sum_{k=0}^{\infty}\frac{[-\alpha]_k[\beta k-\alpha\beta]_n}{k!}\left(\frac{c}{(z-b)^{\beta}}\right)^k,\quad \left|\frac{c}{(z-b)^{\beta}}\right|<1, \quad (4)$$

where

$$[\lambda]_k=\lambda(\lambda+1)\cdots(\lambda+k-1)=\Gamma(\lambda+k)/\Gamma(\lambda),\quad [\lambda]_0=1.$$

2 Solutions to an extended homogeneous Chebyshev's equations by means of N-fractional calculus

We discuss the following type of an extended Chebyshev's equation

$$\varphi_2\cdot(z^2-2bz-1)+\varphi_1\cdot(z-b)-\varphi\cdot\nu^2=0$$

by means of N-fractional calculus.

The above equation is solved by means of N-fractional calculus as follows.

Theorem 1 *Let $\varphi\in F=\{\varphi:0\neq|\varphi_{\nu}|<\infty,\nu\in R\}$,then the homogeneous extended Chebyshev's equation*

$$\varphi_2\cdot(z^2-2bz-1)+\varphi_1\cdot(z-b)-\varphi\cdot\nu^2=0 \quad (1)$$

has particular solutions of the forms,

(i)

$$\varphi=(((z-b)^2-(b^2+1))^{-(\frac{1}{2}+\nu)})_{-(1+\nu)}\equiv\varphi_{[1]}^{*} \quad (2)$$

and

(ii)

$$\varphi=(((z-b)^2-(b^2+1))^{-(\frac{1}{2}-\nu)})_{-(1-\nu)}\equiv\varphi_{[2]}^{*}. \quad (3)$$

Proof.

We set $g=z-b$, $h=z^2-2bz-1$ and operating N^{α} to the both sides of equation (1), we have then

$$(\varphi_2\cdot h)_{\alpha}+(\varphi_1\cdot g)_{\alpha}-(\varphi\cdot\nu^2)_{\alpha}=0, \quad (4)$$

hence

$$\varphi_{2+\alpha} \cdot h + \varphi_{1+\alpha} \cdot (2\alpha + 1) \cdot g + \varphi_\alpha \cdot (\alpha^2 - \nu^2) = 0 \tag{5}$$

since

$$N^\alpha \varphi_m = (\varphi_m)_\alpha = \varphi_{m+\alpha} \quad (m = 2, 1) \tag{6}$$

by our index law, and from Lemma (iv) we have

$$\begin{aligned} N^\alpha(\varphi_1 \cdot g) &= (\varphi_1 \cdot g)_\alpha = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha+1-k)} (\varphi_1)_{\alpha-k} \cdot g_k & (7) \\ &= \varphi_{1+\alpha} \cdot g + \varphi_\alpha \cdot \alpha & (8) \end{aligned}$$

and

$$\begin{aligned} N^\alpha(\varphi_2 \cdot h) &= (\varphi_2 \cdot h)_\alpha & (9) \\ &= \varphi_{2+\alpha} \cdot h + \varphi_{1+\alpha} \cdot 2\alpha g + \varphi_\alpha \cdot \alpha(\alpha - 1). & (10) \end{aligned}$$

Choose α such that $\alpha^2 - \nu^2 = 0$, that is $\alpha = \nu$ *or* $-\nu$. We have then

$$\varphi_{2+\alpha} \cdot h + \varphi_{1+\alpha} \cdot (2\alpha + 1) g = 0. \tag{11}$$

When $\alpha = \nu$, we set

$$\psi = \psi(z) = \varphi_{1+\nu}, \qquad (\varphi = \psi_{-(1+\nu)}) \tag{12}$$

and we obtain

$$\psi_1 \cdot h + \psi \cdot (2\nu + 1) g = 0. \tag{13}$$

Then a particular solution to this linear first order equation is given by

$$\psi = h^{-(\frac{1}{2}+\nu)} \tag{14}$$

Therefore we have

$$\begin{aligned} \varphi &= \psi_{-(1+\nu)} = ((h^{-(\frac{1}{2}+\nu)})_{-(1+\nu)} \\ &= (((z-b)^2 - (b^2+1))^{-(\frac{1}{2}+\nu)})_{-(1+\nu)} = \varphi^*_{[1]}. & (15) \end{aligned}$$

When $\alpha = -\nu$, in the similar way we obtain the second solution

$$\begin{aligned} \varphi = \psi_{-(1-\nu)} &= ((h^{-(\frac{1}{2}-\nu)})_{-(1-\nu)} \\ &= (((z-b)^2 - (b^2+1))^{-(\frac{1}{2}-\nu)})_{-(1-\nu)} = \varphi^*_{[2]}. & (16) \end{aligned}$$

Inversely these functions shown by (15) and (16) satisfy the equation (1) cleary.

Indeed we have

$$\begin{aligned} LHS\ of\ (1) &= \varphi_2 \cdot h + \varphi_1 \cdot g - \varphi \cdot \nu^2 \\ &= \left((\varphi_2 \cdot h)_\alpha + (\varphi_1 \cdot g)_\alpha - \varphi_\alpha \cdot \nu^2\right)_{-\alpha} \\ &= \left((\varphi_{2+\alpha} \cdot h + \varphi_{1+\alpha} \cdot (2\alpha+1)g + \varphi_\alpha \cdot (\alpha^2 - \nu^2)\right)_{-\alpha} \\ &= (\psi_1 \cdot h + \psi \cdot (2\alpha+1)g)_{-\alpha} \\ &= 0 \end{aligned} \tag{17}$$

with applying (12) and (15) or (16).

3 Familiar forms of Solutions of the extended homogeneous Chebyshev's equation

Applying Lemma II, in case of $\alpha = \nu$ we have

$$\varphi = \left(((z-b)^2 - (b^2+1))^{-\nu+\frac{1}{2}}\right)_{-(1+\nu)} = e^{-i\pi(-1-\nu)}(z-b)^{-(\nu+\frac{1}{2})\cdot 2+(1+\nu)}$$

$$\times \sum_{k=0}^{\infty} \frac{[\nu+\frac{1}{2}]_k \Gamma(2k+(\nu+\frac{1}{2})\cdot 2-(1+\nu))}{k!\Gamma(2k+(\nu+\frac{1}{2})\cdot 2)} \left(\frac{(b^2+1)}{(z-b)^2}\right)^k$$

$$= e^{i\pi(1+\nu)}(z-b)^{-\nu} \sum_{k=0}^{\infty} \frac{[\nu+\frac{1}{2}]_k \Gamma(2k+\nu)}{k!\Gamma(2k+2\nu+1)} \left(\frac{(b^2+1)}{(z-b)^2}\right)^k$$

$$= -e^{i\pi\nu}(z-b)^{-\nu} \sum_{k=0}^{\infty} \frac{[\frac{\nu}{2}]_k[\frac{\nu}{2}+\frac{1}{2}]_k \Gamma(\nu)}{k![\nu+1]_k\Gamma(2\nu+1)} \left(\frac{(b^2+1)}{(z-b)^2}\right)^k$$

Now we notice following relations.

$$\Gamma(2k+\nu) = [\nu]_{2k}\Gamma(\nu)$$

$$\Gamma(2k+2\nu+1) = [2\nu+1]_{2k}\Gamma(2\nu+1)$$

$$[\nu]_{2k} = 2^{2k}[\frac{\nu}{2}]_k[\frac{\nu}{2}+\frac{1}{2}]_k$$

$$[2\nu+1]_{2k} = 2^{2k}[\nu+\frac{1}{2}]_k[\nu+1]_k$$

So we can get

$$\varphi = -e^{i\pi\nu}\frac{\Gamma(\nu)}{\Gamma(2\nu+1)}(z-b)^{-\nu} \sum_{k=0}^{\infty} \frac{[\frac{\nu}{2}]_k[\frac{\nu}{2}+\frac{1}{2}]_k}{k![\nu+1]_k} \left(\frac{(b^2+1)}{(z-b)^2}\right)^k$$

$$= -e^{i\pi\nu}\frac{\Gamma(\nu)}{\Gamma(2\nu+1)}(z-b)^{-\nu}{}_2F_1(\frac{\nu}{2}, \frac{\nu}{2}+\frac{1}{2}; \nu+1; \frac{b^2+1}{(z-b)^2}). \tag{1}$$

Here ${}_2F_1(\cdots)$ denote the Gauss's hypergeometric function.

In the case of $\alpha = -\nu$, we have the following form in according to the same way of the case $\alpha = \nu$,

$$\varphi = \left(((z-b)^2 - (b^2+1))^{\nu-\frac{1}{2}}\right)_{\nu-1}$$
$$= -e^{-i\pi\nu}\frac{\Gamma(-\nu)}{\Gamma(1-2\nu)}(z-b)^{\nu}\sum_{k=0}^{\infty}\frac{[-\frac{\nu}{2}]_k[-\frac{\nu}{2}+\frac{1}{2}]_k}{k![-\nu+1]_k}\left(\frac{(b^2+1)}{(z-b)^2}\right)^k$$
$$= -e^{-i\pi\nu}\frac{\Gamma(-\nu)}{\Gamma(1-2\nu)}(z-b)^{\nu}{}_2F_1(-\frac{\nu}{2},-\frac{\nu}{2}+\frac{1}{2};-\nu+1;\frac{b^2+1}{(z-b)^2}). \quad (2)$$

4 Illustrative Example

We show some examples.

(i)We consider the case of $\nu = \frac{1}{2}$ and $b = \frac{1}{2}$. The equation is

$$\varphi_2 \cdot (z^2 - z - 1) + \varphi_1 \cdot (z - \frac{1}{2}) - \varphi \cdot (\frac{1}{2})^2 = 0.$$

Operating $N^{\alpha}$to the equation, we have

$$\varphi_{2+\alpha} \cdot (z^2 - z - 1) + \varphi_{1+\alpha} \cdot (2\alpha+1) \cdot (z - \frac{1}{2}) + \varphi_{\alpha} \cdot (\alpha^2 - (\frac{1}{2})^2) = 0$$

We adopt $\alpha = 1/2$, then

$$\varphi_{2+\frac{1}{2}} \cdot (z^2 - z - 1) + \varphi_{1+\frac{1}{2}} \cdot 2 \cdot (z - \frac{1}{2}) = 0.$$

Setting $\psi = \psi(z) = \varphi_{1+\frac{1}{2}}$, we have the following equation

$$\psi_1 \cdot (z^2 - z - 1) + \psi \cdot (2)(z - \frac{1}{2}) = 0.$$

and the solution

$$\psi = (z^2 - z - 1)^{-1}.$$

Therefore we have the solution as follows,

$$\varphi = (((z^2 - z - 1)^{-1})_{-(1+\frac{1}{2})}$$
$$= (((z - \frac{1}{2})^2 - ((\frac{1}{2})^2 + 1))^{-1})_{-(1+\frac{1}{2})}$$

With applying Lemma II, we have

$$\varphi = \left(((z - \frac{1}{2})^2 - (\frac{5}{4}))^{-1}\right)_{-(\frac{3}{2})}$$
$$= e^{i\pi(\frac{3}{2})}(z - \frac{1}{2})^{-\frac{1}{2}} \times \sum_{k=0}^{\infty}\frac{[1]_k\Gamma(2k+\frac{1}{2})}{k!\Gamma(2k+2)}\left(\frac{\frac{5}{4}}{(z-\frac{1}{2})^2}\right)^k$$

Here we notice the following relations.

$$\Gamma(2k+\frac{1}{2}) = [\frac{1}{2}]_{2k}\Gamma(\frac{1}{2}) = [\frac{1}{2}]_{2k}\sqrt{\pi}$$

$$\Gamma(2k+2) = [2]_{2k}\Gamma(2) = [2]_{2k}$$

$$[1]_k = 1\cdot 2\cdots(1+k-1) = k!$$

Furthermore,

$$[\frac{1}{2}]_{2k} = 2^{2k}[\frac{1}{4}]_k[\frac{1}{4}+\frac{1}{2}]_k$$

$$[2]_{2k} = 2^{2k}[1]_k[\frac{3}{2}]_k = 2^{2k}k![\frac{3}{2}]_k$$

At last we obtain

$$\begin{aligned}\varphi &= e^{i\pi(\frac{3}{2})}\sqrt{\pi}(z-\frac{1}{2})^{-\frac{1}{2}} \times \sum_{k=0}^{\infty}\frac{[\frac{1}{4}]_k[\frac{3}{4}]_k}{k![\frac{3}{2}]_k}\left(\frac{\frac{5}{4}}{(z-\frac{1}{2})^2}\right)^k \\ &= e^{i\pi(\frac{3}{2})}\sqrt{\pi}(z-\frac{1}{2})^{-\frac{1}{2}}{}_2F_1(\frac{1}{4},\frac{3}{4};\frac{3}{2};\frac{\frac{5}{4}}{(z-\frac{1}{2})^2})\end{aligned}$$

(ii) We consider the case of $\nu = \frac{1}{3}$ and $b = \frac{1}{2}$. Then the equation is

$$\varphi_2\cdot(z^2-z-1)+\varphi_1\cdot(z-\frac{1}{2})-\varphi\cdot(\frac{1}{3})^2 = 0.$$

Operating N^α to this equation, we have

$$\varphi_{2+\alpha}\cdot(z^2-z-1)+\varphi_{1+\alpha}\cdot(2\alpha+1)\cdot(z-\frac{1}{2})+\varphi_\alpha\cdot(\alpha^2-(\frac{1}{3})^2) = 0$$

We adopt $\alpha = 1/3$, then

$$\varphi_{2+\frac{1}{3}}\cdot(z^2-z-1)+\varphi_{1+\frac{1}{3}}\cdot 2\cdot(z-\frac{1}{2}) = 0.$$

Setting $\psi = \psi(z) = \varphi_{1+\frac{1}{3}}$, we have the following equation

$$\psi_1\cdot(z^2-z-1)+\psi\cdot(\frac{5}{3})(z-\frac{1}{2}) = 0.$$

and the solution

$$\psi = (z^2-z-1)^{-\frac{5}{6}}.$$

Therefore we have the solution as follows,

$$\begin{aligned}\varphi &= (((z^2-z-1)^{-\frac{5}{6}})_{-(1+\frac{1}{3})} \\ &= (((z-\frac{1}{2})^2-((\frac{1}{2})^2+1))^{-\frac{5}{6}})_{-(1+\frac{1}{3})}\end{aligned}$$

Applying Lemma II,

$$\varphi = \left(((z-\frac{1}{2})^2-(\frac{5}{4}))^{-\frac{5}{6}}\right)_{-(\frac{4}{3})}$$

$$= e^{i\pi(\frac{4}{3})}(z-\frac{1}{2})^{-\frac{1}{3}} \times \sum_{k=0}^{\infty}\frac{[\frac{5}{6}]_k\Gamma(2k+\frac{1}{3})}{k!\Gamma(2k+\frac{5}{3})}\left(\frac{\frac{5}{4}}{(z-\frac{1}{2})^2}\right)^k.$$

We notice the following relations.

$$\Gamma(2k+\frac{1}{3}) = [\frac{1}{3}]_{2k}\Gamma(\frac{1}{3}) = 2^{2k}[\frac{1}{6}]_k[\frac{2}{3}]_k\Gamma(\frac{1}{3})$$

$$\Gamma(2k+\frac{5}{3}) = [\frac{5}{3}]_{2k}\Gamma(\frac{5}{3}) = 2^{2k}[\frac{5}{6}]_k[\frac{4}{3}]_k\Gamma(\frac{5}{3})$$

So we have

$$\varphi \;=\; -e^{i\pi(\frac{1}{3})}\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{5}{3})} \times \sum_{k=0}^{\infty}\frac{[\frac{1}{6}]_k[\frac{2}{3}]_k}{k![\frac{4}{3}]_k}\left(\frac{\frac{5}{4}}{(z-\frac{1}{2})^2}\right)^k$$

$$\times \;=\; -e^{i\pi(\frac{1}{3})}\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{5}{3})}(z-\frac{1}{2})^{-\frac{1}{3}}{}_2F_1(\frac{1}{6},\frac{2}{3};\frac{4}{3};\frac{\frac{5}{4}}{(z-\frac{1}{2})^2}).$$

We can conclude that the familiar form of the solution can write as

$$\varphi(z) = -e^{i\pi\nu}\frac{\Gamma(\nu)}{\Gamma(2\nu+1)}(z-b)^{-\nu}{}_2F_1(\frac{\nu}{2},\frac{\nu}{2}+\frac{1}{2};\nu+1;\frac{b^2+1}{(z-b)^2})$$

where $|(b^2+1)/(z-b)^2| < 1$, including the case of ν be integer formally.

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