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On another proof of Ozaki's theorem and a sufficient condition for univalence

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Abstract

In 1935, S. Ozaki (Sci. Rep. Tokyo Bunrika Daigaku, **2** (1935)) has given the sufficient condition for analytic functions to be at most p-valent in the convex domain. The object of the present paper is to discuss new proof of Ozaki's teorem. A sufficient condition for univalent functions is also considered.

1 Main theorems

Theorem 1 *Let $f(z)$ be analytic in a convex domain D and suppose that*

$$\mathrm{Re}(f^{(p)}(z)) > 0 \quad (z \in D).$$

Then $f(z)$ is at most p-valent in D.

Proof. Applying the mathematical method of redutive absurdity, we prove it. If $f(z)$ is not at most p-valent in D, then there exist $p+1$ points $z_{1,1}$, $z_{1,2}$, $z_{1,3}$, $\cdots$, $z_{1,p}$, $z_{1,p+1}$ which are different each other for which

$$f(z_{1,1}) = f(z_{1,2}) = f(z_{1,3}) = \cdots = f(z_{1,p}) = f(z_{1,p+1}) = 0.$$

Let us number the points in order of multitude of real part of the points, but if some of them have same real part, then let us rotate the z-plane suitably.
Renumbering of $p+1$ points, then without generalization, we can suppose that all the line segments $\overline{z_{1,1}z_{1,2}}$, $\overline{z_{1,2}z_{1,3}}$, $\overline{z_{1,3}z_{1,4}}$, $\cdots$, $\overline{z_{1,p-1}z_{1,p}}$, $\overline{z_{1,p}z_{1,p+1}}$ are not perpendicular with the real axis, and therefore, we can put the following

$$\mathrm{Re}(z_{1,1}) < \mathrm{Re}(z_{1,2}) < \mathrm{Re}(z_{1,3}) < \cdots < \mathrm{Re}(z_{1,p}) < \mathrm{Re}(z_{1,p+1}).$$

Then we have the followings:

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$$
(1)\qquad
\begin{aligned}
\mathrm{Re}\left(\frac{f(z_{1,2})-f(z_{1,1})}{z_{1,2}-z_{1,1}}\right) &= \mathrm{Re}(f'(z_{2,1}))=0,\\
\mathrm{Re}\left(\frac{f(z_{1,3})-f(z_{1,2})}{z_{1,3}-z_{1,2}}\right) &= \mathrm{Re}(f'(z_{2,2}))=0,\\
\mathrm{Re}\left(\frac{f(z_{1,4})-f(z_{1,3})}{z_{1,4}-z_{1,3}}\right) &= \mathrm{Re}(f'(z_{2,3}))=0,\\
&\vdots\\
\mathrm{Re}\left(\frac{f(z_{1,p})-f(z_{1,p-1})}{z_{1,p}-z_{1,p-1}}\right) &= \mathrm{Re}(f'(z_{2,p-1}))=0,\\
\mathrm{Re}\left(\frac{f(z_{1,p+1})-f(z_{1,p})}{z_{1,p+1}-z_{1,p}}\right) &= \mathrm{Re}(f'(z_{2,p}))=0,
\end{aligned}
$$

where

$$z_{2,k}=z_{1,k}+\theta_{1,k}(z_{1,k+1}-z_{1,k})\qquad (0<\theta_{1,k}<1 \text{ and } k=1,2,3,\cdots,p),$$

and the sequence $\{\mathrm{Re}(z_{2,k})\}$ is a strictly increasing sequence.

From step (1), we have

$$
(2)\qquad
\begin{aligned}
\frac{\mathrm{Re}(f'(z_{2,2})-f'(z_{2,1}))}{\mathrm{Re}(z_{2,2}-z_{2,1})} &= \mathrm{Re}\left(\frac{\partial f'(z_{3,1})}{\partial x}\right)=0,\\
\frac{\mathrm{Re}(f'(z_{2,3})-f'(z_{2,2}))}{\mathrm{Re}(z_{2,3}-z_{2,2})} &= \mathrm{Re}\left(\frac{\partial f'(z_{3,2})}{\partial x}\right)=0,\\
&\vdots\\
\frac{\mathrm{Re}(f'(z_{2,p})-f'(z_{2,p-1}))}{\mathrm{Re}(z_{2,p}-z_{2,p-1})} &= \mathrm{Re}\left(\frac{\partial f'(z_{3,p-1})}{\partial x}\right)=0,
\end{aligned}
$$

where

$$z_{3,k}=z_{2,k}+\theta_{2,k}(z_{2,k+1}-z_{2,k})\qquad (0<\theta_{2,k}<1 \text{ and } k=1,2,\cdots,p-1).$$

Then the sequence $\{\mathrm{Re}(z_{3,k})\}$ is also a strictly increasing sequence.

Form step (2), we have

$$\frac{\operatorname{Re}\left(\dfrac{\partial f'(z_{3,2})}{\partial x}-\dfrac{\partial f'(z_{3,1})}{\partial x}\right)}{\operatorname{Re}(z_{3,2}-z_{3,1})} = \operatorname{Re}\left(\frac{\partial^2 f'(z_{4,1})}{\partial x^2}\right)=0,$$

$$\frac{\operatorname{Re}\left(\dfrac{\partial f'(z_{3,3})}{\partial x}-\dfrac{\partial f'(z_{3,2})}{\partial x}\right)}{\operatorname{Re}(z_{3,3}-z_{3,2})} = \operatorname{Re}\left(\frac{\partial^2 f'(z_{4,2})}{\partial x^2}\right)=0,$$

$$\vdots$$

$$\frac{\operatorname{Re}\left(\dfrac{\partial f'(z_{3,p-1})}{\partial x}-\dfrac{\partial f'(z_{3,p-2})}{\partial x}\right)}{\operatorname{Re}(z_{3,p-1}-z_{3,p-2})} = \operatorname{Re}\left(\frac{\partial^2 f'(z_{4,p-2})}{\partial x^2}\right)=0,$$

where

$$z_{4,k} = z_{3,k} + \theta_{3,k}(z_{3,k+1} - z_{3,k}) \qquad (0 < \theta_{3,k} < 1 \text{ and } k = 1, 2, \cdots, p-2)$$

and $\{\operatorname{Re}(z_{4,k})\}$ is a strictly increasing sequence.

Let us continue the same steps as the above, then we have finally the following equality

$$\operatorname{Re}\left(\frac{\partial^{p-1} f'(z_{p+1,1})}{\partial x^{p-1}}\right) = 0,$$

where

$$z_{p+1,1} = z_{p,1} + \theta_{p,1}(z_{p,2} - z_{p,1}) \in D \qquad (0 < \theta_{p,1} < 1).$$

On the other hand, since $f(z)$ is analytic in D, we have

$$\operatorname{Re}\left(\frac{\partial^{p-1} f'(z_{p+1,1})}{\partial x^{p-1}}\right) = \operatorname{Re}(f^{(p)}(z_{p+1,1})) = 0.$$

This contradicts the hypothesis of the theorem and it completes the proof of the theorem. □

Remark In the proof of the above, if $f(z)$ has zero at $z_{1,1}$ of order 2 or $z_{1,1} = z_{1,2}$ and all another zeros are of order 1, then in the step (1), we put

$$\begin{aligned}
\operatorname{Re}(f'(z_{2,1})) &= \operatorname{Re}(f'(z_{1,1})) = \operatorname{Re}(f'(z_{1,2})) = 0,\\
\operatorname{Re}(f'(z_{2,2})) &= 0,\\
\operatorname{Re}(f'(z_{2,3})) &= 0,\\
&\vdots\\
\operatorname{Re}(f'(z_{2,p})) &= 0,
\end{aligned}$$

where

$$z_{2,1} = z_{1,1} = z_{1,2},$$

$$z_{2,k} = z_{1,k} + \theta_{1,k}(z_{1,k+1} - z_{1,k}) \qquad (0 < \theta_{1,k} < 1 \text{ and } k = 2, 3, \cdots, p),$$

the sequence $\{\mathrm{Re}(z_{1,k})\}$ is not a strictly increasing sequence but the sequence $\{\mathrm{Re}(z_{2,k})\}$ is a strictly increasing sequence. Continuing the same steps as the proof of Theorem 1, we have the same conclusion.

For the cases, $f(z)$ has zeros at many points of multiple orders, then applying the same idea as the above, we obtain the same conclusion.

Theorem 2 *Let $f(z)$ be analytic in a convex domain D and suppose that there exists a complex constant α which satisfies*

$$|\arg(-\alpha)| \geqq \frac{\pi}{2}(1+\delta)$$

where $0 \leqq \delta$ and suppose that

$$|\arg(f'(z) - \alpha)| < \frac{\pi}{2}(1+\delta) \qquad (z \in D).$$

Then $f(z)$ is univalent in D.

Proof. If $f(z)$ is not univalent in D, then there exist two points $z_1 \in D$ and $z_2 \in D$, $z_1 \neq z_2$ for which

$$f(z_1) = f(z_2).$$

Then it follows that

$$\begin{aligned}(f(z_2) - \alpha z_2) - (f(z_1) - \alpha z_1) &= \int_{z_1}^{z_2} (f'(z) - \alpha) dz \\ &= (z_2 - z_1) \int_0^1 \{f'(z_1 + t(z_2 - z_1)) - \alpha\}\, dt\end{aligned}$$

and therefore, we have

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} - \alpha = \int_0^1 \{f'(z_1 + t(z_2 - z_1)) - \alpha\}\, dt.$$

Then we have

$$\begin{aligned}\frac{\pi}{2}(1+\delta) \leqq |\arg(-\alpha)| &= \left|\arg\left(\frac{f(z_2) - f(z_1)}{z_2 - z_1} - \alpha\right)\right| \\ &= \left|\arg \int_0^1 \{f'(z_1 + t(z_2 - z_1)) - \alpha\}\, dt\right| \\ &< \frac{\pi}{2}(1+\delta).\end{aligned}$$

This is a contradiction and therefore it completes the proof. □

References

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