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On a remark of strongly convex functions of order β and convex of order α

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Abstract

For analytic functions $f(z)$ in the open unit disk $\mathbb{U}$, two subclasses $\mathcal{C}(\alpha,\beta)$ and $\mathcal{S}^{*}(\alpha,\beta)$ are introduced. The object of the present paper is to investigate a strongly starlikeness of order δ and starlikeness of order $\beta(\alpha)$ of strongly convex functions of order γ and convex of order α.

1 Introduction

Let $\mathcal{S}$ denote the set of functions

$$f(z)=z+\sum_{n=2}^{\infty}a_{n}z^{n}$$

which are analytic and univalent in the open unit disk $\mathbb{U}=\{z:|z|<1\}$.

Suppose that $f(z)\in \mathcal{S}$ satisfies the following conditions

$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)}-\alpha\right)\right|<\frac{\pi}{2}\beta \quad (z\in \mathbb{U}) \tag{1}$$

or

$$\left|\arg\left(\frac{zf'(z)}{f(z)}-\alpha\right)\right|<\frac{\pi}{2}\beta \quad (z\in \mathbb{U}), \tag{2}$$

where $0\leqq\alpha<1$ and $0<\beta\leqq 1$. If $f(z)\in \mathcal{S}$ satisfies (1), then we say that $f(z)$ is strongly convex of order β and convex of order α in $\mathbb{U}$ and we denote by $\mathcal{C}(\alpha,\beta)$ the class of such functions $f(z)$. If $f(z)\in \mathcal{S}$ satisfies (2), then $f(z)$ is said to be strongly starlike of order β and starlike of order α in $\mathbb{U}$ and we also denote by $\mathcal{S}^{*}(\alpha,\beta)$ the class of such functions $f(z)$.

In view of the results by MacGregor [1] and Wilken and Feng [3], it is well known that $f(z)\in \mathcal{C}(\alpha,1)$ implies $f(z)\in \mathcal{S}^{*}(\beta(\alpha),1)$ where

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$$\beta(\alpha) = \begin{cases} \dfrac{1-2\alpha}{2^{2-2\alpha}(1-2^{2\alpha-1})} & (0 \leqq \alpha < 1; \alpha \neq \frac{1}{2}) \\ \dfrac{1}{2\log 2} & (\alpha = \frac{1}{2}). \end{cases}$$

In the present paper, we discuss some properties for $f(z)$ concerning with the classes $\mathcal{C}(\alpha,\beta)$ and $\mathcal{S}^*(\alpha,\beta)$.

2 Main result

To consider our problems, we have to recall here the following lemma due to Nunokawa [2].

Lemma 1. *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathbb{U}$. Suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad (|z| < |z_0|)$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha$$

where $\alpha > 0$. Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha$$

where

$$k \geqq \frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \textit{where} \quad \arg p(z_0) = \frac{\pi}{2}\alpha$$

and

$$k \leqq -\frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \textit{where} \quad \arg p(z_0) = -\frac{\pi}{2}\alpha,$$

where

$$p(z_0)^{1/\alpha} = \pm ia \ (a > 0).$$

Now, we derive

Theorem 1. *If $f(z) \in \mathcal{C}(\gamma,\alpha)$ with $0 \leqq \alpha < 1$, then $f(z) \in \mathcal{S}^*(\delta,\beta(\alpha))$, where $0 < \delta < 1$ and*

(i) *if* $k = \frac{1}{2}\left(a + \frac{1}{a}\right) \geqq 1$ *and*

$$\mathrm{Tan}^{-1}\frac{\delta k}{(\beta(\alpha)+1)(\beta(\alpha)-\alpha)} \geqq \frac{\pi}{2}\delta,$$

then we put

$$\gamma = \delta + \frac{2}{\pi}\mathrm{Tan}^{-1}\frac{R_0 \sin(\Theta - \frac{\pi}{2}\delta)}{1 + R_0 \cos(\Theta - \frac{\pi}{2}\delta)},$$

$$\Theta = \mathrm{Tan}^{-1}\frac{\delta k}{(\beta(\alpha)+1)(\beta(\alpha)-\alpha)},$$

$$R_0 = \frac{\delta}{2(1-\beta(\alpha)^2)}\left\{\left(\frac{1+\delta}{1-\delta}\right)^{\frac{1-\delta}{2}} + \left(\frac{1-\delta}{1+\delta}\right)^{\frac{1+\delta}{2}}\right\},$$

(ii) *if* $k = \frac{1}{2}\left(a + \frac{1}{a}\right) \geqq 1$ *and*

$$\mathrm{Tan}^{-1}\frac{\delta k}{(\beta(\alpha)+1)(\beta(\alpha)-\alpha)} < \frac{\pi}{2}\delta,$$

then we put

$$\gamma = \mathrm{Tan}^{-1}\frac{\delta k}{(\beta(\alpha)+1)(\beta(\alpha)-\alpha)},$$

(iii) *if* $k = \frac{1}{2}\left(a + \frac{1}{a}\right) \leqq -1$ *and*

$$\mathrm{Tan}^{-1}\frac{\delta k}{(\beta(\alpha)+1)(\beta(\alpha)-\alpha)} \leqq -\frac{\pi}{2}\delta,$$

then we put

$$\gamma = -\delta - \frac{2}{\pi}\mathrm{Tan}^{-1}\frac{R_0 \sin(\Theta + \frac{\pi}{2}\delta)}{1 + R_0 \cos(\Theta + \frac{\pi}{2}\delta)},$$

$$\Theta = \mathrm{Tan}^{-1}\frac{\delta k}{(\beta(\alpha)+1)(\beta(\alpha)-\alpha)},$$

$$R_0 = \frac{\delta}{2(1-\beta(\alpha)^2)}\left\{\left(\frac{1+\delta}{1-\delta}\right)^{\frac{1-\delta}{2}} + \left(\frac{1-\delta}{1+\delta}\right)^{\frac{1+\delta}{2}}\right\}$$

(iv) *if* $k = \frac{1}{2}\left(a + \frac{1}{a}\right) \leqq -1$ *and*

$$\mathrm{Tan}^{-1}\frac{\delta k}{(\beta(\alpha)+1)(\beta(\alpha)-\alpha)} > -\frac{\pi}{2}\delta,$$

then we put

$$\gamma = \mathrm{Tan}^{-1}\frac{\delta k}{(\beta(\alpha)+1)(\beta(\alpha)-\alpha)}.$$

Proof. Let us define the function $p(z)$ by

$$p(z)=\frac{zf'(z)}{f(z)}.$$

Then we have that $p(z)$ is analytic in $\mathbb{U}$ and $p(0)=1$. It follows that

$$p(z)+\frac{zp'(z)}{p(z)}=1+\frac{zf''(z)}{f'(z)},$$

and

$$\begin{aligned}1+\frac{zf''(z)}{f'(z)}-\alpha&=p(z)-\beta(\alpha)+\frac{zp'(z)}{p(z)}+\beta(\alpha)-\alpha\\&=(p(z)-\beta(\alpha))\left(1+\frac{z(p(z)-\beta(\alpha))'}{p(z)-\beta(\alpha)}\frac{1}{p(z)}+\frac{\beta(\alpha)-\alpha}{p(z)-\beta(\alpha)}\right).\end{aligned}$$

Therefore, we see that

$$\begin{aligned}&\arg\left(1+\frac{zf''(z)}{f'(z)}-\alpha\right)\\&=\arg(p(z)-\beta(\alpha))+\arg\left(1+\frac{z(p(z)-\beta(\alpha))'}{p(z)-\beta(\alpha)}\frac{1}{p(z)}+\frac{\beta(\alpha)-\alpha}{p(z)-\beta(\alpha)}\right).\end{aligned}$$

If there exists a point z_0, $|z_0|<1$ such that

$$|\arg(p(z)-\beta(\alpha))|<\frac{\pi}{2}\delta \quad (|z|<|z_0|)$$

and

$$|\arg(p(z_0)-\beta(\alpha))|=\frac{\pi}{2}\delta,$$

then, let us put

$$q(z)=\frac{p(z)-\beta(\alpha)}{1-\beta(\alpha)},\ q(0)=1.$$

Applying Lemma 1, we have that

$$\frac{z_0q'(z_0)}{q(z_0)}=\frac{z_0q'(z_0)}{p(z_0)-\beta(\alpha)}=i\delta k$$

where

$$q(z_0)^{\frac{1}{\delta}}=\left(\frac{p(z_0)-\beta(\alpha)}{1-\beta(\alpha)}\right)^{\frac{1}{\delta}}=\pm ia \quad (a>0)$$

and

$$k\geqq\frac{1}{2}\left(a+\frac{1}{a}\right)\geqq 1 \text{ when } \arg(p(z)-\beta(\alpha))=\frac{\pi}{2}\delta$$

and

$$k\leqq-\frac{1}{2}\left(a+\frac{1}{a}\right)\leqq -1 \text{ when } \arg(p(z)-\beta(\alpha))=-\frac{\pi}{2}\delta.$$

At first, let us consider the case $\arg(p(z_0)-\beta(\alpha)) = \frac{\pi}{2}\delta$, then it follows that

$$
\begin{aligned}
&\arg\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}-\alpha\right)\\
&= \arg(p(z_0)-\beta(\alpha))\left(1+\frac{z_0 p'(z_0)}{p(z_0)-\beta(\alpha)}\frac{1}{p(z_0)}+\frac{\beta(\alpha)-\alpha}{p(z_0)-\beta(\alpha)}\right)\\
&= \frac{\pi}{2}\delta + \arg\left(1+\frac{i\delta k}{(\beta(\alpha)+(ia)^{\delta})(1-\beta(\alpha))}+\frac{\beta(\alpha)-\alpha}{(1-\beta(\alpha))(ia)^{\delta}}\right)\\
&\geqq \frac{\pi}{2}\delta + \arg\left(1+\frac{i\delta k}{(\beta(\alpha)+1)(1-\beta(\alpha))(ia)^{\delta}}+\frac{\beta(\alpha)-\alpha}{(1-\beta(\alpha))(ia)^{\delta}}\right)\\
&= \frac{\pi}{2}\delta + \arg\left(1+e^{-i\frac{\pi}{2}\delta}\left(\frac{i\delta k}{(\beta(\alpha)+1)(1-\beta(\alpha))a^{\delta}}+\frac{\beta(\alpha)-\alpha}{(1-\beta(\alpha))a^{\delta}}\right)\right).
\end{aligned}
$$

Next, let us consider

$$
\begin{aligned}
\arg\left(\frac{i\delta k}{(\beta(\alpha)+1)(1-\beta(\alpha))a^{\delta}}+\frac{\beta(\alpha)-\alpha}{(1-\beta(\alpha))a^{\delta}}\right) &= \mathrm{Tan}^{-1}\left(\frac{\frac{\delta k}{(\beta(\alpha)+1)(1-\beta(\alpha))a^{\delta}}}{\frac{\beta(\alpha)-\alpha}{(1-\beta(\alpha))a^{\delta}}}\right)\\
&= \mathrm{Tan}^{-1}\left(\frac{\delta k}{(\beta(\alpha)-\alpha)(\beta(\alpha)+1)}\right).
\end{aligned}
$$

On the other hand, applying the some method in the result by Nunokawa [2], we have

$$
\begin{aligned}
&\left|\frac{\beta(\alpha)-\alpha}{(1-\beta(\alpha))a^{\delta}}+\frac{i\delta k}{(\beta(\alpha)+1)(1-\beta(\alpha))a^{\delta}}\right| \qquad (3)\\
&> \left|\frac{\delta}{2(1-\beta(\alpha)^2)a^{\delta}}\left(a+\frac{1}{a}\right)\right|\\
&= \frac{\delta}{2(1-\beta(\alpha))a^{\delta}}\left(a^{1-\delta}+\frac{1}{a^{1+\delta}}\right)\\
&\geqq \frac{\delta}{2(1-\beta(\alpha)^2)}\left(\left(\frac{1+\delta}{1-\delta}\right)^{\frac{1-\delta}{2}}+\left(\frac{1+\delta}{1-\delta}\right)^{\frac{1+\delta}{2}}\right)\\
&= R_0
\end{aligned}
$$

say.

Therefore, for the case

$$\arg(p(z_0)-\beta(\alpha)) = \frac{\pi}{2}\delta$$

and

$$\mathrm{Tan}^{-1}\left(\frac{\delta k}{(\beta(\alpha)-\alpha)(\beta(\alpha)+1)}\right) \geqq \frac{\pi}{2}\delta,$$

we have

$$\begin{aligned}
&\arg\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}-\alpha\right)\\
&=\arg(p(z_0)-\beta(\alpha))+\arg\left(1+\frac{z_0 p'(z_0)}{p(z_0)-\beta(\alpha)}\frac{1}{p(z_0)}+\frac{\beta(\alpha)-\alpha}{p(z_0)-\beta(\alpha)}\right)\\
&\geqq \frac{\pi}{2}\delta+\arg\left(1+R_0 e^{i\left(\Theta-\frac{\pi}{2}\delta\right)}\right)\\
&=\frac{\pi}{2}\delta+\mathrm{Tan}^{-1}\frac{R_0\sin\left(\Theta-\frac{\pi}{2}\delta\right)}{1+R_0\cos\left(\Theta-\frac{\pi}{2}\delta\right)},
\end{aligned}$$

where

$$\Theta=\mathrm{Tan}^{-1}\frac{\delta k}{(\beta(\alpha)+1)(\beta(\alpha)-\alpha)}.$$

This is the contradiction for the condition of the theorem.

On the other hand, for the case

$$\arg(p(z_0)-\beta(\alpha))=\frac{\pi}{2}\delta$$

and

$$\Theta=\mathrm{Tan}^{-1}\left(\frac{\delta k}{(\beta(\alpha)-\alpha)(\beta(\alpha)+1)}\right)<\frac{\pi}{2}\delta,$$

putting $a\to+0$ in (3), we easily have

$$\begin{aligned}
&\arg\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}-\alpha\right)\\
&=\arg(p(z_0)-\beta(\alpha))+\arg\left(1+\frac{z_0 p'(z_0)}{p(z_0)-\beta(\alpha)}\frac{1}{p(z_0)}+\frac{\beta(\alpha)-\alpha}{p(z_0)-\beta(\alpha)}\right)\\
&\geqq \frac{\pi}{2}\delta+\arg e^{i\left(\Theta-\frac{\pi}{2}\delta\right)}\\
&=\Theta\\
&=\mathrm{Tan}^{-1}\frac{\delta k}{(\beta(\alpha)-\alpha)(\beta(\alpha)+1)}.
\end{aligned}$$

This is also the contradiction for the theorem.

For the case

$$\arg(p(z_0)-\beta(\alpha))=-\frac{\pi}{2}\delta$$

and

$$\mathrm{Tan}^{-1}\left(\frac{\delta k}{(\beta(\alpha)-\alpha)(\beta(\alpha)+1)}\right)\leqq-\frac{\pi}{2}\delta,$$

where

$$k\leqq-\frac{1}{2}\left(a+\frac{1}{a}\right)$$

and

$$\left(\frac{p(z_0)-\beta(\alpha)}{1-\beta(\alpha)}\right)^{\frac{1}{\delta}}=-ia\quad(a>0),$$

or for the case

$$\arg(p(z_0)-\beta(\alpha)) = -\frac{\pi}{2}\delta$$

and

$$\mathrm{Tan}^{-1}\left(\frac{\delta k}{(\beta(\alpha)-\alpha)(\beta(\alpha)+1)}\right) < -\frac{\pi}{2}\delta,$$

where

$$k \leqq -\frac{1}{2}\left(a+\frac{1}{a}\right)$$

and

$$\left(\frac{p(z_0)-\beta(\alpha)}{1-\beta(\alpha)}\right)^{\frac{1}{\delta}} = -ia \quad (a>0),$$

applying the same method as the above, we have the contradiction for the theorem. Therefore the proof of the theorem is completed. □

Remark 1. We have to say that Theorem 1 is not sharp. To consider the sharp result, we need another method. Therefore we leave this problem for our discussion.

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