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On generic automorphisms of a tree structure

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Abstract

We give a theory T with the strict order property such that for some automorphism σ_0 of a prime model M_0 of T, the theory

$$T + \text{“}\sigma \text{ is an automorphism”} + \text{“}\sigma | M_0 = \sigma_0\text{”}$$

is model complete. Note that $T +$ “σ is an automorphism” has no model companion if T has the strict order property [3]. This seems to have some resemblance with the theory of the rings of Witt Vectors carrying the Frobenius automorphism [1].

We consider each natural number n as the set $\{0, 1, \ldots, n-1\}$.

Consider a structure $(M_0, <)$ with

$$M_0 = \{f : n \to n+1 \mid n < \omega,\ f(i) < i+1 \text{ for } i < n\},$$

and $f < g$ if g is a proper extension of f as a map for $f, g \in M_0$.

For each $f \in M_0$ with $\operatorname{dom} f = n$, let f^s be a map such that

$$f^s(i) = (f(i) + 1) \bmod (i+1)$$

for $i < n$. Then the map $s : M_0 \to M_0$ defined by $s(f) = f^s$ is an automorphism of $(M_0, <)$. ϵ denotes the least element of M_0 (i.e., ϵ is the empty sequence). Let $<_1$ be a definable relation on M_0 defined by the formula

$$x < y \wedge \forall z\, \neg(x < z < y).$$

Let T_0 be the theory of $(M_0, <, <_1)$. Note that for any model M of T_0, $\operatorname{acl}_M(\emptyset) = M_0$. The root (the least element) of M_0 will be denoted by ϵ.

Proposition 1. *Let M be a model of T_0. Then the following sentenses are valid in M:*

(1) $\forall x \, \exists y \quad x <_1 y$.

(2) $\forall x, y \quad x < y \to \exists z \; x <_1 z \leq y$.

(3) $\forall x, y \quad x < y \to \exists z \; x \leq z <_1 y$.

(4) $\forall x, y, z \quad x, y \leq z \; \to \; x < y \vee x = y \vee y < x$.

(5) $\forall x, y \, \exists u, v \quad x \not\leq y \to u <_1 v \leq x \wedge u \leq y \wedge v \not\leq y$.

(6) *Let n be any natural number. If $x < y$ and x has (at least) n childs then y has (at least) $n+1$ childs.*

Theorem 2. *The theory*

$$T_0 \cup \{\sigma \text{ is a } <\text{-automorphism extending } s\}$$

in the language $\{<, <_1, \sigma\} \cup M_0$ has a model companion. In fact, it is model complete.

We fix models $M \subset M'$ of T and assume that σ is a $<$-automorphism of M' extending s and M is σ-invariant.

Lemma 3. *If $a, b \in M$ then $\inf_M\{a, b\} = \inf_{M'}\{a, b\}$.*

Proof. Let $c = \inf_M\{a, b\}$. If $c = a$ or $c = b$ then there is nothing to prove.

Suppose $c < a, b$. Then we can choose c_a, $c_b \in M$ such that $c <_1 c_a \leq a$, $c <_1 c_b \leq b$, and c_a is incomparable with c_b. Now, we show that $c = \inf_{M'}\{a, b\}$. Let $d \in M' - M$ be such that $d < a, b$. Then d is comparable with both c_a and c_b. Only the case $d < c_a$, c_b is possible. Therefore, $d < c$. □

Definition 4. Suppose $a, b \in M' - M$. We say that a and b are *dependent over M* if there is $c \in M' - M$ such that $c \leq a$ and $c \leq b$. We call such c a *witness* of the dependence. a and b are dependent over M if and only if $\inf\{a, b\} \in M' - M$.

We say that a and b are independent over M if a and b are not dependent over M.

Lemma 5. *The dependence over M is an equivalence relation on $M' - M$.*

Proof. The reflexivity and the symmetry are trivial. We show the transitivity. Suppose b and c are dependent over M with a witness u, and c and d are dependent over M with a witness v. Since $u \leq c$ and $v \leq c$, u and v are comparable. Without loss of generality, we can assume that $u \leq v$. Then $u \leq v \leq d$. Therefore, b and d are dependent over M with a witness u. □

Lemma 6. *If $b \in M' - M_0$ then b and $\sigma^m b$ are independent over M_0 for any integer $m \neq 0$.*

Proof. Let $m \neq 0$ be an integer and $b \in M' - M$. Choose $f < b$ such that $f \in M_0$ and $\operatorname{dom} f \supset m$. Then f and $s^m f$ are imcomparable and also $s^m f < \sigma^m b$.

Suppose there is $a \in M' - M_0$ such that $a \leq b$ and $a \leq \sigma^m b$. f and a are comparable by $f < b$ and $a \leq b$. Since f has a finite distance from the root, we have $f < a$. Similarly, $s^m f < a$. Therefore, f and $s^m f$ are comparable. A contradiction. □

Corollary 7. *If $a, b \in M' - M$ are dependent over M then a and $\sigma^m b$ are independent over M for any integer $m \neq 0$.*

Proof. Suppose $a, b \in M' - M$ are dependent over M and a and $\sigma^m b$ are dependent over M for some integer $m \neq 0$. Suppose $c \leq a$, $c \leq b$ with $c \in M' - M$, and $d \leq a$, $d \leq \sigma^m b$ with $d \in M' - M$.

Since $c, d \leq a$, c and d are comparable. Therefore, $\min\{c, d\} \leq \inf\{b, \sigma^m b\}$, and hence $\inf\{b, \sigma^m b\} \in M' - M$ contradicting Lemma 6. □

Definition 8. Suppose $a, b \in M' - M$. We say that a and b are *quasi-connected over* M if there is $c \in M'$ such that

(1) $M' \models c \leq a, b$,

(2) $M' \models c \leq y \leq a$ implies $y \in M' - M$, and

(3) $M' \models c \leq y \leq b$ implies $y \in M' - M$.

We call c a *witness* of this property. Note that if a and b are quasi-connected over M then it is dependent over M.

Lemma 9. *The quasi-connectedness over M is an equivalence relation on $M' - M$.*

Proof. The reflexivity and the symmetry are trivial. We show the transitivity. Suppose b and c are quasi-connected over M with a witness u and c and d are quasi-connected over M with a witness v. Since $u \leq c$ and $v \leq c$, u and v are comparable. Without loss of generality, we can assume that $u \leq v$. We show that u is a witness for quasi-connectedness of b and d over M. If $u \leq w \leq b$ then $w \in M' - M$ since u is a witness for quasi-connectedness of b and c.

Suppose $u \leq w \leq d$. Then w and v are comparable. If $w \leq v$ then $u \leq w \leq c$ and thus $w \in M' - M$. If $v < w$ then $v \leq w \leq d$ and thus $w \in M' - M$. □

Lemma 10. *Suppose that B is a finite subset of $M' - M$ quasi-connected over M, $a_1, \ldots, a_m \in M$ and for each a_i there is $b_i \in B$ such that $b_i < a_i$. Then there is $b \in B$ such that $b < \inf\{a_1, \ldots, a_m\}$.*

Proof. Let $a = \inf\{a_1, \ldots, a_m\}$ in M. Then $a = \inf\{a_1, \ldots, a_m\}$ in M' by Lemma 3.

Let $b = \inf B$ in M'. We have $b \in M' - M$ because B is quasi-connected over M. Since b is a lower bound for $\{a_1, \ldots, a_m\}$, we have $b \leq a$. Choose $b_1 \in B$ such that $b_1 < a_1$. Then b_1 and a are comparable. If $a \leq b_1$ then $b \leq a \leq b_1$, but this cannot happen since there is no element $y \in M$ such that $b \leq y \leq b_1$. Therefore, $b_1 < a$. □

Lemma 11. (1) *Suppose $M' \models a <_1 b$ with $a \in M$ and $b \in M' - M$. Then there is no $a' \in M$ such that $M' \models b < a'$.*

(2) *If $b \in M' - M$ then there is no $a \in M$ such that $M' \models b <_1 a$.*

Proof. (1) Suppose $M' \models a <_1 b < a'$ with $a, a' \in M$ and $b \in M' - M$. Then there must be $a'' \in M$ such that $M \models a <_1 a'' < a'$, and thus $M' \models a <_1 a'' < a'$. But this cannot happen because $b \neq a''$.

(2) Suppose there is $b \in M' - M$ and $a \in M$ such that $M' \models b <_1 a$. Since $M' \models \epsilon < a$, we have $M \models \epsilon < a$. Therefore, $M \models a' <_1 a$ for some $a' \in M$ and thus $M' \models a' <_1 a$. But this cannot happen because $b \neq a'$. □

Definition 12. Suppose C and D are subsets of M'. We write $C < D$ if there is $c \in C$ such that $c \leq d$ for any $d \in D$.

Definition 13. A finite subset X of $M' - M$ is called *canonical* if the following conditions are satisfied:

(1) For any $x, y \in X$, whenever x and $\sigma^m(y)$ with $m \in \mathbb{Z}$ are dependent over M then $m = 0$;

(2) if $x, y \in X$ are dependent over M then there is $z \in X$ witnessing the dependence; and

(3) if $x, y \in X$ are quasi-connected over M then there is $z \in X$ witnessing the quasi-connectedness.

Definition 14. Let B be a subset of M'. $\langle B \rangle_\sigma$ denotes the set $\{\sigma^m(b) \mid b \in B,\ m \in \mathbb{Z}\}$.

Lemma 15. *For any finite subset $X \subset M' - M$ there is a canonical subset $Z \subset M' - M$ such that $X \subset \langle Z \rangle_\sigma$.*

Proof. We prove the statement by induction on the number of elements in X. It is trivial if $|X| = 0$. Suppose $X = \{a\} \cup X'$ with $|X'| < |X|$. By the induction hypothesis, there is a cononical subset Y' of $M' - M$ such that $X' \subset \langle Y' \rangle_\sigma$.

We split the proof into the following cases.

Case 1. $\sigma^m a$ and b are quasi-connected over M for some $b \in Y'$ and an integer m.

Let b_0 be the least element in Y' which is quasi-connected to $\sigma^m a$ over M. Let $c = \inf\{\sigma^m a, b_0\}$. We claim that $Y = Y' \cup \{\sigma^m a,\ c\}$ is canonical and has the desired property.

Let C_{b_0} be the quasi-connected component of Y containing b_0 and D_{b_0} be the dependent component of Y containing b_0. It is easy to see that $\{c\} \cup C_{b_0}$ is a tree. $\{c\} \cup D_{b_0}$ is also a tree. Let d be the least element of D_{b_0}. Since $c \leq b_0$ and $d \leq b_0$, c and d are comparable. Therefore, $\{c\} \cup D_{b_0}$ is a tree.

Now, supppose that $\sigma^{m+l}a$ and $b \in Y'$ are dependent over M. Then $\sigma^l b_0$ and $\sigma^{m+l}a$ are dependent over M and thus $\sigma^l b_0$ and $b \in Y'$ are dependent over M. Since Y' is canonical, we have $l = 0$.

Case 2. Case 1 does not hold but $\sigma^m a$ and b are dependent over M for some $b \in Y'$ and an integer m.

Let b_0 be the least element in Y' which is dependent to $\sigma^m a$ over M. Choose a witness $c \in M' - M$ of dependence of b_0 and $\sigma^m a$. $Y = Y' \cup \{\sigma^m a, c\}$ is canonical and has the desired property. The argument is the same as that for Case 1.

Case 3. There is no integer m and $b \in Y'$ such that $\sigma^m a$ and b are dependent over M. In this case, $Y = Y' \cup \{a\}$ is canonical and has the desired property. □

Lemma 16. *Suppose* $\{t_1, \dots, t_n\} \subset M' - M$ *is canonical. Then any formula in* $\mathrm{qftp}_{\{<,\sigma\}}(t_1, \dots, t_n/M)$ *is realised in* M.

Proof. Suppose $\{t_1, \dots, t_n\} \subset M' - M$ is canonical. Let t be the tuple $(t_1, \cdots, t_n)$ and $\varphi(x)$ a formula with $x = (x_1, \dots, x_n)$ belonging to $\mathrm{qftp}_{\{<,<_1,\sigma\}}(t/M)$. Let N be a natural number such that if $\sigma^m(x_i)$ occurs in $\varphi(x)$ then $m \leq N$. Let A be a finite subset of M such that $\varphi(x)$ is over A.

By adding finitely many points of M to A if necessary, we can assume the following:

- If C is a quasi-connected component of t then $\{a\} < C$ for some $a \in A$;
- if C and C' are two quasi-connected components of t with $C < C'$ then there is $a \in A$ such that $C < \{a\} < C'$;
- if C is a quasi-connected component of t and there is $a \in M$ and $c \in M' - M$ quasi-connected to C over M such that $a <_1 c$ then $a \in A$ and $c \in C$;
- if C is a quasi-connected component of t such that $\{a \in A \mid C < \{a\}\}$ is non-empty then $\inf\{a \in A \mid C < \{a\}\} \in A$;
- if $a \in A$ is comparable with t_i for some i then $\sigma^m(a) \in A$ for $m \leq N$; and
- if $a \in A$ is comparable with $\sigma^m(t_i)$ for some i and a natural number $m \leq N$ then $\sigma^{-m}(a) \in A$.

We can assume that $t = C_1\hat{}\cdots\hat{}C_l$ where each C_i is an enumeration of a quasi-connected component of t.

Let a_i be the maximum element in A such that $\{a_i\} < C_i$ and b_i be the minimum element in A such that $C_i < \{b_i\}$. Such a_i exists by the assumption on A and such b_i exists if there is $b \in A$ such that $C_i < \{b\}$ by Lemma 10 and the assumption on A.

Suppose that there are infinitely many elements d of M connected to a_i such that $a_i < d < C_i$. Choose $a_i' \in M$ connected to a_i with the following properties:

- If $x \in A$ and $M \models \sigma^m b_i \not\leq x$ with $0 \leq m \leq N$ then $M \models \sigma^m a_i \not\leq x$; and
- if C' is a quasi-connected component of t such that $C_i \not< C'$ then $\{a_i'\} \not< C'$.

In the case that b_i exists, choose a tuple C_i' from M such that $\mathrm{qftp}_{\{<,<_1\}}(C_i/a_i', b_i) = \mathrm{qftp}_{\{<,<_1\}}(C_i'/a_i', b_i)$. Then we have $\mathrm{qftp}_{\{<,<_1\}}(\sigma^m C_i/A) = \mathrm{qftp}_{\{<,<_1\}}(\sigma^m C_i'/A)$ for $m = 0, 1, \ldots, N$.

In the case that there is no such b_i for C_i, choose a tuple C_i' from M such that $\mathrm{qftp}_{\{<,<_1\}}(C_i/a_i') = \mathrm{qftp}_{\{<,<_1\}}(C_i'/a_i')$. Then we have $\mathrm{qftp}_{\{<,<_1\}}(\sigma^m C_i/A) = \mathrm{qftp}_{\{<,<_1\}}(\sigma^m C_i'/A)$ for $m = 0, 1, \ldots, N$.

Suppose $C_{i_1}, \ldots, C_{i_k}$ are quasi-connected and $a <_1 \inf C_{i_j}$ for $j = 1, \ldots, k$. In this case, there is no $x \in A$ such that $C_{i_j} < \{x\}$ by Lemma 11. We can choose $c_{i_j}' \in M - A$ for $j = 1, \ldots, k$ which are pairwise distinct such that $M \models a_i <_1 c_{i_j}'$ and $M \models \sigma^m c_{i_j}' \not\leq x$ for $x \in A$ and m with $0 \leq m \leq N$. Choose a tuple C_{i_j}' for $j = 1, \ldots, k$ from M such that $\mathrm{qftp}_{\{<,<_1\}}(C_{i_j}, \inf C_{i_j}) = \mathrm{qftp}_{\{<,<_1\}}(C_{i_j}', c_{i_j}')$. Then $\mathrm{qftp}_{\{<,<_1\}}(C_{i_j}/A) = \mathrm{qftp}_{\{<,<_1\}}(C_{i_j}'/A)$.

Let $t' = C_1'\hat{\ } \cdots \hat{\ } C_l'$.

Claim 1. $\mathrm{qftp}_{\{<,<_1\}}(t\hat{\ }\sigma t\hat{\ }\sigma^2 t\hat{\ } \ldots \hat{\ }\sigma^N t/A) = \mathrm{qftp}_{\{<,<_1\}}(t'\hat{\ }\sigma t'\hat{\ }\sigma^2 t'\hat{\ } \ldots \hat{\ }\sigma^N t'/A)$

□

Proof of Theorem 2. We show that $(M, <, \sigma|M)$ is existentially closed in $(M', <, \sigma)$. Choose a finite tuple $(t_1, \ldots, t_n)$ from $M' - M$ and let $\varphi(x_1, \ldots, x_n)$ be a quantifier-free formula of $\{<, \sigma\} \cup M$ realised by $(t_1, \ldots, t_n)$. By Lemma 15, we can choose $t_1', \ldots, t_n' \in M' - M$ such that $t_i = \sigma^{k_i}(t_i')$ for each i with some $k_i \geq 0$ and the set $\{t_1', \ldots, t_n'\}$ is canonical. We have

$$M' \models \varphi(\sigma^{k_1}(t_1'), \ldots, \sigma^{k_n}(t_n')).$$

By Lemma 16, we can choose $t_1'', \ldots, t_n'' \in M$ such that

$$M \models \varphi(\sigma^{k_1}(t_1''), \ldots, \sigma^{k_n}(t_n'')).$$

Therefore, $\varphi(x_1, \ldots, x_n)$ is realised in M. □

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