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Completeness and The Number of Types For Infinitary Logic

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Abstract

In Infinitary Logic, the topological space $S(L_F, T)$ is not compact. Morley showed $S(L_F, T)$ is analytic where L_F is countable[2]. In this paper I show that $S(L_F, T)$ is completely metrizable where L_F is countable.

1 Introduction

In Infinitary Logic, compactness fail. For example, $\{x \neq c_i\}_{i<\omega} \cup \{\bigvee_{i<\omega}(x = c_i)\}$ is finitely satisfiable but inconsistent. By this fact, we can't get saturated models and arbitrary large models in general. How do we construct a large model? The model existence theorem is a way to construct some models(see [1]). In this paper, I show a basic property of infinitary logic. In many cases you can use this property instead of the model existence theorem. In section 2, I define L_F a fragment of $L_{\kappa,\omega}$. This definition has little deferences from some text books[1], but you can easily understand that it is sufficiently general. In section 3, I translate L_F to $L(\tau)$, a language with first order logic. We'll see that any class of models of a theory of L_F is equivalent to a subclass of an elementary class which language is $F(\tau)$. This subclass is characterized by a set of types. In section 4, I show that if language is countable then $S(L_F, T)$ is completely metrizable.

2 Preliminaries

First I define $L_{\kappa,\omega}$ and a fragment of $L_{\kappa,\omega}$.

Definition 1 Suppose L is the set of all (first order) L-formulas.

1. Let $\{p_i, f_j, c_k\}$ be a set of new symbols. Then We'll write $L(\{p_i, f_j, c_k\})$ as the set of all (first order) formulas of the expanded language which is added $\{p_i, f_j, c_k\}$ to the language L.

2. $L_{\kappa,\omega}$ is the smallest set of formulas such that

   (a) $L \subset L_{\kappa,\omega}$.

   (b) $L_{\kappa,\omega}$ is closed under finite boolean combination and finite quantification.

   (c) If $\Phi(\bar{x}) \subset L_{\kappa,\omega}$ ($|\Phi| < \kappa$, $|\bar{x}| < \omega$), then $\bigwedge \Phi(\bar{x}) \in L_{\kappa,\omega}$.

3. We say L_F is a fragment of $L_{\kappa,\omega}$ iff $L \subset L_F \subset L_{\kappa,\omega}$ and it is closed under finite boolean combinations and finite quantifications, subformulas, and finitely exchanging of terms(i.e. if $\phi(\bar{t}) \in L_F$, $\bar{t}'$ is L-term, then $\phi(\bar{t}') \in L_F$).

I note that every formula has only finitely many variables.(There may be infinitely many occurrences.) In the following, we fix a fragment $L_F \subset L_{\kappa,\omega}$.

3 Translation

3.1 Translation of L_F into $L(\tau)$(first order logic)

I construct a first order language $L(\tau)$ corresponding to L_F.

Definition 2 We define τ be a set of new predicate symbols P_Φ.

1. $\tau = \{P_\Phi(\bar{x}) \mid \bigwedge \Phi(\bar{x}) \in L_F\}$.

2. $\phi^* \in L(\tau)$ is defined in each $\phi \in L_F$ as follows.

   (a) If $\phi \in L$ then $\phi^* = \phi$.

   (b) If ϕ is $\phi_1 \wedge \phi_2(\neg\phi_1, \exists x\phi_1)$, then ϕ^* is defined by $\phi_1^* \wedge \phi_2^*(\neg\phi_1^*, \exists x\phi_1^*)$.

   (c) If $\phi = \bigwedge \Phi(\bar{x})$, then $\phi^* = P_\Phi(\bar{x})$.

Remark 3 The map $* : L_F \to L(\tau)$ is injective but not surjective. For example, suppose $\bigwedge \Phi(y)$ be a L_F-formula and $t(x)$ be a L-term. Let $\Phi'(x) = \Phi(t(x))$. There is a predicate symbol $P_\Phi(y)$ and we can take a $L(\tau)$-formula $P_\Phi(t(x))$. But $(\bigwedge \Phi'(x))^*$ is just $P_{\Phi'}(x)$. So there is no L_F-formula ψ such that $\psi^* = P_\Phi(t(x))$.

Next I define suitable subclass of $L(\tau)$-structure STR. Each member of STR omits a set of $L(\tau)$-types Γ and interprets P_Φ like $\bigwedge \Phi$. In section 3.2, we'll see that STR is "suitable".

Definition 4 Suppose M is a L-structure.

1. M^* is a $L(\tau)$-structure expanded M such that $M^* \models P_\Phi(\bar{a})$ if and only if $M \models \bigwedge \Phi(\bar{a})$ for all $P_\Phi \in L(\tau)$.

2. $\Gamma = \{q(\bar{y}) | q = \{\neg P_\Phi(\bar{y})\} \cup \Phi(\bar{y})^*, P_\Phi(\bar{y}) \in L(\tau)\}$.

3. $T_1 = \{\forall\bar{x}(P_\Phi(\bar{x}) \to \phi^*(\bar{x})) | P_\Phi \in L(\tau), \phi \in \Phi\}$

4. $T_2 = \{\forall\bar{x}(P_\Phi(\bar{t}(\bar{x})) \leftrightarrow P_{\Phi'}(\bar{x})) | \bar{t}$ is L-term, $\bigwedge\Phi(\bar{y}), \bigwedge\Phi'(\bar{x}) \in L_F$, $\bigwedge\Phi'(\bar{x}) = \bigwedge\Phi(\bar{t}(\bar{x}))\}$.

5. $\mathrm{STR} = \{N | N$ is a $L(\tau)$-structure, N omits $\Gamma, N \models T_1.\}$

Let $* : M \mapsto M^*$ be the map in Definition 4. It is easy to show the map is a injection. Moreover, the image of the map is a subset of STR. Actually, if $M \models \bigwedge\Phi(\bar{a})$, then $M \models \phi(\bar{a})$ for all $\phi \in \Phi$. This implies $M^* \models T_1$. On the other hand, if $M \models \phi(\bar{a})$ for all $\phi \in \Phi$, then $M \models \bigwedge\Phi(\bar{a})$. this implies M^* omits Γ.
The next remark is important. This claims that $* : L_F \to L(\tau)$ is not a bijection but it seems a bijection under T_2.

Remark 5 For all $L(\tau)$-formula $\phi(\bar{x})$, there is a L_F-formula $\psi(\bar{x})$ such that $T_2 \models \forall\bar{x}(\phi(\bar{x}) \leftrightarrow \psi(\bar{x})^*)$.

Proof: By induction on $\phi(\bar{x})$. If $\phi(\bar{x}) = P_\Phi(\bar{t}(\bar{x}))$, then we can take $\psi(\bar{x}) = P_{\Phi'}(\bar{x})$ where $\Phi'(\bar{x}) = \Phi(\bar{t}(\bar{x}))$. By T_2, ϕ is equivalent to ψ. The other cases are straightforward. ∎

3.2 Interpreting as a subclass of $L(\tau)$-structures

Proposition 6 Suppose N is a L-structure. If N is in STR, then $N\lceil_L \models \phi(\bar{a})$ if and only if $N \models \phi^*(\bar{a})$ for all $\phi \in L_F$, $\bar{a} \in N$.

Proof: By induction on ϕ. Suppose $\phi = \bigwedge\Phi(\bar{x})$. Let $N\lceil_L \models \bigwedge\Phi(\bar{a})$. By definition, this means $N\lceil_L \models \psi(\bar{a})$ for all $\psi \in \Phi$. By induction hypothesis, $N \models \psi^*(\bar{a})$ for all $\psi \in \Phi$. Since N is in STR, N omits Γ(Definition 4). Then $N \models P_\Phi(\bar{a})$. Conversely, let $N \models P_\Phi(\bar{a})$. Because N is in STR, $N \models T_1$. So we get $N \models \phi^*(\bar{a})$ for all $\phi \in \Phi$. By induction hypothesis, $N\lceil_L \models \phi(\bar{a})$ for all $\phi \in \Phi$. Therefor $N\lceil_L \models \bigwedge\Phi(\bar{a})$. The other cases are straightforward. ∎

Corollary 7 Suppose $\phi \in L_F$, $\Sigma \subset L_F$.

1. $\mathrm{Im}(*) = \mathrm{STR}$, $*^{-1} = \lceil_L$.

2. $N \in \mathrm{St} \Rightarrow N \models T_2$.

3. $\Sigma \models \phi \iff$ For all $M \in \mathrm{STR}$, if $M \models \Sigma^*$ then $M \models \phi^*$.

Proof: 2. We want to show that $N \models P_{\Phi'}(\bar{a}) \leftrightarrow P_\Phi(\bar{t}(\bar{a}))$ for all $\bar{a} \in N$. Let $N \models P_{\Phi'}(\bar{a})$. Then $N\lceil_L \models \bigwedge\Phi'(\bar{a})$ by proposition 6. Take $\bar{b} = \bar{t}(\bar{a}) \in N$. Since $\Phi'(\bar{a}) = \Phi(\bar{t}(\bar{a}))$, we get $N\lceil_L \models \bigwedge\Phi(\bar{b})$. Again by proposition 6, $N \models P_\Phi(\bar{b})$. This implies $N \models P_\Phi(\bar{t}(\bar{a}))$. The other direction is the same. ∎

Proposition 6 and Corollary 7 say that you can consider the model theory of STR instead of the model theory of L_F.

4 Complete metric

4.1 G_δ

Definition 8 Suppose T is a set of L_F-sentences.

1. $\Sigma(\bar{x}) \subset L_F$ is an n-type with respect to T if $|\bar{x}| = n$ and there are a L-structure M and elements $\bar{a} \in M$ such that $M \models \phi(\bar{a})$ for all $\phi(\bar{x}) \in \Sigma(\bar{x})$.

2. $S_n(L_F, T)$ is the set of all complete(in L_F) n-types w.r.t. T.

In model theory for first order logic, the space of types $S_n(T)$ is compact. Morley showed $S_n(L_F, T)$ is analytic where L_F is countable[2]. A topological space is analytic if it is a image of continuous function of a Borel set. In this section, I introduce G_δ subsets. Clearly every G_δ subset is a Borel set then it is analytic. We will see $S_n(L_F, T)$ is a G_δ subset of a stone space.

Definition 9 Let S be a topological space, and $A \subset S$. Then A is called a G_δ subset of S if there are countably many open sets $O_i (i < \omega)$ such that $A = \cap_{i<\omega} O_i$.

The next fact is well known. For example, see [4].

Fact 10 Let S be completely metrizable and $A \subset S$. Then A is G_δ if and only if A is completely metrizable.

4.2 $S_n(L_F, T)$ is G_δ

First, I claim that we can consider $S_n(L_F, T)$ as a subset of $S_n(T^*)$.

Lemma 11 If $\Sigma(\bar{x}) \subset L_F$ is finitely satisfiable, then $\Sigma^* \cup T_1 \cup T_2$ is also finitely satisfiable. If $\Sigma(\bar{x}) \subset L_F$ is finitely satisfiable and complete in L_F, then $\Sigma^* \cup T_1 \cup T_2$ has a unique completion in $L(\tau)$.

Proof: If $M \models \phi(\bar{a})$, then $M^* \models \phi^*(\bar{a})$ by definition of $*$. Moreover, $M^* \in$ STR.. Then $M \models \cup T_1 \cup T_2$ by Corollary 7. (Remark 5 implies the uniqueness of completion.) ∎

Recall Γ is the set of $L(\tau)$-types(See Definition 4). If $\Sigma(\bar{x}) \subset L_F$ is consistent, then there is a model $M \models \Sigma(\bar{a})$. Then M^* must be omits Γ and $M^* \models \Sigma^*(\bar{a}) \cup T_1$. Conversely, If there is a model $N \models \Sigma^*(\bar{a}) \cup T_1$ which omits Γ, then $N \lceil_L \models \Sigma(\bar{a})$. This fact implies that $S_n(L_F, T)$ is a G_δ subset of a stone space by Omitting Types Theorem. We'll see this in Proposition 12 and Theorem 14.

Proposition 12 Suppose L_F is countable and $p(\bar{x})$ is a set of L_F-formulas. Let $p(\bar{x})$ be finitely satisfiable and complete in L_F. The following are equivalent.

1. $p(\bar{x})$ is consistent.

2. For every $q(\bar{x}, \bar{y}) \in \Gamma$, q is nonisolated w.r.t. $p(\bar{x})^* \cup T_1 \cup T_2$.

3. For every $\bigwedge \Phi(\bar{x},\bar{y}), \psi(\bar{x},\bar{y}) \in L_F$, either of (a) or (b) holds.
   (a) $\forall \bar{y}(\psi(\bar{x},\bar{y}) \rightarrow \bigwedge \Phi(\bar{x},\bar{y})) \in p(\bar{x})$
   (b) $\exists \bar{y}(\psi(\bar{x},\bar{y}) \wedge \neg\phi(\bar{x},\bar{y})) \in p(\bar{x})$ for some $\phi \in \Phi$.

Proof:
$(2 \rightarrow 1)$
By omitting types, we can take a model $M \models p^*(\bar{a}) \cup T_2$ which omits Γ. Then $M\lceil_L \models p(\bar{a})$ by Proposition6.
$(1 \rightarrow 3)$
Notice $\{\psi \rightarrow \phi | \phi \in \Phi\} \models \psi \rightarrow \bigwedge \Phi$. So, if (b) does't hold then (a) holds by the consistency and the completeness of $p(\bar{x})$.
$(3 \rightarrow 2)$
Since $p(\bar{x})$ is complete, $p^*(\bar{x}) \supset T_1$. Because of Lemma11, we can assume $p^*(\bar{x}) \cup T_2$ is a complete consistent in $L(\tau)$. Let $q(\bar{x},\bar{y}) = \{\neg P_\Phi(\bar{x},\bar{y})\} \cup \Phi^*(\bar{x},\bar{y})$ be isolated w.r.t. $p^*(\bar{x}) \cup T_2$. Then we can find a $L(\tau)$-formula $\psi'(\bar{x},\bar{y})$ such that ψ' isolates q. By Remark 5, there is a L_F-formula $\psi(\bar{x},\bar{y})$ such that $T_2 \models \psi^* \leftrightarrow \psi'$. By 3., either $\forall \bar{y}(\psi^*(\bar{x},\bar{y}) \rightarrow P_\Phi(\bar{x},\bar{y})) \in p^*(\bar{x})$ or $\exists \bar{y}(\psi^*(\bar{x},\bar{y}) \wedge \neg\phi^*(\bar{x},\bar{y})) \in p^*(\bar{x})$ for some $\phi^* \in \Phi^*$ holds. But ψ^* isolates q. Then $p(\bar{x})^* \cup T_2$ must be inconsistent. ∎

Corollary 13 Suppose L_F and $p(\bar{x})$ satisfy the assumption of Proposition12. Let L_F satisfy following condition (a)(b).
(a) $\psi, \bigwedge \Phi \in L_F \Rightarrow \bigwedge\{\psi \rightarrow \phi\}_{\phi \in \Phi} \in L_F$.
(b) $\bigwedge \Phi(x,\bar{y}) \in L_F \Rightarrow \bigwedge\{\forall x \phi(x,\bar{y})\}_{\phi \in \Phi} \in L_F$.
Then TFAE.

1. $p(\bar{x})$ is consistent.
2. For all $\bigwedge \Phi(\bar{x}) \in L_F$, if $\neg \bigwedge \Phi(\bar{x}) \in p(\bar{x})$ then there is $\phi \in \Phi$ such that $\neg\phi(\bar{x}) \in p(\bar{x})$. ∎

Theorem 14 Suppose $T \subset L_F$, and $|L_F| \leq \omega$.
Then $S_n(L_F, T)$ is completely metrizable.

Proof: First I prove at $T = \emptyset$. Let $D_n = \{p(\bar{x}) | p(\bar{x})$ is complete(in L_F) and finitely satisfiable, $|x| \leq n.\}$(I remark p may not be consistent). Then D_n will be a stone space and $S_n(L_F, \emptyset)$ is a subset of D_n. Because D_n is a second countable stone space, it is completely metrizable. Let's show $S_n(L_F, \emptyset)$ is a G_δ subset of D_n. By Proposition 12, we can take

$$S_n(L_F, \emptyset) = \cap_{\bigwedge \Phi(\bar{x},\bar{y}), \psi(\bar{x},\bar{y}) \in L_F} \cup_{\phi \in \Phi} O_{\forall \bar{y}(\psi \rightarrow \bigwedge \Phi)) \vee \exists \bar{y}(\psi \wedge \neg\phi))}.$$

So $S_n(L_F,)$ is completely metrizable by Fact 10. If $T \neq \emptyset$, we can take $S_n(L_F, T) = S_n(L_F, \emptyset) \cap \cap_{\phi \in T} O_\phi$. ∎

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