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On Coheir Sequences

– Indiscernible Array, Tree and Forest –

Akito Tsuboi
Institute of Mathematics
University of Tsukuba

1 Preliminaries

We study indiscernibility. Indiscernible sequences are important objects in model theory. They are essentially used when defining certain important notions in model theory, e.g. dividing, forking and simplicity are all defined via indiscernible sequences. A coheir sequence is a special type of an indiscernible sequence.

Indiscernible array and tree are also important objects, and they generalize the notion of indiscernible sequence. Roughly, we say that an infinite matrix $(a_{ij})_{ij}$ is an indiscernible array if both the columns and the rows of the matrix form an indiscernible sequence. Similarly a set $(a_\eta)_{\eta\in I}$ with the index set I having a tree order structure is called an indiscernible tree if it has some indiscerniblility condition.

In this note, we discuss when we can find an indiscernible array (or an indiscernible tree) with a given property. More precisely, we are interested in the following type of questions:

1. Let $X=(x_{ij})$ be a set of variables. Let $\Gamma(X)$ be a set of conditions (written by formulas whose free variables are among X). Is it possible to realize Γ by an indiscernible array?

2. Let $Y=(y_\eta)_{\eta in I}$ be a set of variables, where I is a tree. Let $\Delta(Y)$ be a set of conditions. Is it possible to realize Δ by an indiscernible tree?

First we give some general results concerning these questions. Later we apply our results to the study of theories with $\kappa_{\mathrm{inp}}(T)<\infty$.

Let us recall some basic definitions.

Definition 1 1. Let $M \subset A$. We say that $p(x) \in S(A)$ is a coheir extension of $p|A$, if $p(x)$ is finitely satisfiable in M.

2. Let $p(x)$ be a finitely satisfiable type in M. We say that $I = (a_i)_{i \in \omega}$ is a coheir sequence over M in $p(x)$ if the following hold for all i,

   (a) $\mathrm{tp}(a_{i+1}/Aa_0, ..., a_i)$ is a coheir extension of $p(x)$.

   (b) $\mathrm{tp}(a_{i+1}/Aa_0, ..., a_i) \supset \mathrm{tp}(a_i/Aa_0, ..., a_{i-1})$.

The following are easily verified.

Fact 2 *1. If I is a coheir sequence over M in $p \in S(A)$ then I is an indiscernible sequence over A in p.*

2. A coheir sequence can be extended to an arbitrary length.

2 Existence of Indiscernible Array

Definition 3 Let I and J be indiscernible sequences over A.

1. We say that $(a_{ij})_{i \in \Gamma, j \in \Delta}$ is an array of type (I, J) if every $I_i = (a_{ij})_{j \in \Delta}$ is isomorphic to I over A and every $J_j = (a_{ij})_{i \in \Gamma}$ is isomorphic to J over A.

2. Moreover, we say that the array is an indiscernible array over A, if

   (a) $(I_i)_{i \in \Gamma}$ is an indiscernible sequence over A, and

   (b) $(J_j)_{j \in \Delta}$ is an indiscernible sequence over A.

Remark 4 1. If there is an array of type (I, J), then there is an A-indiscernible array of type (I, J). This can be shown by an easy argument using Ramsey's theorem.

2. If $(a_{ij})_{i,j \in \omega}$ is an A-indiscernible array, then for any increasing $\eta, \nu \in \omega^\omega$, we have $\mathrm{tp}((a_{\eta(i),\nu(j)})_{i,j \in \omega}) = \mathrm{tp}((a_{ij})_{i,j \in \omega})$

3. Let I be any indiscernible sequence over A. Then there is an indiscernible array of type (I, I). This can be shown by extending I to an indiscernible sequence of length ω^2.

Proposition 5 *Let $p(x) \in S(M)$. Let I be an M-indiscernible sequence in $p(x)$. Then there is an array $(a_{ij})_{i,j\in\omega}$ and an M-coheir sequence J such that $(a_{ij})_{i,j\in\omega}$ is an M-indiscernible array of type (I, J). Moreover, we can choose $(a_{ij})_{i,j\in\omega}$ so that $(a_{i,\eta(i)})_{i\in\omega} \cong_M J$ holds for any $\eta \in \omega^\omega$.*

Proof: Let $I_0^* = (b_j)_{j\in\omega^2}$ be an extension of I with the order type ω^2. We prepare two sets of variables $X_0 = \{x_i : i \in \omega^2\}$ and $X_1 = \{y_i : i \in \omega^2\}$. Let $\Gamma(X_0, X_1)$ be the following set of formulas.

$$\text{“}X_0 \cong_M X_1 \cong_M I_0^*\text{”} \cup \bigcup_{\substack{F,G\subset\omega^2 \\ \varphi(x_F,y_G)\in L(M)}} \{\neg\varphi(x_F, y_G) : \varphi(b_F, y_G) \text{ not satisfiable in } M\}.$$

Choose $I_1^* = (c_j)_{j\in\omega^2}$ such that $\mathrm{tp}(I_1^*/MI_0^*)$ is a coheir extension of $\mathrm{tp}(I_0^*/M)$. Then I_0^*, I_1^* satisfies $\Gamma(X_0, X_1)$. Moreover, $(b_{n_j})_{j\in\omega^2}(c_{n_j})_{j\in\omega^2}$ satisfies Γ, whenever $\{n_j : j \in \omega^2\}$ is an increasing sequence. So, by a compactness argument using Ramsey's theorem, we can assume that $(b_j c_j)_{j\in\omega^2}$ forms an indiscernible sequence. By continuing this process, we can find copies $I_i^* = (b_{ij})_{j\in\omega^2}$ of I_0^* such that

1. $(I_i^*)_{i\in\omega}$ is a coheir sequence over M,
2. $(J_j^*)_{j\in\omega^2}$ is an indiscernible sequence over M, where $J_j^* = (b_{ij})_{i\in\omega}$.

For $i, j \in \omega$, we put

$$\begin{aligned} a_{ij} &= b_{i,\omega i+j}, \\ I_i &= (a_{ij})_{j\in\omega}, \\ J_j &= (a_{ij})_{i\in\omega}. \end{aligned}$$

Claim A *$(I_i)_{i\in\omega}$ is an M-coheir sequence.*

By condition 1, clearly $\mathrm{tp}(I_{i+1}/MI_0\cdots I_i)$ is finitely satisfiable. Again by condition 1, $\mathrm{tp}(I_{i+1}/MI_0\cdots I_{i-1}) = \mathrm{tp}((b_{i,\omega i+j})_{j\in\omega}/MI_0\cdots I_{i-1})$. By condition 2, $\mathrm{tp}((b_{i,\omega i+j})_{j\in\omega}/MI_0\cdots I_{i-1}) = \mathrm{tp}(I_i/MI_0\cdots I_{i-1})$. Thus $\mathrm{tp}(I_{i+1}/MI_0\cdots I_{i-1}) = \mathrm{tp}(I_i/MI_0\cdots I_{i-1})$, and hence $(I_i)_{i\in\omega}$ is an M-coheir sequence.

Similarly, we can show that $(J_j)_{j\in\omega}$ is an M-indiscernible sequence. Further, by condition 2 above, the type of $(b_{i,\nu(i)})_{i\in\omega}$ over M is fixed for all strictly increasing sequence $\nu \in (\omega^2)^\omega$. However, for any $\eta \in \omega^\omega$, the ordinal $\omega i + \eta(i)$ increases as i increases. Hence the moreover part holds.

Definition 6 Let $O \subset \omega^{<\omega}$.

1. Let $L_0 = \{<_{\mathrm{len}}, <_{\mathrm{ini}}, <, \cap, Q\}$ and $L_1 = L_0 \cup \{P_n : n \in \omega\}$. We consider the following structure on O:

   (a) $P_n(\eta) \iff \mathrm{len}(\eta) = n$ i.e. the length of η is n. $(n \in \omega)$.

   (b) $\eta <_{\mathrm{len}} \nu \iff \mathrm{len}(\eta) < \mathrm{len}(\nu)$.

   (c) $\eta <_{\mathrm{ini}} \nu \iff \eta$ is a proper initial segment of ν.

   (d) $\eta < \nu \iff \eta$ is less than ν in the lexicographic order.

   (e) $\eta \cap \nu =$ the longest common initial segment of η and ν.

2. Let $X, Y \subset O$ be two finite subsets. We write $X \sim_1 Y$ if X and Y have the same atomic type with respect to L_1. We write $X \sim_0 Y$ if X and Y have the same atomic type with respect to L_0.

3. Let H be an infinite subset of ω and let $\{h_i : i \in \omega\}$ be the enumeration of H in increasing order. For a sequence $\eta = \langle \eta(0), ..., \eta(l-1)\rangle$ of length l, we define $\sigma_H(\eta)$ of length $h_{l-1}+1$ by

$$\sigma_H(\eta)(i) = \begin{cases} \eta(j) & \text{if } i = h_j, \\ 0 & \text{otherwise.} \end{cases}$$

   In different notation,

$$\sigma_H(\eta) = \langle 0^{h_0}, \eta(0)\rangle ^\frown \langle 0^{h_1-h_0-1}, \eta(1)\rangle ^\frown \cdots ^\frown \langle 0^{h_{l-1}-h_{l-2}-1}, \eta(l-1)\rangle,$$

   where 0^l denotes the l-time iteration of 0. σ_H is an order preserving mapping with respect to $<_{\mathrm{len}}$, $<_{\mathrm{ini}}$ and $<$.

Remark 7 1. Let us consider the case $h_0 = 1$ and $h_1 = 3$. For simplicity, we write η^* for $\sigma_H(\eta)$. Then $\langle 0,1\rangle \cap \langle 0,0\rangle = \langle 0\rangle \cap \langle 0\rangle$, but $\langle 0,1\rangle^* \cap \langle 0,0\rangle^* = \langle 0,0,0,1\rangle \cap \langle 0,0,0,0\rangle = \langle 0,0,0\rangle$ and $\langle 0\rangle^* \cap \langle 0\rangle^* = \langle 0,0\rangle \cap \langle 0,0\rangle = \langle 0,0\rangle$. So the mapping σ_H does not preserve the operation $\cap$.

2. For $\eta_i, ..., \eta_n$, there are $i \leq j \leq n$ such that $\eta_1 \cap \cdots \cap \eta_n = \eta_i \cap \eta_j$. So L_0-atomic formulas are assumed to have the form $\eta \cap \nu = \eta_1 \cap \nu_1$ or $\eta \cap \nu <^* \eta_1 \cap \nu_1$, where $<^*$ is either one of $<$, $<_{\text{len}}$, $<_{\text{ini}}$.

Lemma 8 *Let $h, k : \omega \to \omega \setminus \{0\}$ be two increasing functions satisfying $h(i+1) - h(i) > 1$ and $k(i+1) - k(i) > 1$ for all i. Let $H = \mathrm{ran} f$ and $K = \mathrm{ran} k$. Then, for any $X, Y \subset \omega^{<\omega}$ with $X \sim_0 Y$, we have $\sigma_H(X) \sim_0 \sigma_K(Y)$.*

Proof: We show that the L_0-atomic type of $\sigma_H(X)$ is determined by the L_0-atomic type of X without using specific properties of H. We simply write η^* for $\sigma_H(\eta)$. Let φ be a given atomic $L_0(X^*)$-formula. We describe how the validity of φ is determined. Since other cases are treated similarly, we only consider typical cases. First we consider the formula $\eta^* \cap \nu^* = \eta_1^* \cap \nu_1^*$, where $\eta, \nu, \eta_1, \nu_1 \in X$. By the definition of σ_H, we have

$$\eta^* \cap \nu^* = \begin{cases} (\eta \cap \nu)^{*\frown} 0^m & \text{if } \eta \perp \nu, \\ (\eta \cap \nu)^* & \text{otherwise,} \end{cases}$$

where $m = h(\mathrm{len}(\eta \cap \nu)) - h(\mathrm{len}(\eta \cap \nu) - 1) - 1 > 0$, and $\eta \perp \nu$ is the formula expressing that η and ν are not $<_{\text{ini}}$-comparable. In particular, we have $\mathrm{len}(\eta^* \cap \nu^*) = h(\mathrm{len}(\eta \cap \nu))$ if $\eta \perp \nu$ and $\mathrm{len}(\eta^* \cap \nu^*) = h(\mathrm{len}(\eta \cap \nu) - 1) + 1$ otherwise. From this we see

$$\eta^* \cap \nu^* = \eta_1^* \cap \nu_1^* \iff \eta \cap \nu = \eta_1 \cap \nu_1 \text{ and } \begin{cases} \eta \perp \nu \text{ and } \eta_1 \perp \nu_1 \\ \text{or} \\ \neg(\eta \perp \nu) \text{ and } \neg(\eta_1 \perp \nu_1). \end{cases}$$

The right-hand side does not depend on H, and whether or not it holds is determined by the L_0-atomic type of X. We also see

$$\eta^* \cap \nu^* <_{\text{ini}} \eta_1^* \cap \nu_1^* \iff \eta \cap \nu <_{\text{ini}} \eta_1 \cap \nu_1 \text{ or } \begin{cases} \neg(\eta \perp \nu) \text{ and } \eta_1 \perp \nu_1 \\ \text{and} \\ \eta \cap \nu = \eta_1 \cap \nu_1 \end{cases}$$

$$\eta^* \cap \nu^* <_{\text{len}} \eta_1^* \cap \nu_1^* \iff \eta \cap \nu <_{\text{len}} \eta_1 \cap \nu_1 \text{ or } \begin{cases} \neg(\eta \perp \nu) \text{ and } \eta_1 \perp \nu_1 \\ \text{and} \\ \eta \cap \nu \leq_{\text{len}} \eta_1 \cap \nu_1 \end{cases}$$

$$\eta^* \cap \nu^* < \eta_1^* \cap \nu_1^* \iff \eta^* \cap \nu^* <_{\text{ini}} \eta_1^* \cap \nu_1^* \text{ or } \eta \cap \nu < \eta_1 \cap \nu_1.$$

Again the right-hand side is determined by the L_0-atomic type of X. So the L_0-type of X^* is completely determined by that of X.

Definition 9 Let $A = \{a_\eta : \eta \in \omega^{<\omega}\}$.

1. We say that A is an indiscernible tree over B if whenever $X \sim_0 Y$ then $a_X \cong_B a_Y$, where $a_X = \{a_\eta : \eta \in X\}$.

2. We say that A is a weakly indiscernible tree over B if whenever $X \sim_1 Y$ then $a_X \cong_B a_Y$.

3. We say that an indiscernible tree A over B is a strongly indiscernible tree, if whenever H is an infinite subset of ω, then the subtree determined by H is B-isomorphic to A, i.e. $A \cong_B \sigma_H(A)$, where $\sigma_H(A) = (a_{\sigma_H(\eta)})_{\eta\in\omega^{<\omega}}$. .

Proposition 10 *Let $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ be a set of $L(B)$-formulas with free variables among x_η's. Suppose that Γ has the subtree property in the following sense: If A realizes Γ then any subtree of the form $\sigma_H(A)$ also realizes Γ. Then the following conditions are equivalent:*

1. Γ is realized by a weakly indiscernible tree over B,

2. Γ is realized by an indiscernible tree over B,

3. Γ is realized by a strongly indiscernible tree over B.

Proof: The implications $3 \Rightarrow 2 \Rightarrow 1$ are trivial. To prove $1 \Rightarrow 3$, assume $(a_\eta)_{\eta\in\omega^{<\omega}}$ is a weakly indiscernible tree satisfying Γ. We prove that the strong indiscernibility is attained by modifying $(a_\eta)_{\eta\in\omega^{<\omega}}$. Our aim is to prove the finite satisfiability of the union $\Gamma \cup \Delta \cup \Theta$:

$$\Delta((x_\eta)_{\eta\in\omega^{<\omega}}) = \bigcup_{\varphi\in L(M)} \{\varphi(x_X) \leftrightarrow \varphi(x_Y) : X, Y \subset \omega^{<\omega}, X \sim_0 Y\},$$

$$\Theta((x_\eta)_{\eta\in\omega^{<\omega}}) = \bigcup_{\varphi\in L(M)} \{\varphi(x_X) \leftrightarrow \varphi(x_{\sigma_K(X)}) : X \subset \omega^{<\omega},\ K \subset \omega,\ |K| = \omega\}.$$

Let $Z \subset \omega^{<\omega}$ be a finite set and $n = |\{\mathrm{len}(\eta) : \eta \in Z\}|$. For an n-element subset F of ω, by the weak indiscernibility, the L-type of a_X with $X \sim_0 Z$ and $\{\mathrm{len}(\eta) : \eta \in Z\} = F$ depends only on F (does not depend on the specific choice of X). Fix L-formulas $\varphi_1, ..., \varphi_k$ with free variables $x_0, ..., x_{|X|-1}$. By the above argument, the following function $f : [\omega]^n \to 2^k$ is well-defined:

$$f(F) = (f_1(F), ..., f_k(F)),$$

$$f_i(F) = \begin{cases} 1 & \varphi_i(a_X) \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

By Ramsey's theorem, we can get a homogeneous set $H = \{h_i : i \in \omega\} \subset \omega$ for this f. We can assume that $h_{i+1} - h_i > 1$ for every i.

Now we define the tree $(b_\eta)_{\eta\in\omega^{<\omega}}$ by

$$b_\eta = \sigma_H(a_\eta) = a_{\sigma_H(\eta)}.$$

By the subtree property of Γ, $(b_\eta)_{\eta\in\omega^{<\omega}}$ satisfies Γ. So what is left to show is that $(b_\eta)_{\eta\in\omega^{<\omega}}$ satisfies the restriction of $\Delta \cup \Theta$ to the formulas $\varphi_1, ..., \varphi_k$. But this holds by Lemma 8.

Corollary 11 *Let $p(x) \in S(M)$. Let I be an M-indiscernible sequence in p. Let $J = (b_j)_{j\in\omega}$ be an M-indiscernible sequence in p such that $\mathrm{tp}(b_0 \cdots b_n/Mb_{n+1})$ does not divide over M for every n. Then there is a tree $(a_\eta)_{\eta\in\omega^{<\omega}}$ such that*

1. *$(a_{\nu|j})_{j\in\omega} \cong_M J$ for every path ν.*
2. *$(a_{\eta\frown i})_{i\in\omega} \cong_M I$ for every $\eta \in \omega^{<\omega}$.*

Furthermore we can choose $(a_\eta)_{\eta\in\omega^{<\omega}}$ as a strongly indiscernible tree over M.

Proof: The conditions 1 and 2 are expressed by a set of formulas, say Γ. The existence of $(a_\eta)_{\eta\in\omega^{<\omega}}$ satisfying $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ is easily proven. By Theorem 2.6 of [Sh, AP., §2], since the length of J can be made arbitrarily long, we may assume that the tree $(a_\eta)_{\eta\in\omega^{<\omega}}$ is a weakly indiscernible tree. It is easy to see that $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ has the subtree property. So, by proposition 10, we can choose a strongly indiscernible tree $(a_\eta)_{\eta\in\omega^{<\omega}}$ satisfying Γ.

Proposition 12 *Let $p(x) \in S(M)$. Let $I = (a_\eta)_{\eta\in\omega^{<\omega}}$ be an indiscernible tree in p. Then there are copies $I_i = (a_{i,\eta})_{\eta\in\omega^{<\omega}}$ of I such that*

1. *$(I_i)_{i\in\omega}$ forms an M-indiscernible sequence,*
2. *$(a_{i,\eta(i)})_{i\in\omega}$ is an M-indiscernible sequence, for any sequence $\eta \in (\omega^{<\omega})^\omega$, and the type of $(a_{i,\eta(i)})_{i\in\omega}$ over M is fixed.*

Proof: A similar method as used in proposition 5 can be applied. Let $O = \lambda^{<\lambda}$, where $\lambda \geq \omega^2$. Preparing $X = (x_\eta)_{\eta\in O}$ of variables, we consider the set $\Gamma(X)$ consisting of

3. "$(x_\eta)_{\eta\in\omega^{<\omega}} \cong_M I$",

4. $\bigcup_{\varphi\in L(M)}\{\varphi(x_F) \leftrightarrow \varphi(x_G) : F, G \subset O, F \sim_0 G\}$.

Since I is an indiscernible tree, Γ is consistent. So we can find $I^* = (b_\eta)_{\eta\in O}$ satisfying Γ. We put

$$a_{i,\eta} = b_{\mu_i{}^\frown\eta},$$

where μ_i is the constant sequence $\langle \omega i, \omega i, \cdots\rangle$ of length ωi. By letting $I_i = (a_{i,\eta})_{\eta\in\omega^{<\omega}}$, we show that the requirements are fullfilled. First note that $O_0 = \omega^{<\omega}$ and $O_i = \{\mu_i{}^\frown\eta : \eta \in O_0\}$ have the same L_0-atomic type. So I_i's are all isomorphic to I over M. Moreover, $(O_i)_{i\in\omega}$ is an L_0-atomic indiscernible sequence. From this we know that $(I_i)_{i\in\omega}$ is an M-indiscernible sequence. Further, for any sesquence $\eta \in {O_0}^\omega$, $(\mu_i{}^\frown\eta(i))_{i\in\omega}$ is an L_0-atomic indiscernible sequence and the L_0-atomic type is fixed. Hence, $(a_{i,\eta(i)})_{i\in\omega}$ is an M-indiscernible sequence, for any $\eta \in \omega^\omega$, and the type of $(a_{i,\eta(i)})_{i\in\omega}$ over M is fixed.

3 Application to theories with $\kappa_{\mathrm{inp}}(T) < \infty$

In this section we assume $\kappa_{\mathrm{inp}}(T) < \infty$.

Fact 13 *The following are equivalent:*

1. *$\varphi(x, a)$ divides over M;*

2. *There is a coheir sequence J over M with $a \in J$ such that $\{\varphi(x, b) : b \in J\}$ is inconsistent.*

Proof: Suppose that $\varphi(x, a)$ divides over M and choose an M-indiscernible sequence I in $\mathrm{tp}(a/M)$ such that $\{\varphi(x, b) : b \in I\}$ is inconsistent. Then, by Proposition 5, choose an indiscernible array $(a_{ij})_{i,j\in\omega}$ such that $I_i = (a_{ij})_{j\in\omega}$ is isomorphic to I and that each $(a_{i\eta(i)})_{i\in\omega}$ is a coheir sequence. By $\kappa_{\mathrm{inp}}(T) < \infty$, there must be a path η such that $\{\varphi(x, a_{i,\eta(i)}) : i \in \omega\}$ is consistent.

Proposition 14 *(Chernikov-Kaplan) $\theta(x)$ divides over M if and only if $\theta(x)$ forks over M.*

Proof: For this, we show that

(*) if each of $\varphi(x,a)$ and $\psi(x,a)$ divides over M, then $\varphi(x,a) \vee \psi(x,a)$ divides over M.

Choose M-indiscernible sequences I witnessing the dividing of φ and $J = (b_i)_{i\in\omega}$ witnessing the dividing of ψ. By Fact 13 and compactness, we can assume that $\mathrm{tp}(b_0 \cdots b_n/Mb_{n+1})$ does not fork over M $(n \in \omega)$. By Corollary 11, there is a strongly indiscernible tree $A = (a_\eta)_{\eta\in\omega^{<\omega}}$such that

1. $(a_{\nu|j})_{j\in\omega} \cong_M J$ for every path ν.

2. $(a_{\eta^\frown i})_{i\in\omega} \cong_M I$ for every $\eta \in \omega^{<\omega}$.

By Proposition 12, there are $A_i = (a_{i,\eta})_{\eta\in\omega^{<\omega}}$ $(i \in \omega)$ such that for any $\eta \in \omega^\omega$, $I_\eta = (a_{i,\eta(i)})_{i\in\omega}$ is an M-indiscernible sequence and that I_η's are mutually isomorphic over M.

Claim A *For any η, $\{\varphi(x,b) : b \in I_\eta\}$ is inconsistent.*

Assume otherwise. Since I_η's are all isomorphic, $\{\varphi(x,b) : b \in I_\eta\}$ is consistent, for any η. Now we consider sequences

$$I_i = (a_{i,\langle j\rangle})_{j\in\omega} \quad (i \in \omega).$$

By the condition 2 and the the indiscernibility, we have $I_i \cong_M I$. So, for some k, every $\{\varphi(x,b) : b \in I_i\}$ is k-inconsisitent. On the other hand, by our assumption, $\{\varphi(x, a_{i,\eta(i)}) : i \in \omega\}$ is consistent, for any $\eta \in \omega^\omega$. So we would have $\kappa_{\mathrm{inp}}(T) = \infty$, a contradiction.

For the same reason as in the above claim, we can show that $\{\psi(x,b) : b \in I_\eta\}$ is inconsistent. Thus we see that $\{\varphi(x,b) \vee \psi(x,b) : b \in I_\eta\}$ is inconsistent.

4 Examples

Example 15 Let T be the theory of the binary tree $M = (2^{<\omega}, <)$ of height ω. Let $\mathcal{M}$ be an elementary extension of M and choose $\eta \in \mathcal{M} \setminus M$. η is a $\{0,1\}$-sequence of infinite length, say α. For simplicity, we assume $\eta(i) = 0$ for any $i < \alpha$. We consider the formula $\eta < x$ with the free variable x. First we show that this formula divides over M: For $n \in \omega$, let η_n be the n-th predecessor of η and define ν_n by

$$\nu_n = \eta_n{}^\frown \underbrace{\langle 1, ..., 1\rangle}_{n}$$

The set $\{\nu_n < x : n \in \omega\}$ is 2-inconsistent. So $\eta < x$ divides over M.

Moreover, by compactness, there is an indiscernible sequence I such that

$$\{\nu < x : \nu \in I\}$$

is 2-inconsistent. It is not hard to show that I is a coheir sequence. Notice that two elements from I are not comparable. On the other hand, there is a coheir sequence $J = \{\iota_n : n \in \omega\}$ such that

$$\iota_{n+1} < \iota_n \quad (n \in \omega).$$

Then $\{\iota_n < x : n \in \omega\}$ is consistent.

Example 16 (Essentially in [3]) Let us consider the following example without the independence property, which is essentially given by Shelah. Let $M = (M, R)$ be a connected graph having the following properties:

1. M has no cycle;
2. Every point in M has infinitely many R-neighbors.

Let P be the 3-ary relation defined by:

$$P(a, b, c) \iff \text{every path between } a \text{ and } c \text{ passes through } b.$$

We consider M as a $\{P, R\}$-structure. Choose $a, b \in M$ with $R(a, b)$, then we have

$$x = x \vdash P(x, a, b) \vee P(x, b, a).$$

We claim that $P(x, a, b)$ 2-divides over $\emptyset$. Choose infinitely many distinct a_n $(n \in \omega)$ with $R(a_n, b)$. Then $\{P(x, a_n, b) : n \in \omega\}$ is 2-inconsistent, since there is no cycle in M. From this we conclude that the formula $x = x$ forks over $\emptyset$.

Let $d \in M$ be an arbitrary element. Since M is a connected cycle-less graph, we have $P(d, a, b)$ or $P(d, b, a)$. We assume $P(d, a, b)$ holds. If $\mathrm{tp}(ab/d) = \mathrm{tp}(a'b'/d)$ then we have d is a common solution of $P(x, a, b)$ and $P(x, a', b')$. So $P(x, a, b)$ does not divide over d. In case $P(d, b, a)$, the formula $P(x, b, a)$ does not divide over d.

Example 17 (Forklore?) Let R be a binary relation symbol. A symmmetric irreflexive R-structure will be referred as a graph. We say that three points a, b, c in a graph form a triangle, if $R(a,b) \wedge R(b,c) \wedge R(c,a)$ holds. Let M be a random graph without triangles. To define M more precisely, let K be the class of all finite graphs without triangles. K has the amalgamation property, so there is a unique K-generic structure M. The theory $T = \mathrm{Th}(M)$ is not simple, but we have $\kappa_{\mathrm{inp}}(T) < \infty$.

We work in a large elementary extension of M. We consider the formula $R(x,a)$. Let $I = \{a_i : i \in \omega\}$ be an indiscernible sequence over A with $a_0 = a$. By the indiscernibility, there are no R-edges in I. So we can find b satisfying $\{R(x, a_i) : a_i \in I\}$, since b does not make a triangle. This shows that $R(x,a)$ does not divide A.

Now we see that $R(x, a_0) \wedge R(x, a_1)$ divides over A. Let $J = \{a_{0j}a_{1j} : j \in \omega\}$ be an A-indiscernible sequence such that $R(a_{ij}, a_{kl}) \iff i \neq k \text{ and } j \neq l$ for any i, j, k, l. Suppose that $\{R(x, a_{0j}) \wedge R(x, a_{1j}) : j \in \omega\}$ is satisfied by an element d. Then there would be a triangle d, a_{00}, a_{11}. This is a contradiction.

References

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