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An application of the decomposition theorem

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Abstract

In this paper we present an application of the decomposition theorem for abstract elementary classes. We give a list of conditions implying that any two models of cardinality λ which are $L_{\infty,\lambda}$-equivalent are isomorphic (for a large enough λ). A similar version was proved by Shelah for first order theories in [5].

In [2], Rami Grossberg and Olivier Lessmann proposed a number of axioms in order to lift and generalize the decomposition theorem, first proved by Shelah in [5], to abstract elementary classes (hereafter AEC). In this paper we present an application of this abstract version of the theorem. We show that if an AEC satisfies a similar setting to the one proposed in [2] then any two models of cardinality λ which are $L_{\infty,\lambda}$-equivalent are isomorphic (for a large enough λ). At least two main differences between [2] and the approach here outlined are important to mention. Firstly, the choice of axioms is slightly different. Secondly, an additional condition is added to the definition of decomposition. Although these differences will not change the application here discussed, they were needed to reach a detailed and gapless proof of the abstract version of the decomposition theorem. For more details about this see [3].

The notation will be standard. We work in an AEC $(\mathcal{K}, \prec)$ with the amalgamation property and arbitrary large models. This enables us to fix a $\bar{\kappa}$-universal and strongly $\bar{\kappa}$-model-homogeneous (hence $\bar{\kappa}$-Galois saturated) model $\mathfrak{C} \in \mathcal{K}$ for big enough cardinal $\bar{\kappa}$. Every set and structure is assumed to be respectively a subset and a substructure of $\mathfrak{C}$ of cardinality less than $\bar{\kappa}$. Types, which are called in this abstract framework *Galois types*, are denoted by $gt(a/M)$ (the galois type of a over M) and correspond simply to orbits in $\mathrm{Aut}(\mathfrak{C})$, that is, $gt(a/M)$ is the set of all $b \in \mathfrak{C}$ such that there is $f \in \mathrm{Aut}(\mathfrak{C})$ such that $f \upharpoonright M = id$ and $f(a) = b$. We first present the axiomatic setting and start with an axiom that defines an independence relation as a relation between triplets of subsets of $\mathfrak{C}$. We denote this relation by $A \underset{C}{\downarrow} B$.

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Axiom 1 (Independence).

- [def] ***(Definition)*** $A \underset{C}{\downarrow} B \Leftrightarrow A \underset{C}{\downarrow} B \cup C$
- [tri] ***(Triviality)*** *If* $A \not\subseteq M$ *then* $A \underset{M}{\not\downarrow} A$
- [fin] ***(Finite Character)*** $A \underset{C}{\downarrow} B$ *if and only if* $A_0 \underset{C}{\downarrow} B_0$ *for all finite* $A_0 \subseteq A$ *and* $B_0 \subseteq B$.
- [mon] ***(Monotonicity)*** *Let* $C \subseteq C' \subseteq B$ *and* $B' \subseteq B$ *such that* $A \underset{C}{\downarrow} B$. *Then* $A \underset{C'}{\downarrow} B'$.
- [loc] ***(Local Character)*** *Let* $(M_\alpha : \alpha < \lambda)$ *be an* $\prec$*-increasing and continuous sequence of models such that* $\bigcup_{\alpha<\lambda} M_\alpha = M$. *Then, for every* a *there is* $\alpha < \lambda$ *such that* $a \underset{M_\alpha}{\downarrow} M$.
- [tra] ***(Transitivity)*** *Let* $M_0 \subseteq M_1 \subseteq N$. *Then* $A \underset{M_1}{\downarrow} N$ *and* $A \underset{M_0}{\downarrow} M_1 \Leftrightarrow A \underset{M_0}{\downarrow} N$.
- [sym] ***(Symmetry)*** $A \underset{M}{\downarrow} B \Leftrightarrow B \underset{M}{\downarrow} A$
- [inv] ***(Invariance)*** *Let* f *be an* $\mathcal{K}$*-embedding with* $A \cup B \cup C \subseteq dom(f)$. *Then* $A \underset{C}{\downarrow} B \Leftrightarrow f(A) \underset{f(C)}{\downarrow} f(B)$

With respect to this independence we define independent sets as usual and we extend this concept for trees as follows (this corresponds to what is usually known as an independent system or (in Shelah's terminology) as a system in complete amalgamation.

Definition 0.1. *Let* T *be a rooted tree. We say* $\langle M_\eta : \eta \in T\rangle$ *is an independent tree (itree), if* $M_\sigma \subseteq M_\tau$ *whenever* $\sigma \leq \tau$ *and for all* $\eta \in T$*:*

$$M_\eta \underset{M_{\eta^-}}{\downarrow} \bigcup_{\eta \not\leq \sigma} M_\sigma$$

Here a tree is a partial order $(T, <)$ such that for every $a \in T$ the set $a_< = \{b \in T : b < a\}$ is well ordered. For a tree T we say that $\langle M_\eta : \eta \in T\rangle$ is over N if $\bigcup_{\eta \in T} M_\eta \subseteq N$. For $U \subseteq T$, we let $M_U = \bigcup_{\eta \in U} M_\eta$. Notice that in this case M_U is not necessarily a model.

Axiom 2 (Existence of Prime Models). *1. There is a prime model over the* $\emptyset$.

2. For $a \in N - M$, *there is* $M' \prec N$ *prime over* $M \cup a$ *(usually also denote by* $M(a)$*).*

3. If $\langle M_\eta : \eta \in T\rangle$ *is an independent system over* N, *then there is* $M' \prec N$ *prime over* $\bigcup_{\eta \in T} M_\eta$.

Axiom 3 (Dominance). [dom] *If* $A \underset{M}{\downarrow} B$ *and* M_1 *is prime over* $M \cup B$ *then* $A \underset{M}{\downarrow} M_1$.

Definition 0.2. *Let $p \in S(M)$.*

(1) Let $N \prec M$. We say p is independent from M over N, denoted by $p \underset{N}{\downarrow} M$, if $a \underset{N}{\downarrow} M$ for all a realizing p.

(2) p is stationary if for all N such that $M \prec N$, there is a unique extension $p_N \in S(N)$ of p such that $p_N \underset{M}{\downarrow} N$. We say p_N is a free extension of p.

(3) Let $q \in S(N)$. We say p is orthogonal to q, denoted by $p \perp q$, if $a \underset{M_1}{\downarrow} b$ for all M_1 containing $M \cup N$ and all $a \models p_{M_1}$ and $b \models q_{M_1}$.

(4) p is orthogonal to N, denoted by $p \perp N$, if $p \perp q$ for all $q \in S(N)$.

(5) If $M_0 \prec M_1, M_2$, we write $gt(M_1/M_0) \perp M_2$ if and only if $gt(a/M_0) \perp M_2$ for all $a \in M_1 - M_0$.

(6) Assume p is stationary. Then, p is regular if for all N containing M and $q \in S(N)$ extending p, either $q = p_N$ or $q \perp p$.

The remaining axioms capture the desired behavior of Galois types:

Axiom 4 (Uniformity). *If $gt(A/M) = gt(B/M)$ and both $A \underset{M}{\downarrow} N$ and $B \underset{M}{\downarrow} N$, then $gt(A/N) = gt(B/N)$.*

Axiom 5 (Existence of Stationary Types). *Let M be a model. Then $p \in S(M)$ is stationary.*

Axiom 6 (Existence of Regular Types). *If $M \subseteq N$ and $M \neq N$, then there exists a regular type $p \in S(M)$ realized in $N - M$.*

Axiom 7 (Perpendicularity). *Let $M \prec N$ and $p \in S(N)$ be regular. Then $p \perp M$ if and only if $p \perp q$ for all regular types $q \in S(M)$. Moreover $gt(M_1/M_0) \perp M_2$ if and only if for all regular types $p \in S(M_0)$ realized in $M_1 - M_0$ we have that $p \perp M_2$.*

Axiom 8 (Equivalence). *Let $p, q \in S(M)$ be regular and $a \notin M$ realize p. Then q is realized in $M(a) - M$ if and only if $p \not\perp q$.*

The main consequences of the above listed axioms can be found in [3]. As in the first order version of the decomposition theorem, the NDOP property plays a crucial role.

Definition 0.3. *$\mathcal{K}$ has the NDOP (non-dimensional order property) if for every $M_0, M_1, M_2 \in \mathcal{K}$ such that $M_1 \underset{M_0}{\downarrow} M_2$ the following holds: for all M' prime over $M_1 \cup M_2$, and for every regular type $p \in S(M')$ either $p \not\perp M_1$ or $p \not\perp M_2$.*

Finally we give the definition of a decomposition:

Definition 0.4. *We say that $\langle M_\eta, a_\eta : \eta \in T\rangle$ is a decomposition of M if it satisfies the following properties*

(1) $ht(T) \leq \omega$.

(2) $\langle M_\eta : \eta \in T \rangle$ *is a tree over* M.

(3) *If* η^{--} *exists, then* $gt(M_\eta / M_{\eta^-}) \perp M_{\eta^{--}}$.

(4) *For all* $\eta \in T$, $(M_\sigma : \sigma^- = \eta, \sigma \in T)$ *is independent over* M_η.

(5) *Let* r *be the root of* T. *Then* M_r *is prime over* $\emptyset$. *Moreover,* M_η *is prime over* $M_{\eta^-} \cup a_\eta$.

(6) *If* η^- *exists,* $gt(a_\eta / M_{\eta^-})$ *is a regular type.*

(7) *For all* $\eta \in T$, $\rho^- = \eta$ *and* $\sigma^- = \eta$, *either* $gt(a_\rho / M_\eta) = gt(a_\sigma / M_\eta)$ *or* $gt(a_\rho / M_\eta) \perp gt(a_\sigma / M_\eta)$.

Condition (7) corresponds to the condition added to the original definition of decomposition in [2].

Definition 0.5. *A model* M *is minimal over* A *if prime models exist over* A *and whenever* $N \subseteq M$ *is prime over* A, *then* $N = M$.

Theorem 0.6 (Decomposition Theorem). *Suppose* $\mathcal{K}$ *has the NDOP and satisfies axioms 1-8. Then for every* $M \in \mathcal{K}$ *there is a decomposition* $\langle M_\eta, a_\eta : \eta \in T \rangle$ *of* M *such that* M *is prime and minimal over* $\bigcup_{\eta \in T} M_\eta$.

The proof of the theorem is basically an application of Zorn's lemma with respect to the natural order between decompositions. For the set of axioms here outlined a proof can be found in [3]. The proof shows also that M is prime and minimal over any maximal decomposition of M. We state this result as a corollary.

Corollary 0.7. *For every maximal decomposition* $\langle M_\eta, a_\eta : \eta \in T \rangle$ *of* M, M *is prime and minimal over* $\bigcup_{\eta \in T} M_\eta$.

In order to prove our application we need to prove first some lemmas relying on special properties such as the uniqueness of prime (primary) models.

Lemma 0.8. *Given* $(a_i : i \leq n)$, *there are* M_i *prime (primary) over* $M \cup a_i$ *such that* $M_i \prec N$ *for all* $i \leq n$, *where* N *is prime (primary) over* $M \cup \bigcup_{i \leq n} a_i$.

Proof: Fix $i \leq n$. Let $M(a_i)$ be prime over $M \cup a_i$. Since N contains $M \cup a_i$, there is a $\mathcal{K}$-embedding $f : M(a_i) \to N$ fixing $M \cup a_i$ pointwise. Let $f(M(a_i)) = M_i$. Then, M_i is prime over $f(M \cup a_i) = M \cup a_i$, and $M_i \prec N$, which is what we wanted. $\square$

Lemma 0.9. *Assume that prime (primary) models are unique. Then, if* $(a_i : i < \alpha)$ *is independent over* M, $(M(a_i) : i < \alpha)$ *is also independent over* M, *where* $M(a_i)$ *is the prime (primary) model over* $M \cup a_i$.

Proof: By finite character we may assume that $\alpha = n < \omega$. Let N be prime over $M \cup \{a_j : j \leq n, j \neq i\}$. By the previous lemma and the uniqueness of prime (primary) models we may assume that $M(a_j) \prec N$ for all $j \leq n$ and $j \neq i$. Then we have that

$$a_i \underset{M}{\downarrow} \bigcup_{i \neq j} a_j \overset{dom}{\Rightarrow} a_i \underset{M}{\downarrow} N \overset{sym+dom}{\Rightarrow} N \underset{M}{\downarrow} M(a_i) \overset{mon+sym}{\Rightarrow} M(a_i) \underset{M}{\downarrow} \bigcup_{i \neq j} M(a_j) \square$$

Now we are ready to prove the main theorem.

Theorem 0.10. *Let $M_1, M_2 \in \mathcal{K}$ of cardinality $\lambda > LS(\mathcal{K}) + 2^\omega$ with the NDOP. Then if M_1 and M_2 are $L_{\infty,\lambda}$ equivalent, they are isomorphic.*

Proof: We build a sequence of decompositions $\mathcal{S}^l_n = \langle N^l_\eta, a^l_\eta : \eta \in T_n \rangle$ for M_l where $l = 1, 2$ satisfying the following properties:

(1) T_n is a subtree of ${}^{n\geq}\lambda$.

(2) $\mathcal{S}^l_m \leq_{\mathcal{D}} \mathcal{S}^l_n$ for $m < n$ (where $\leq_{\mathcal{D}}$ is the natural order between decompositions).

(3) $\mathcal{S}^l_n$ is a maximal decomposition up to height n, i.e., there is no $\eta \in T_n$ such that $ht(\eta) < n$ and for some $\nu = \eta^\frown\alpha$, $\nu \notin T_n$, there are N^l_ν and a^l_ν such that $\langle N^l_\rho, a^l_\rho : \rho \in T_n \cup \{\nu\} \rangle$ is a decomposition for M_l.

(4) F_n is an elementary embedding from $\bigcup_{\eta \in T_n} N^1_\eta$ to M_2, mapping N^1_η onto N^2_η.

(5) $(M_1, c)_{c \in N^1_\eta} \equiv_{\infty,\lambda} (M_2, F_n(c))_{c \in N^1_\eta}$.

(6) $F_m \subseteq F_n$ for $m < n$.

Assume first that such sequences exist. Let $T = \bigcup_{i<\omega} T_i$ and $\mathcal{S}_l = \langle N^l_\eta, a^l_\eta : \eta \in T \rangle$ the "limit" of all these decompositions. It is not to dificult to see that $\mathcal{S}_l$ is a decomposition for M_l (for a proof see [3]). We now show that it is a maximal decomposition. Suppose not. Then there is $\eta \in T$ such that for some $\nu = \eta^\frown\alpha$, $\nu \notin T$, there are N^l_ν and a^l_ν such that $\langle N^l_\rho, a^l_\rho : \rho \in T \cup \{\nu\} \rangle$ is a decomposition for M_l. Let $ht(\eta) = n$. Then, we have that $\langle N^l_\rho, a^l_\rho : \rho \in T_{n+1} \cup \{\nu\} \rangle$ is a decomposition for M_l extending T_{n+1}, which contradicts (3). This shows that both $\mathcal{S}_l$ are maximal. Hence, by corollary 0.7, we know that M_l is prime and minimal over $\bigcup_{\eta \in T} N^l_\eta$. In addition, we have that $F = \bigcup_{n<\omega} F_n$ is an elementary map with domain $\bigcup_{\eta \in T} N^1_\eta$. Hence, there is an elementary embedding $F' : M_1 \to M_2$ extending F. By the minimality of M_2, this embedding must be onto, hence M_1 and M_2 are isomorphic. It remains to show how to build sequences satisfying (1)-(5).

We build the sequences by induction on n. For $n = 0$, let N^1_0 be the prime model over the $\emptyset$ and $a^1_0 \in M_1$ (wlog we can pick any element). We have that $|N^1_0| = LS(\mathcal{K}) < \lambda$ (the Löwenheim-Skolem cardinal of the AEC). Since $M_1 \equiv_{\infty,\lambda} M_2$, there is a back-and-forth set for M_1 and M_2. Hence, there is a partial isomorphism g with domain N^1_0. We let N^2_0 be the image of N^1_0 under g. Trivially this implies that

$$(M_1, c)_{c \in N^1_0} \equiv_{\infty,\lambda} (M_2, g(c))_{c \in N^1_\eta}$$

Clearly N_0^2 is prime over $\emptyset$. Let a_0^2 be any element in M_2. Then, we let $T_0 = \{0\}$, $\mathcal{S}_0^l = \langle N_0^l, a_0^l\rangle$ and $F_n = g$. We have that $\mathcal{S}_0^l$ is a decomposition for M_l and they satisfy condition (3) since there is just one decomposition of height 1 (by the definition of decomposition). By the choice of F_0 we have both conditions (4) and (5). This completes the base case.

Assume $\mathcal{S}_n^l$ and F_n have been defined. We proceed to define $\mathcal{S}_{n+1}^l$ and F_{n+1}. Notice that the new elements in T_{n+1} must be above those $\eta \in T_n$ such that $ht(\eta) = n$, otherwise condition (3) is contradicted. Thus, let be $\eta \in T_n$ such that $ht(\eta) = n$. First, we notice that it is enough to have an ordinal α and a sequence $(a_\beta : \beta < \alpha)$ such that:

(a) $(a_\beta^l : \beta < \alpha)$ is independent over N_η^l.

(b) $gt(a_\beta^l/N_\eta^l)$ is regular and $gt(a_\beta^l/N_\eta^l) \perp N_{\eta^-}^l$ if η^- exists.

(c) For $\beta < \beta' < \alpha$, either $gt(a_\beta^l/N_\eta^l) = gt(a_{\beta'}^l/N_\eta^l)$ or $gt(a_\beta^l/N_\eta^l) \perp gt(a_{\beta'}^l/N_\eta^l)$.

(d) $(a_\beta^l : \beta < \alpha)$ is maximal with respect to (a), (b) and (c).

(e) There is an isomorphism $F_\beta : N_\beta^1 \longrightarrow N_\beta^2$, where N_β^l is the prime model over $N_\eta^l \cup a_\beta^l$, such that $F_\beta \restriction N_\beta^1 = F_n \restriction N_\beta^1$.

(f) $(M_1, c)_{c\in N_\beta^1} \equiv_{\infty,\lambda} (M_2, F_\beta(c))_{c\in N_\beta^1}$.

Assume that for each $\eta \in T_n$ of height n we can find an ordinal α_η satisfying the above conditions. Then, we let

$$T_{n+1} = T_n \cup \{\eta^\frown\beta : \beta < \alpha_\eta, \eta \in T_n, ht(\eta) = n\}$$

$$\mathcal{S}_{n+1}^l = \langle N_\rho^l, a_\rho^l : \rho \in T_{n+1}\rangle$$

where $N_{\eta^\frown\beta}^l = N_\beta^l$ and $a_{\eta^\frown\beta}^l = a_\beta^l$. Condition (1) is trivially satisfied. Condition (2) follows from (a)-(c) and lemma 0.9. Condition (3) follows from (d). We define F_{n+1} as follows:

$$F_{n+1} = \bigcup\{\bigcup_{\beta<\alpha_\eta} F_\beta : \eta \in T_n, ht(\eta) = n\}$$

We fist show this function is well-defined. Let $\beta < \alpha_\eta$ and $\gamma < \alpha_{\eta'}$. Let $x \in N_{\eta^\frown\beta}^1 \cap N_{\eta'^\frown\gamma}^1$. Since $\mathcal{S}_{n+1}^1$ is a decomposition, we must have that $x \in N_\nu$ for $\nu < \eta$ and $\nu < \eta'$ (otherwise, we contradict triviality of independence). Hence, by (e) we have that $F_\beta \restriction N_\nu^1 = F_n \restriction N_\nu^1 = F_\gamma \restriction N_\nu^1$, and it is well defined in this case. Now assume towards a contradiction that there is $x \in M_1 - (N_{\eta^\frown\beta}^1 \cap N_{\eta'^\frown\gamma}^1)$ such that $x \in \mathrm{dom}(F_\beta) \cap \mathrm{dom}(F_\gamma)$. Then, again, since $\mathcal{S}_{n+1}^l$ is a decomposition we have that

$$N_{\eta^\frown\beta}^1 \underset{N_\nu}{\downarrow} N_{\eta'^\frown\gamma}^1$$

And by monotonicity we have that

$$x \underset{N_\nu}{\downarrow} x$$

which again contradicts triviality of independence. Therefore the function is well-defined. Finally, properties (e) and (f) imply conditions (4)-(6). We proceed then to find α and $(a^l_\beta : \beta < \alpha)$ for $\eta \in T_n$ of height n.

We let

$$I_l = \{a \in M_l : gt(a/N^l_\eta) \text{ is regular}, gt(a/N^l_\eta) \perp N^l_{\eta^-} \text{ if } \eta^- \text{ exists}\}$$

If $I_l = \emptyset$, we set $\alpha = 0$ and we do not add any elements above η. Hence assume that $I_l \neq \emptyset$. We consider a family $\mathcal{J}_l$ of subsets of $J \subseteq I_l$ defined by

$$J \in \mathcal{J}_l \Leftrightarrow \text{ for all } a, a' \in J \text{ either } gt(a/N^l_\eta) = gt(a'/N^l_\eta) \text{ or } gt(a/N^l_\eta) \perp gt(a'/N^l_\eta)$$

It is easy to see that $(\mathcal{J}_l, \subseteq)$ is non-empty and closed under unions of chains (if two elements of the union realize different non-orthogonal types, there is a subset in the chain to which they belong, which is a contradiction). Hence, by Zorn's lemma, we let J_l be maximal elements in $\mathcal{J}_l$ for $l = 1, 2$. Without loss of generality, we can assume that the types being realized by elements in J_1 and J_2 are the same (here we employ a back-and-forth argument using condition (5) of the induction hypothesis). Now we build by induction on γ an ordinal γ^* and a sequence of subsets $J^\gamma_l \subseteq J_l$ such that:

- the elements in J^γ_l satisfy the same $L_{\infty,\lambda}$-type.
- the dimension of J^γ_l over $(\bigcup_{\beta<\gamma} J^\beta_l, N_\eta)$ is less than λ.
- γ^* is the first ordinal where we cannot continue this sequence.

Since all the types realized in J_l are orthogonal, by definition of γ^*, the dimension of J_l over $(\bigcup_{\beta<\gamma^*} J^\beta_l, N^l_\eta)$ is either λ or zero. Assume it is λ and let $(b^l_i : i < \lambda)$ be a maximal independent set over $(\bigcup_{\beta<\gamma^*} J^\beta_l, N^l_\eta)$. Then we define $(a^l_\beta : \beta < \lambda)$ by induction on $\beta < \lambda$ as follows:

(i) $(a^l_\xi : \xi \leq \beta)$ is independent over $(\bigcup_{\beta<\gamma^*} I^\beta_l, N^l_\eta)$.

(ii) $(M_1, c, a^1_\beta)_{c \in N^1_\eta} \equiv_{\infty,\lambda} (M_1, F_n(c), a^2_\beta)_{c \in N^l_\eta}$

(iii) $b^1_\beta \in (a^1_\xi : \xi \leq 2\beta)$

(iv) $b^2_\beta \in (a^2_\xi : \xi \leq 2\beta + 1)$

Assume that $(a^l_\xi : \xi < 2\gamma)$ has been defined satisfying the previous conditions. For notational simplicity assume $2\gamma = \beta$. Take $a^1_\beta = b^1_\beta$. If there is $\xi < \beta$ such that $a^1_\xi = b_\beta$, then we define $a^2_\beta = a^2_\xi$. If not, then let b^2_ζ for $\zeta \geq \beta$, such that $gt(b^1_\beta/N^1_\eta) = gt(b^2_\zeta/N^2_\eta)$. The fact that they satisfy the same $L_{\infty,\lambda}$-type over N^l_η guarantees condition (ii). That there is such ζ is granted by the following argument. Suppose there is no such b^2_ξ with the same type. Then, using the back-and-forth set from condition (5) of our induction hypothesis there is $c \in J_2$

such that $gt(b^1_\delta/N^1_\eta) = gt(c/N^2_\eta)$. Since $gt(c/N^2_\eta) \neq gt(b^2_\beta/N^2_\eta)$ for all $\beta < \lambda$, by the definition of J_2, the type $gt(c/N^2_\eta)$ is ortogonal to $gt(b^2_i/N^2_\eta)$ for all $i < \lambda$. Hence $(b^2_i : i < \lambda) \cup \{c\}$ is still independent over $(\bigcup_{\beta<\gamma^*} J^\beta_l, N^l_\eta)$, which is a contradiction. This implies there is such a ζ. For $\beta = 2\gamma + 1$, we start setting $a^2_\beta = b^2_\beta$, and proceed to define a^1_β in an analogous way. The maximality of J_l and the sequence $(b_i : i < \lambda)$ guarantee properties (a)-(d). If the dimension is zero, we do not add any elements to the tree and this completes the construction. Finally, condition (ii) implies that the map $F_n \cup \{a^1_\beta, a^2_\beta\}$ is an elementary map. Hence there is an embedding from $F_\beta : N^1_\beta \to N^2_\beta$. But then $F_\beta(N^1_\beta)$ is also primary over $N^2_\eta \cup a^2_\beta$, so by the uniqueness of primary models we have that F_β is an isomorphism. This completes the proof of conditions (e) and (f), and the construction. □

Examples of this application are the following: in the first order case, a supersable theory with Galois types as usual types with the independence relation induced by non-forking; an ω-stable theory taking Galois types as strong types. For the non-elementary case, we have the class of $(D, \aleph_0)$-homogeneous models of a totally transcendental good diagram D. The satisfaction of the listed properties for this last class has been proved by Olivier Lessman in a sequence of papers [4], [1], [2] (some of them co-authored with Rami Grossberg). In those papers a proof of the main gap theorem for this class is reached. It is worth it to mention that the main gap theorem has been proved for all those classes that satisfy the conditions here listed. Hence a natural, but unsolved question, is if the conclusion of our theorem is a sufficient condition for a class to satisfy the main gap theorem. We left this as an open question.

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