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ON 3-AMPLENESS IN ROSY THEORIES

徳山工業高等専門学校・一般科目 米田郁生 (IKUO YONEDA)
GENERAL EDUCATION, TOKUYAMA COLLEGE OF TECHNOLOGY

ABSTRACT. In rosy theories we introduce a geometric notion of independence, strong non-3-ampleness, and we show that strong non-3-ampleness implies non-3-ampleness, and non-2-ampleness(=CM-triviality) implies strong non-3-ampleness.

1. INTRODUCTION

There is a simple characterization of CM-triviality. By using the characterization, we could show that any rosy CM-trivial theory has weak canonical bases, and CM-triviality in the real sort implies geometric elimination of imaginaries [Y]. We want to know whether these results can be extended in case of non-3-ampleness or not, so our first motivation is to find a simple characterization of non-3-ampleness like in case of CM-triviality. This paper is organized as follows. In the second section, we review the definiton of CM-triviality(=non-2-ampleness) and non-n-ampleness for each $n < \omega$ in rosy theories. In the third section, trying to find a simple charactrization, we offer another geometric notion (we call it strong non-3-ampleness). We show that strong non-3-ampleness implies non-3-ampleness, and non-2-ampleness(=CM-triviality) implies strong non-3-ampleness. But, for now there are no examples of non-3-ample and 2-ample structures. We also raise up some problems on non-3-ampleness.

Our notation is standard. Let T be a rosy theory. (i.e. having a good independence relation $\mathop{\smile\!\!\!\!\!\!|}\,$) We work in $\mathcal{M}^{\mathrm{eq}}$, the eq-structure, consisting of imaginary elements, where $\mathcal{M}$ is a sufficiently saturated model of T. $\bar{a}, \bar{b}, \ldots \subset_\omega \mathcal{M}$ denote finite sequences in $\mathcal{M}^{\mathrm{eq}}$. $A, B, \ldots$ denote small subsets of $\mathcal{M}^{\mathrm{eq}}$ and AB denotes $A \cup B$. For $a \in \mathcal{M}^{\mathrm{eq}}$ and $A \subset \mathcal{M}^{\mathrm{eq}}$, we write $a \in \mathrm{acl}^{\mathrm{eq}}(A)$ if the orbit of a by automorphisms fixing A pointwise is finite. In rosy theories [A], we have that $a \mathop{\smile\!\!\!\!\!\!|}_b c$ implies $\mathrm{acl}^{\mathrm{eq}}(ab) \cap \mathrm{acl}^{\mathrm{eq}}(bc) = \mathrm{acl}^{\mathrm{eq}}(b)$.

2. REVIEW OF ROSY CM-TRIVIALITY AND NON-n-AMPLENESS

CM-triviality is a geometric notion of the nonforking independence relation. In 1988, it is introduced by Hrushovski where he disproves Zilber's conjecture on strongly minimal sets [H]. CM-triviality forbids a point-line-plane incident system. Hrushovski also offered three characterizations of CM-triviality in stable theories. The following is the simplest characterization for rosy CM-triviality.

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Definition 2.1. A rosy theory T is CM-trivial, if $\bar{a} \perp_{A \cap \mathrm{acl}^{\mathrm{eq}}(\bar{a}, B)} B$ holds for any $\bar{a}, A, B \subset \mathcal{M}^{\mathrm{eq}}$ such that $\bar{a} \perp_A B$ and A, B are algebraically closed.

The Weak Caninical Base $\mathrm{wcb}(\bar{a}/B)$ of $\mathrm{tp}(\bar{a}/B)$ has the following properties, where B is an algebraically closed subset of $\mathcal{M}^{\mathrm{eq}}$:

- $\bar{a} \perp_{\mathrm{wcb}(\bar{a}/B)} B$
- $\mathrm{wcb}(\bar{a}/B)$ is algebraically closed.
- $\bar{a} \perp_A B \Rightarrow \mathrm{wcb}(\bar{a}/B) \subseteq \mathrm{acl}^{\mathrm{eq}}(A) \subseteq \mathcal{M}^{\mathrm{eq}}$

The weak canocal base is the smallest algebraically closed subset C of B such that $\bar{a} \perp_C B$. As in [P2], rosy theories do not necessarily have weak canonical bases. But any rosy CM-trivial theory has weak canonical bases, so we have the following characterization [Y].

Fact 2.2. Let T be rosy. The following are equivalent.

(1) T is CM-trivial.

(2) T has weak canonical bases and $\mathrm{wcb}(\bar{a}/A) \subseteq \mathrm{wcb}(\bar{a}/B)$ holds for any $\bar{a}, A, B \subset \mathcal{M}^{\mathrm{eq}}$ such that $\mathrm{acl}^{\mathrm{eq}}(\bar{a}, A) \cap B = A$ with $A = \mathrm{acl}^{\mathrm{eq}}(A)$ and $B = \mathrm{acl}^{\mathrm{eq}}(B)$.

We use the following notations to briefly write the definition of n-ampleness.

$$a \wedge b := \mathrm{acl}^{\mathrm{eq}}(a) \cap \mathrm{acl}^{\mathrm{eq}}(b)$$

$$a_{<i} := a_0, a_1, \ldots, a_{i-1}$$

$$a_{<0} := \emptyset$$

Definition 2.3. T is not n-ample, if $a_n \perp_c a_0$ holds for any $c, a_0, a_1, \ldots, a_n$ such that $ca_{<i}a_i \wedge ca_{<i}a_{i+1} = \mathrm{acl}^{\mathrm{eq}}(ca_{<i})$, $a_i \perp_{a_{i-1}c} a_{<i}$ for $i = 1, 2, \ldots n$, where c looks like constants.

The following is proved in [P1].

Remark 2.4. (1) one-basedness ($a \perp_{a \wedge b} b$ for any a, b) is equivalent to non-1-ampleness: $ca_0 \wedge ca_1 = \mathrm{acl}^{\mathrm{eq}}(c)$ $(a_1 \perp_{a_0} a_0)$ implies $a_1 \perp_c a_0$.

(2) CM-triviality ($a \perp_b c \Rightarrow a \perp_{b \wedge ac} c$) is equivalent to non-2-ampleness: $ca_0 \wedge ca_1 = \mathrm{acl}^{\mathrm{eq}}(c), ca_0a_1 \wedge ca_0a_2 = \mathrm{acl}^{\mathrm{eq}}(ca_0)$
$a_2 \perp_{a_1} a_0$, $a_1 \perp_{a_0} a_0$ imply $a_2 \perp_c a_0$.

(3) Non-n-ampleness implies non-$(n+1)$-ampleness for each $n < \omega$.

3. STRONG NON-3-AMPLENESS

Definition 3.1. A rosy theory T is not 3-ample, if $a_3 \perp_c a_0$ holds for any $c, a_0, a_1, a_2, a_3 \subset \mathcal{M}^{\mathrm{eq}}$ such that $a_0c \wedge a_1c = \mathrm{acl}^{\mathrm{eq}}(c), a_0a_1c \wedge a_0a_2c = \mathrm{acl}^{\mathrm{eq}}(a_0c)$ $a_0a_1a_2c \wedge a_0a_1a_3c = \mathrm{acl}^{\mathrm{eq}}(a_0a_1c)$, $a_3 \perp_{a_2c} a_0a_1, a_2 \perp_{a_1c} a_0$.

The following remark is a non-3-ample's version of Fact 2.2 under assuming the existence of weak canonical bases.

Remark 3.2. If T has weak canonical bases, then the following are equivalent.

(1) T is not 3-ample.

(2) $\mathrm{wcb}(a_3/ca_0) \subseteq \mathrm{acl}^{\mathrm{eq}}(\mathrm{wcb}(a_3/ca_0a_1a_2)c)$ holds for any $a_0, a_1, a_2, a_3, c \subset \mathcal{M}^{\mathrm{eq}}$ such that $a_0c \wedge a_1c = \mathrm{acl}^{\mathrm{eq}}(c), a_0a_1c \wedge a_0a_2c = \mathrm{acl}^{\mathrm{eq}}(a_0c)$, $a_0a_1a_2c \wedge a_0a_1a_3c = \mathrm{acl}^{\mathrm{eq}}(a_0a_1c)$ $a_3 \perp_{a_2c} a_0a_1, a_2 \perp_{a_1c} a_0$

Proof. (1) $\Rightarrow$ (2): Clear.
(1) $\Leftarrow$ (2): We may assume $c = \emptyset$. As $a_3 \perp_{a_2} a_0a_1$ and $\mathrm{wcb}(a_3/a_0) \subseteq \mathrm{wcb}(a_3/a_0a_1a_2)$, we have $\mathrm{wcb}(a_3/a_0) \subseteq a_0 \wedge a_2$. On the other hand, as $a_2 \perp_{a_1} a_0$, we have $a_0 \wedge a_2 \subseteq a_0a_1 \wedge a_1a_2 \subseteq \mathrm{acl}^{\mathrm{eq}}(a_1)$. As $a_0 \wedge a_1 = \mathrm{acl}^{\mathrm{eq}}(\emptyset)$, we have $\mathrm{wcb}(a_3/a_0) \subseteq a_0 \wedge a_2 \subseteq a_0 \wedge a_1 = \mathrm{acl}^{\mathrm{eq}}(\emptyset)$. □

Now we consider the following notion.

Definition 3.3. We say that T is strongly non-3-ample, if $a_3 \perp_{a_0a_2 \wedge a_1a_2 \wedge a_0a_1a_3} a_0$ holds for any a_0, a_1, a_2, a_3 such that $a_3 \perp_{a_2} a_0a_1, a_2 \perp_{a_1} a_0$

The definition of non-3-ampleness has three condition algebraically closed set and two conditions on independency. On the other hand, the definition of strong non-3-ampleness has only one condition on algebraically closed set and two conditions on independency, so it is simpler than that of non-3-ampleness.

Proposition 3.4. *Strong non-3-ampleness implies non-3-ampleness.*

Proof. Suppose that $a_0 \wedge a_1 = \mathrm{acl}^{\mathrm{eq}}(\emptyset), a_0a_1 \wedge a_0a_2 = \mathrm{acl}^{\mathrm{eq}}(a_0), a_0a_1a_2 \wedge a_0a_1a_3 = \mathrm{acl}^{\mathrm{eq}}(a_0a_1)$, $a_3 \perp_{a_2} a_0a_1, a_2 \perp_{a_1} a_0$, and let $b := a_0a_2 \wedge a_1a_2 \wedge a_0a_1a_3$.

We need to show $b = \mathrm{acl}^{\mathrm{eq}}(\emptyset)$.

Claim 1 $b \subseteq \mathrm{acl}^{\mathrm{eq}}(a_1)$: As $a_2 \perp_{a_1} a_0$, we have $a_0a_1 \wedge a_1a_2 = \mathrm{acl}^{\mathrm{eq}}(a_1)$. Then we have

$$\begin{aligned} b &= a_0a_2 \wedge a_1a_2 \wedge a_0a_1a_3 \\ &\subseteq a_0a_1a_2 \wedge a_1a_2 \wedge a_0a_1a_3 \\ &= (a_0a_1a_2 \wedge a_0a_1a_3) \wedge a_1a_2 \\ &= a_0a_1 \wedge a_1a_2 = \mathrm{acl}^{\mathrm{eq}}(a_1) \end{aligned}$$

Claim 2 $b \subseteq \mathrm{acl}^{\mathrm{eq}}(a_0)$: As $b \subseteq \mathrm{acl}^{\mathrm{eq}}(a_1)$,

$$\begin{aligned} b &\subseteq a_1 \wedge a_0a_2 \wedge a_1a_2 \wedge a_0a_1a_3 \\ &\subseteq a_1 \wedge a_0a_2 \subseteq a_0a_1 \wedge a_0a_2 = \mathrm{acl}^{\mathrm{eq}}(a_0) \end{aligned}$$

By two claims, we have $b \subseteq a_0 \wedge a_1 = \mathrm{acl}^{\mathrm{eq}}(\emptyset)$.

□

Remark 3.5. Assume that $\mathrm{acl}^{\mathrm{eq}}(A(B \wedge C)) = AB \wedge AC$ for any $A, B, C \subseteq \mathcal{M}^{\mathrm{eq}}$. (We usually have that $\mathrm{acl}^{\mathrm{eq}}(A(B \wedge C)) \subseteq AB \wedge AC$.) Then non-3-ampleness coincides with strong non-3-ampleness.

Proof. Let $b = a_0a_2 \wedge a_1a_2 \wedge a_0a_1a_3$. As $b \subseteq a_0a_2 \wedge a_1a_2$, we have

$$a_3 \underset{a_2b}{\perp} a_0a_1, a_2 \underset{a_1b}{\perp} a_0.$$

So we need to show

(1) $a_0b \wedge a_1b = \mathrm{acl}^{\mathrm{eq}}(b)$
(2) $a_0a_1b \wedge a_0a_2b = \mathrm{acl}^{\mathrm{eq}}(a_0b)$
(3) $a_0a_1a_2b \wedge a_0a_1a_3b = \mathrm{acl}^{\mathrm{eq}}(a_0a_1b)$

The proof of $a_0b \wedge a_1b = \mathrm{acl}^{\mathrm{eq}}(b)$:

$$\begin{aligned} a_0b \wedge a_1b &\subseteq (a_0a_2 \wedge a_0a_1a_2 \wedge a_0a_1a_3) \wedge (a_0a_1a_2 \wedge a_1a_2 \wedge a_0a_1a_3) \\ &\subseteq a_0a_2 \wedge a_1a_2 \wedge a_0a_1a_3 = \mathrm{acl}^{\mathrm{eq}}(b) \end{aligned}$$

The proof of $a_0a_1b \wedge a_0a_2b = \mathrm{acl}^{\mathrm{eq}}(a_0b)$: We use our assumption in the last equation.

$$\begin{aligned} a_0a_1b \wedge a_0a_2b &\subseteq (a_0a_1a_2 \wedge a_0a_1a_3) \wedge (a_0a_2 \wedge a_0a_1a_2 \wedge a_0a_1a_2a_3) \\ &\subseteq a_0a_2 \wedge a_0a_1a_3 = \mathrm{acl}^{\mathrm{eq}}(a_0b) \end{aligned}$$

The proof of $a_0a_1a_2b \wedge a_0a_1a_3b = \mathrm{acl}^{\mathrm{eq}}(a_0a_1b)$: We also use our assumption in the last equation.

$$\begin{aligned} a_0a_1a_2b \wedge a_0a_1a_3b &\subseteq (a_0a_1a_2 \wedge a_0a_1a_2a_3) \wedge (a_0a_1a_2a_3 \wedge a_0a_1a_3) \\ &\subseteq a_0a_1a_2 \wedge a_0a_1a_3 = \mathrm{acl}^{\mathrm{eq}}(a_0a_1b) \end{aligned}$$

□

Remark 3.6. Non-2-ampleness(=CM-triviality) implies strong non-3-ampleness.

Proof. We will show that $a_3 \perp_{a_2} a_0a_1$ implies $a_3 \perp_{a_0a_2 \wedge a_1a_2 \wedge a_0a_1a_3} a_0a_1$. Put $A = \mathrm{acl}^{\mathrm{eq}}(a_0a_2)$ and $B = \mathrm{acl}^{\mathrm{eq}}(a_1a_2)$. Clearly we have $a_3 \perp_{a_2} AB$. As $\mathrm{acl}^{\mathrm{eq}}(a_2) \subseteq A \cap B \subseteq AB$, we have $a_3 \perp_{A\cap B} AB$. In particular, we have $a_3 \perp_{A\cap B} a_0a_1$. By CM-triviality, we get $a_3 \perp_{a_0a_2 \wedge a_1a_2 \wedge a_0a_1a_3} a_0a_1$, as desired.

□

We have the following implications.
non-1-ampleness($\Leftrightarrow$ one-basedness) $\Rightarrow$ non-2-ampleness($\Leftrightarrow$ CM-triviality) $\Rightarrow$ strong non-3-ampleness $\Rightarrow$ non-3-ampleness $\Rightarrow$ non-4-ampleness $\Rightarrow \cdots$

In [E] an n-ample (relational) structure M_n is constructed for each $n < \omega$, but it is unknown whether M_n is not $(n+1)$-ample. For now, n-ample and non-$(n+1)$-ample structure is not discoverd for each $n \geq 2$. (Generic relational structures are 1-ample and non-2-ample.)

Problem 3.7. *It is shown that free group is 2-ample* [P3]. *Is free group non-3-ample? (We need the characterization of non-forking in the free group to check non-3-ampleness.)*

Problem 3.8. *Does non-3-ample theory have weak canonical bases? (I think No.) We need to check Adler's criterion* [A] *:* $a \perp_B C, a \perp_C B \Rightarrow a \perp_{B\cap C} BC$ *for any* a, B, C *such that* B *and* C *are algebraically closed subsets of* $\mathcal{M}^{\mathrm{eq}}$.

Problem 3.9. *Is strong non-3-ampleness with weak canonical bases equivalent to CM-triviality?*

Let $T = \mathrm{Th}(\mathbb{R}, +, <, \pi|_{(-1,1)}(*))$, where $\pi|_{(-1,1)}(x) := \pi \cdot x$ for $x \in (-1,1)$. T is an o-minimal theory with elimination of imaginaries. As T does not have weak canonical bases, T is 2-ample. And T does not interpret fields by [LP][PS], so it is possible that T is not n-ample for some $n < \omega$

Problem 3.10. *Is T non-3-ample? (We need the charactrization of non-forking in T.)*

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General Education, Tokuyama College of Technology, Gakuendai, Shunan 745-8585, Japan
E-mail address: yoneda@tokuyama.ac.jp