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INDISCERNIBLE SEQUENCES AND ARRAYS IN VALUED FIELDS

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1. Introduction

In his work on the classification for first-order theories Shelah introduced a hierarchy of combinatorial properties, so called dividing lines. It includes stable theories, which are quite well understood by now, dependent theories, simple theories, $NSOP$,etc.

The one which we are concerned with here is NTP_2. It first appears in [She90] and is a common generalization of both simple and NIP theories. It has attracted a certain attention recently. In particular, a uniform analog is introduced by Adler in [Adl07], theory of forking and dividing for such theories is developed in [CK08] and a general treatment is given in [Che10].

One of the central tasks of model theory is to characterise algebraic structures (e.g. groups and fields) depending on their place in the classification hierarchy. Here we will be concerned with valued fields and Ax-Kochen-type statements, that is that a certain property of the valued field can be determined by looking just at the value group and the residue field. A classical theorem of Delon ([Del79]) shows that if the residue field is NIP, then the whole valued field is NIP. Historically it also mentioned that the value group is NIP, but later Gurevitch and Schmidt demonstrated that any ordered abelian group is NIP. More recent result of the similar type are [She09] and [DGL09].

Here we prove an analogue for the NTP_2 property. This provides new algebraic examples of essentially NTP_2 theories, e.g. the ultraproduct of p-adics over all prime p's.

2. Preliminaries

As usual we will be working in a monster model $\mathbb{M}$. We will be assuming some knowledge of the basic stability theoretic techniques and things like indiscernible sequences.

2.1. Burden and NTP_2.

Definition 1. Given a (partial) type $p(x) \in S(A)$ we say that $(I_\alpha = (a_i^\alpha)_{i<\omega}, \phi_\alpha(x, y_\alpha), k_\alpha)_{\alpha<\kappa}$ is a *dividing pattern* in p of depth κ if

- $\{\phi_\alpha(x, a_i^\alpha)\}_{i<\omega}$ is k_α-inconsistent for each α
- $p(x) \cup \{\phi_\alpha(x, a_{f(\alpha)}^\alpha\}_{\alpha<\kappa}$ is consistent for every $f : \kappa \to \omega$

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We let $bdn(p)$ (*burden* of p) be the supremum of depths of all dividing patterns in p, and if it doesn't exist let $bdn(p) = \infty$, and let $\kappa_{inp}(T) = bdn(x = x)$. Notice that $\kappa_{inp}(T) < \infty \iff \kappa_{inp}(T) < |T|^+$, and is this case T is said to be NTP_2. T is called ***strong*** if there is no dividing pattern of infinite depth. It is observed in [Adl07] that in simple theories burden coincides with weight.

We say that $(I_\alpha)_{\alpha<\kappa}$ is an indiscernible array if I_α is an indiscernible sequence over $(I_\beta)_{\beta<\kappa,\neq\alpha}$ for each α. A standard use of the Erdös-Rado type theorem shows that one can calculate burden of a type by looking only at those definable patterns for which $(I_\alpha)_{\alpha<\kappa}$ is an indiscernible array.

Remark 2. $bdn(p) \geq \kappa \iff$ there is an indiscernible array $(I_\alpha)_{\alpha<\kappa}$ and ϕ_α, k_α such that
- $\{\phi_\alpha(x, a_i^\alpha)\}_{i<\omega}$ is k_α-inconsistent for each α
- $p \cup \{\phi_\alpha(x, a_0^\alpha)\}_{\alpha<\kappa}$ is consistent

Later we will need the following

Fact 3. [Che10] *1) T is NTP_2 (strong) $\iff$ every formula in one variable is NTP_2 (there is no deviding pattern of infinite depth with $|x| = 1$).*

2) Set of formulas with NTP_2 is closed under disjunctions (but not under conjunctions, this complicates proof a bit).

2.2. **Valued fields.** We will be condsidering valued fields in the Denef-Pas language, that is for us $\bar{K} = (K, \Gamma, k)$ with sort Γ for the value group, sort k for the residue field, $v : K \to \Gamma$ for the valuation map, and $ac : K \to k$ for the angular component.

The following fact is well-known

Fact 4. *Let $\bar{K} = (K, \Gamma, k)$ be a valued field eliminating field quantifiers in the Denef-Pas language, and $T = Th(\bar{K})$.*

1) If $M \models T$ and $p(x) \in S_1(M)$ then $p(x) \equiv \{\chi(v(x - c)) : \chi \in L_\Gamma(M), c \in M\} \cup \{\rho(ac(x - c)) : \rho \in L_k(M), c \in M\}$.

2) Every formula $\phi(x, \bar{c})$ is equivalent to one of the form $\bigvee_{i<n}(\chi_i(x) \wedge \rho_i(x))$ where $\chi_i = \bigwedge \chi_j^i(v(x - c_j^i), \bar{d}_j^i)$ with $\chi_j^i(x, \bar{d}_j^i) \in L(\Gamma)$ and $\rho_i = \bigwedge \rho_j^i(ac(x - c_j^i), \bar{e}_j^i)$ with $\rho_j^i(x, \bar{e}_j^i) \in L(k)$.

3) k and Γ are stably embedded.

Proof. 2) follows from 1) by compactness (I don't know if this equivalence is uniform, that is the resulting formula doesn't depend on $\bar{c}$, but won't matter for the application). □

Example 5. Henselian valued fields of characteristic $(0, 0)$ eliminate field quantifier by the theorem of Pas [Pas89].

3. Indiscernible sequences and arrays in valued fields

Some lemmas about the behavior of $v(x)$ and $ac(x)$ on indiscernible sequences, contained in the proof of [She09, Claim 1.17].

Lemma 6. *Let $(c_i)_{i\in I}$ be indiscernible. Consider function $(i, j) \mapsto v(c_j - c_i)$ with $i < j$. It satisfies one of the following*

1) it is strictly increasing depending only on i (we will call such a sequence "increasing")

2) it is strictly decreasing depending only on j ("decreasing")
3) it is constant ("constant")

Proof. a) Suppose for some i the sequence $(v(c_j - c_i))_{j>i}$ is increasing with j, say for some $i < j_1 < j_2$ we have $v(c_{j_2} - c_i) > v(c_{j_1} - c_i)$. Then $v(c_{j_1} - c_i) = v(c_{j_2} - c_{j_1}) = v(c_\infty - c_{j_1})$, and so by indiscernibility $v(c_j - c_i) = v(c_\infty - c_j) \implies (v(c_j - c_i))_{i<j}$ is constant for any j

b) Suppose for some j the sequence $(v(c_j - c_i))_{i<j}$ is decreasing with $i \implies (v(c_j - c_i))_{i<j}$ is constant for any i.

c) Suppose for some i the sequence $(v(c_j - c_i))_{i<j}$ depends on j. Then by b) $(v(c_j - c_i))_{i<j}$ is constant for any j (if it was increasing we get $v(c_j - c_{i_2}) > v(c_j - c_{i_1})$ for any $i_1 < i_2 < j$. Then $v(c_j - c_{i_1}) = v(c_{i_2} - c_{i_1})$, so by indiscernibility $v(c_j - c_i) = v(c_\infty - c_i)$ - doesn't depend on j, a contradiction).

d) Analogously if for some j sequence $(v(c_j - c_i))_{i<j}$ depends on $i \implies (v(c_j - c_i))_{i<j}$ is constant for any i.

If c) then $(v(c_j - c_i))_{i<j}$ is decreasing with j. Otherwise we have $v(c_{j_2} - c_{-\infty}) = v(c_{j_2} - c_{j_1}) > v(c_{j_1} - c_{-\infty})$, so $v(c_{j_1} - c_{-\infty}) = v(c_{j_2} - c_{-\infty})$ - a contradiction.

If d) then $(v(c_j - c_i))_{i<j}$ is increasing with i.

If neither c) nor d) - we are left with the constant case 3). □

Lemma 7. *Let $(c_i)_{i\in I}$ be an indiscernible sequence with increasing valuation . Then for any a there is some $h \in \bar{I} \cup \{\infty\}$ (Dedekind closure of I) such that*

for $i < h$: $v(a - c_i) = v(c_\infty - c_i)$, $ac(a - c_i) = ac(c_\infty - c_i)$ and $v(c_\infty - c_i) < v(a - c_\infty)$

for $i > h$: $v(a-c_i) = v(a-c_\infty)$, $ac(a-c_i) = ac(a-c_\infty)$ and $v(c_\infty-c_i) > v(a-c_\infty)$

Proof. Suppose there are $i_1 < i_2$ with $v(c_\infty - c_{i_1}) = v(a - c_\infty)$ and $v(c_\infty - c_{i_2}) = v(a - c_\infty)$ - but this contradicts that c_i is increasing. So there can be at most one such point, call it h.

So let $v(c_\infty - c_h) = v(a - c_\infty)$, and let $i > h$. Then $v(c_\infty - c_i) > v(c_\infty - c_h) = v(a - c_\infty)$, so $v(a - c_i) = v(a - c_\infty)$. And since $v((a - c_i) - (a - c_\infty)) = v(c_\infty - c_i) > v(a - c_\infty)$ we conclude $ac(a - c_i) = ac(c_\infty - c_i)$.

Let $i < h$. Then $v(c_\infty - c_i) < v(c_\infty - c_h) = v(a - c_\infty)$, so $v(a - c_i) = v(c_\infty - c_i)$. And since $v((a - c_i) - (c_\infty - c_i)) = v(a - c_\infty) > v(c_\infty - c_i)$ we conclude $ac(a - c_i) = ac(c_\infty - c_i)$. □

Now we proceed to the main result

Theorem 8. *k is $NTP_2 \implies K$ is NTP_2*

Proof. By Fact 3, 1) we only have to show that every formula with $|x| = 1$ is NTP_2. By Fact 4, Fact 3, 2) and stable embeddedness of Γ, k it is enough to show that every formula of the following form is NTP_2

$\phi(x, \bar{y}) = \chi(v(x - y_0), ..., v(x - y_n), \bar{y}_\chi) \wedge \rho(ac(x - y_0), .., ac(x - y_n), \bar{y}_\rho)$

where $\chi \in L_\Gamma$, $\rho \in L_k$ (we also use indiscernibility of the array to get same formula equivalent to the original one for all parameters in the array, since apriori that equivalence is not uniform).

First we prove it for $n = 1$. So let $(\bar{d}_i^\alpha)_{i\in I, \alpha<\kappa}$ (where $\bar{d}_i^\alpha = c_i^\alpha \bar{d}_{\chi,i}^\alpha \bar{d}_{\rho,i}^\alpha$) be an indiscernible array witnessing that ϕ has TP_2. Take $a \models \{\phi(x, \bar{d}_0^\alpha)\}_{\alpha<\kappa}$.

Case 1: (c_0^α) is increasing. Let $h \in \kappa \cup \{\infty\}$ be as given by Lemma 7. As at least one of $(0, h)$ and (h, ∞) must be infinite we can replace κ by it.

Case 1.1: $v(a - c_0^\alpha) = v(c_0^\infty - c_0^\alpha)$, $ac(a - c_0^\alpha) = ac(c_0^\infty - c_0^\alpha)$. But then actually $c_0^\infty \models \{\phi(x, \bar{d}_0^\alpha)\}_{\alpha<\kappa}$, and by indiscernibility of the array $c_0^\infty \models \{\phi(x, \bar{d}_i^\alpha)\}_{i\in I, \alpha<\kappa}$ - a contradiction.

Case 1.2: $v(a - c_0^\alpha) = v(a - c_0^\infty)$, $ac(a - c_0^\alpha) = ac(a - c_0^\infty)$ and $v(a - c_0^\infty) < v(c_0^\infty - c_0^\alpha)$.

Consider $\phi_\chi(x_\chi, \bar{e}_{\chi,i}^\alpha) := \chi(x_\chi, \bar{d}_{\chi,i}^\alpha) \wedge x_\chi < v(c_0^\infty - c_i^\alpha)$ with $\bar{e}_{\chi,i}^\alpha = \bar{d}_{\chi,i}^\alpha \cup v(c_0^\infty - c_i^\alpha)$. Since $(\bar{e}_{\chi,i}^\alpha)_{i\in I,\alpha<\kappa}$ is still an indiscernible array, $\phi_\chi \in L_\Gamma$ and Γ is NTP_2 we find some $a_\chi \in \Gamma$ satisfying $\{\phi_\chi(x_\chi, \bar{e}_{\chi,i}^\alpha)\}_{\alpha<\kappa,i\in I}$.

Analogously letting $\phi_\rho(x_\rho, \bar{d}_{\rho,i}^\alpha) := \rho(x_\rho, \bar{d}_{\rho,i}^\alpha)$ by NTP_2 of k we find some $a_\rho \in k$ satisfying $\{\phi_\rho(x_\rho, \bar{d}_{\rho,i}^\alpha)\}_{\alpha<\kappa,i\in I}$.

Take $a \in K$ satisfying $v(a - c_0^\infty) = a_\chi \wedge ac(a - c_0^\infty) = a_\rho$, then it satisfies $\{\phi(x, \bar{d}_i^\alpha)\}_{i\in I,\alpha<\kappa}$ - a contradiction.

Case 2: (c_0^α) is decreasing - reduces to the first case by reversing the order of rows.

Case 3: (c_0^α) is constant.

If $v(a - c_0^\alpha) < v(c_0^\infty - c_0^\alpha)$ $(= v(c_0^\beta - c_0^\alpha))$ for some α then $v(a - c_0^\alpha) = v(a - c_0^\beta) = v(a - c_0^\infty)$ for any β, and $ac(a - c_0^\alpha) = ac(a - c_0^\infty)$ for all α's and it falls under case 1.2.

Next, there can be at most one α with $v(a - c_0^\alpha) > v(c_0^\infty - c_0^\alpha)$ (if also $v(a - c_0^\beta) > v(c_0^\infty - c_0^\beta)$ for some $\beta > \alpha$ then $v(c_0^\beta - c_0^\alpha) = v(a - c_0^\alpha) > v(c_0^\infty - c_0^\alpha)$, a contradiction). Throw the corresponding row away and we are left with the case $v(a - c_0^\alpha) = v(c_0^\infty - c_0^\alpha) = v(c_0^\infty - c_0^{-\infty})$ for all α. Then $ac(a - c_0^\alpha) = ac(a - c_0^\infty) - ac(c_0^\infty - c_0^\alpha)$. Let $\phi_\rho(x_\rho, \bar{e}_{\rho,i}^\alpha) := \rho(x_\rho - ac(c_0^\infty - c_0^\alpha), \bar{d}_{\rho,i}^\alpha)$ with $\bar{e}_{\rho,i}^\alpha = \bar{d}_{\rho,i}^\alpha \cup ac(c_0^\infty - c_0^\alpha)$. Again since k is NTP_2 we find a_ρ satisfying $\{\phi_\rho(x_\rho, \bar{e}_{\rho,i}^\alpha)\}_{\alpha<\kappa,i\in I}$. Also notice that by indiscernibility $\models \{\chi(v(c_0^\infty - c_0^{-\infty}), \bar{d}_{\chi,i}^\alpha\}_{i\in I,\alpha<\kappa}$. Let a satisfy $v(a - c_0^\infty) = v(c_0^\infty - c_0^{-\infty}) \wedge ac(a - c_0^\infty) = a_\rho$, it does the job.

Now we show that the general case reduces to $n = 1$. So suppose that $\phi(x, \bar{y}) = \chi(v(x - y_0), ..., v(x - y_{n-1}), \bar{y}_\chi) \wedge \rho(ac(x - y_0), .., ac(x - y_{n-1}), \bar{y}_\rho)$ has TP_2 for some n. Let $S = \{\gamma_0, ..., \gamma_N\}$ enumerate all possible order types of n points. Then tautologically $\phi(x, \bar{y}) \equiv \phi(x, \bar{y}) \wedge \bigvee_{\gamma\in S} \gamma(v(x - y_0), ..., v(x - y_{n-1}))$. And by Fact 3, 2) $\phi(x, \bar{y}) \wedge \gamma(v(x - y_0), ..., v(x - y_{n-1}))$ has TP_2 for some $\gamma \in S$. Let $\bar{d}_i^\alpha = c_{0,i}^\alpha ... c_{n-1,i}^\alpha \bar{d}_{\chi,i}^\alpha \bar{d}_{\rho,i}^\alpha$ be an indiscernible array witnessing it and $a \models \{\phi'(x, \bar{d}_0^\alpha)\}_{\alpha<\kappa}$. W.l.o.g. γ says $v(x - y_0) \geq ... \geq v(x - y_{n-1})$ and let $n' \leq n - 1$ be maximal such that $v(x - y_0) = v(x - y_i) \in \gamma$ for all $i < n'$.

Then $v(a - c_{i,0}^\alpha) = v(c_{i,0}^\alpha - c_{0,0}^\alpha)$, $ac(a - c_{i,0}^\alpha) = ac(c_{i,0}^\alpha - c_{0,0}^\alpha)$ for $i \geq n'$ and $v(a - c_{i,0}^\alpha) = v(a - c_{0,0}^\alpha)$, $ac(a - c_{i,0}^\alpha) = ac(c_{i,0}^\alpha - a_{0,0}^\alpha) - ac(a - c_{0,0}^\alpha)$ for $i < n'$. So letting $\phi'(x, \bar{y}')$ be $\phi(x, \bar{y}')$ with

- $v(x - y_i)$ replaced by $v(x - y_0)$ for all $i < n'$ and by $v(y_i - y_0)$ for all $i \geq n'$
- $ac(x - y_i)$ replaced by $ac(y_i - y_0) - ac(x - y_0)$ for all $i < n'$ and by $ac(x - y_0)$ for all $i \geq n'$

and noticing that $(\bar{e}_i^\alpha) = (\bar{d}_i^\alpha \cup v(c_{i,0}^\alpha - c_{0,0}^\alpha) \cup ac(c_{i,0}^\alpha - c_{0,0}^\alpha))_{i\in I,\alpha<\kappa}$ is still an indiscernible array, $a \models \{\phi'(x, \bar{e}_0^\alpha)\}_{\alpha<\kappa}$ and $\{\phi'(x, \bar{e}_i^\alpha)\}_{i\in I}$ is still inconsistent for each $\alpha < \kappa$ we conclude that $\phi'(x, \bar{y}')$ has TP_2, contradicting the case $n = 1$. □

Remark 9. One could give an alternative proof of the Delon's theorem using the same lemmas.

Example 10. Let $K = \prod_{p \text{ prime}} \mathbb{Q}_p/\mathfrak{U}$ with $\mathfrak{U}$ a non-principal ultrafilter. Then k is pseudo-finite, so has IP. And Γ has SOP of course. By an observation of Scanlon valuation ring is definable in K in the pure field language, so K has both IP and SOP in the pure field language. However by the theorem it is NTP_2, even in a larger Denef-Pas language.

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