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Remarks on the model theory of analytic Zariski structures

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Abstract

We survey here Zilber's attempts of developing the model theory of analytic Zariski structures.

1 Introduction

The notion of analytic Zariski structures is introduced by Zilber in order to study analytic structures from model theoretic point of view. This is a natural generalization of the notion of Zariski geometry (structure) also introduced by Zilber earlier.

In case of Zariski geometries, the model theory is of first-order and carried out in $L_{\omega\omega}$. Main theorem states that the theory of a Zariski geometry is strongly-minimal and hence categorical in every uncountable cardinal.

In case of analytic Zariski structures, however, the situation differs dramatically: since the notion of *analyticity* needs non-Noetherian topology, we need $L_{\infty\omega}(Q)$ to develop its full model theory.

In this note, we first review Zilber's idea of showing the categoricity for non-elementary classes called *quasi-minimal excellent.*

2 Quasi-minimal excellent classes and categoricity

It is well known that strongly-minimal theories T are $\aleph_1$-categorical. The proof goes like this: the strong-minimality of T gives rise to a pregeometry to each model M. Then a notion of *independence* can be introduced to M. Hence a basis of M is also defined. Note that any basis of a structure has the same cardinality as the structure.

Consider two structures M_0, M_1 of the same uncountable cardinality. Suppose that both B_0, B_1 are basis of M_0, M_1 respectively. Any bijection between B_0 and B_1 can be extended to an isomorphism from M_0 to M_1.

The essence of the argument above can be formalized as three properties as follows:

- pregeometry needed for the notion of independence and in particular the basis
- homogeneity, more specifically ω-homogeneity for constructing isomorphisms between models by back-and-forth argument.
- ω-stability, i.e., there are not too many types otherwise the uncountable categoricity breaks down.

This analysis seems to lead Zilber to the following definition of quasi-minimal excellent class.

Definition 1 (G-monomorphism) Suppose H and H' are structures and G is their common substructure. A partial mapping $\phi : H \to H'$ is called a G-monomorphism if $\phi \restriction_G = \mathrm{id}$ and it preserves quantifier-free formulas over G.

Definition 2 (Quasi-minimal excellent, [Z03]) A class $\mathcal{C}$ of structures are called *quasi-minimal excellent* if

1. $\mathcal{C}$ is equipped with a pregeometry, i.e., for any $H \in \mathcal{C}$ there is a closure notion cl such that (H, cl) forms a pregeometry.
2. ω-homogeneity over a submodel, i.e.,
3. Any finite subset X of $\mathrm{cl}(C)$ where $C \subseteq H \in \mathcal{C}$ is *special* is defined over a finite subset C_0 of C. Here C is special if there is cl-independent $A \subseteq H$ and $A_1, cdots, A_k \subseteq A$ such that

$$C = \bigcup_{i=1}^{k} \mathrm{cl}(A_i)$$

Remark 3 In his definition above, Zilber does not assume the pregeometry to satisfy the exchange principle nor the countable closure property.

Theorem 4 (Thm 1, [Z03]) Let $\mathcal{C}$ be a quasi-minimal excellent class, $H, H' \in \mathcal{C}$ both with the countable closure property, $A \subseteq H, A' \subseteq H'$, independent and $\mathrm{cl}(A) = H$, $\mathrm{cl}(A') = H'$. Suppose that there is a bijection $\psi_0 : A \to A'$. Then ψ_0 extends to an isomorphism $\psi : H \to H'$. ■

From this theorem we immediately have:

Corollary 5 If the class $\mathcal{C}$ satisfies the same assumptions of the above theorem and also the exchange property, then $\mathcal{C}$ is categorical in every uncountable categoricity. ■

If in addition a quasi-minimal class is axiomatisable by an $L_{\omega_1\omega}$-sentence, we have a more precise theorem:

Theorem 6 (Thm 2, [Z03]) Suppose a quasi-minimal class $\mathcal{C}$ is axiomatisable by an $L_{\omega_1\omega}$-sentence and the relations $y \in \mathrm{cl}(x_1, \cdots, x_n)$ are $L_{\omega_1\omega}$-definable for all n. Suppose further that there is an $H \in \mathcal{C}$ containing an infinite cl-independent subset.

Then for any uncountable cardinal κ there is a structure $H_\kappa \in \mathcal{C}$ of cardinality κ with the countable closure property. A structure with these properties is unique in $\mathcal{C}$ provided $\mathcal{C}$ satisfies the exchange property. ■

3 Categoricity of algebraically closed fields with pseudo-exponentiation

The notion of quasi-minimal excellent classes is not an abstract non-sense. We have a very beautiful and successful example, that is a class $\mathcal{K}_{\mathrm{ex}}$ of algebraically closed fields of pseudo-exponentiation.

$\mathcal{K}_{\mathrm{ex}}$ is a class of algebraically closed fields of characteristic zero equipped with a pseudo-exponentiation ex:

- Any $F \in \mathcal{K}_{\text{ex}}$ is an algebraically closed field of characteristic zero. $\text{ex} : F \to F^{\times}$, $\text{ex}(x+y) = \text{ex}(x) \cdot \text{ex}(y)$.
- $\ker(\text{ex}) \simeq \mathbb{Z}$.
- Each $F \in \mathcal{K}_{\text{ex}}$ is strongly exponentially-algebraically closed, i.e., for any ex-irreducible free, ex-normal variety V in $2n$ variables ex-defined over a finite $C \subset F$, with $\dim V = n$, there is a generic over C solution of V in F. Roughly speaking this means that any finite system of exponentially-algebraic equations has a solution if the system of equations do not violate the property above for ex.

Lemma 7 (Thm 5.13, [Z05]) If F_1, F_2 are strongly exponentially-algebraically closed and of infinite cl-dimension, then F_1 and $_2$ are $L_{\omega_1\omega}$-equivalent. ∎

For the uncountable categoricity of $\mathcal{K}_{\text{ex}}$, we need to show that some element F in the class have the countable closure property. This can be shown that there is an infinite-dimensional countable member in the class $\mathcal{K}_{\text{ex}}$.

Lemma 8 The class $\mathcal{K}_{\text{ex}}$ is axiomatized by an $L_{\omega_1\omega}(Q)$-sentence, where Q stands for a quantifier expressing there are uncountably many. ∎

Theorem 9 (Thm 5.16, [Z05]) $\mathcal{K}_{\text{ex}}$ is quasi-minimal excellent with countable closure property. Hence, for any uncountable cardinality κ there is a unique, up to isomorphism, structure $F \in \mathcal{K}_{\text{ex}}$ of cardinality κ. ∎

Remark 10 In Zilber's original paper [Z05], the class $\mathcal{K}_{\text{ex}}$ above is described as $\mathcal{EC}^*_{\text{st,ccp}}$ where "st" stands for standard kernel and "ccp" for the countable closure property.

Remark 11 Zilber realized that a theorem of J. Ax which is a solution to a function field version of the famous Schanuel conjecture is a key to the proof of showing that $\mathbb{C}_{\text{exp}}$, the complex field with the complex exponentiation, has the countable closure property, (Lemma 5.12, [Z05]). However showing that $\mathbb{C}_{\text{exp}} \in \mathcal{K}_{\text{ex}}$ is extreamly hard.

4 Analytic Zariski structures and categoricity

Analytic Zariski structures are structures with topology defined such that certain closed sets capture the notion of *analiticity.* Those closed sets should reflect properties of *genuine analytic subsets.*

4.1 Language for analytic Zariski structures

To develop the model theory of such structures there are two ways to define the natural language of topological structures.

1. Start with a topological space M with certain properties. Then consider a language having all predicate symbols corresponding to closed sets. This is the style of [HZ96].
2. Start with a first-order structure M. Consider a collection $\mathcal{C}$ of first-oder definable with parameters subsets of M^n for each n. The collection $\mathcal{C}$ has certain properties enable to define a topological structure. Then consider a language having all predicate symbols to all sets in $\mathcal{C}$. Zilber calls this structure a topological structure $(M, \mathcal{C})$. He then defines the *natural language* for topological structure having predicate symbol for each closed set in $\mathcal{C}$.

We follow the second approach in this note.

4.2 Axioms for analytic Zariski structures

There are four sorts of axioms; the first group is for the topology of the underlying set, the second group for the property of dimensions, the third for the analytic sets, and the last one is for the analytic rank.

(L1) closed sets are closed under arbitrary intersections

(L2) closed sets are closed under finite unions

(L3) the domain of the structure is closed

(L4) the graph of the equality is closed

(L5) any singleton of the domain is a closed

(L6) Cartesian products of closed sets are closed

(L7) closed sets are closed under permutations of coordinates

(L8) for any closed set and any point the fiber over the point is closed

(L1) through (L8) define the notion of topological structures. Next Zilber introduces the notion of dimension for projective sets. Here projective sets are finite unions of projections of certain closed sets.

Remark 12 (L1) is very confusing; Zilber is assuming here that for any $C_0, \cdots \in \mathcal{C}$, there is a set $C \in \mathcal{C}$ such that $C = \bigcap_{i=0}^{\infty} C_i$.

Definition 13 (projective) $\mathcal{C}$-constructibe sets are finite unions of sets S with $S \subseteq_{cl} U \subseteq_{op} M^n$. Projective sets are finite unions of projections $\mathrm{pr}S$ where $S \subseteq_{cl} U \subseteq_{op} M^n$. To each non-empty projective set, a non-negative integer called its dimension is attached.

(SI) for any irreducible set $S \subseteq_{cl} U \subseteq_{op} M^n$ and its closed subset $S' \subseteq S$, if $\dim S' = \dim S$ then $S' = S$,

(DP) for a nonempty projective S, $\dim S = 0$ if and only if S is at most countable,

(CU) If $S = \bigcup_{i \in \mathbb{N}} S_i$ with all S_i projective, then $\dim S = \max\{\dim S_i : i \in \mathbb{N}\}$,

(WP) given an irreducible $S \subset_{cl} U \subseteq_{op} M^n$ and $F \subset_{cl} V \subseteq_{op} M^{n+k}$ with the projection $\mathrm{pr} : M^{n+k} \to M^n$ such that $\mathrm{pr}F \subseteq S$ and $\dim prF = \dim S$, then there exists $D \subseteq_{op} S$ such that $D \subseteq \mathrm{pr}F$.

(AF) for any irreducible $S \subseteq_{cl} U \subseteq_{op} M^n$ and a projection map $\mathrm{pr} : M^n \to M^m$,

$$\dim S = \dim \mathrm{pr}(S) + \min_{a \in \mathrm{pr}(S)} \dim(\mathrm{pr}^{-1}(a) \cap S)$$

(FC) for any irreducible $S \subseteq_{cl} U \subseteq_{op} M^n$ and a projection map $\mathrm{pr} : M^n \to M^m$, there exists $V \subseteq_{op} \mathrm{pr}S$ (relatively open) such that

$$\min_{a \in \mathrm{pr}(S)} \{\dim(\mathrm{pr}^{-1}a) \cap S))\} = \dim(\mathrm{pr}^{-1}(v) \cap S)$$

for any $v \in \mathrm{pr}(V) \cap \mathrm{pr}(S)$.

WP above is an acronym for the weak properness and it works as a kind of quantifier elimination. Topological structures with the function dim satisfying the above properties (SI) through (WP) is called *topological structures with good dimension.*

Definition 14 (Analytic subsets, [Z10]) A subset S such that $S \subseteq_{cl} U \subseteq_{op} M^n$ is called *analytic* if for each $a \in S$ there is an open set $V_a \subseteq_{op} U$ with $a \in V_a$ and $S \cap V_a$ is the union of finitely many relatively closed irreducible subsets.

Analytic subsets should satisfy the following properties;

(INT) For any open subset U, the intersection of analytic subsets of U is analytic in U.

(CMP) For any open subset U, an analytic subset S of U and $a \in S$, there are analytic subsets $S_a, S'_a \subseteq U$ such that

1. S_1 is a finite union of irreducible analytic subsets of U
2. $a \in S_a \setminus S'_a$, and $S = S_a \cup S'_a$

(CC) For any open subset U and any analytic subset S of U, there are at most countably components of S such that S is its union.

Definition 15 (Analytic Rank, [Z10]) To each subset $S \subseteq_{cl} U \subseteq_{op} M^n$, we define **the analytic rank** of S in U which is a natural number satisfying:

1. $\mathrm{ark}_U(S) = 0$ if and only if $S = \emptyset$;
2. $\mathrm{ark}_U(S) \leq k + 1$ if and only if there is a set $S' \subseteq_{cl} S$ such that $\mathrm{ark}(S') \leq k$ and with the set $S^0 = S \setminus S'$ being analytic in $U \setminus S'$.

4.3 Quasi-minimal excellent class of analytic Zariski structures

Suppose M is an analytic Zariski structure. M should capture some aspects of *analyticity* from model theoretic point of view. Naturally we try to form a class of structures associated with the structure M

We now review Zilber's idea of constructing a quasi-minimal excellent class of structures associated with the structure M.

4.3.1 Core substructure of analytic Zariski structures

Definition 16 (Defn 6.3.1, [Z10]) Let $(M, \mathcal{C})$ be a topological structure. Suppose M_0 is a non-empty subset of M and $\mathcal{C}_0$ a subfamily of $\mathcal{C}$. We say that $(M_0, \mathcal{C}_0)$ is a **core substructure** if

1. if $\{(x_1, \cdots, x_n)\} \in \mathcal{C}_0$ then each $x_i \in M_0$ $(i = 1, \cdots n)$
2. $\mathcal{C}_0$-closed sets are closed under finite intersection
3. $\mathcal{C}_0$ satisfies (L1)-(L7), and (L8) with $a \in M_0^k$
4. $\mathcal{C}_0$ satisfies (WP), (AF), (FC) and (AS)
5. for any $\mathcal{C}_0$-constructible $S \subseteq_{an} U \subseteq_{op} M^n$, every irreducible component S_i of S is $\mathcal{C}_0$-constructible
6. for any non-empty $\mathcal{C}_0$-constructible $U \subseteq M$, $U \cap M_0 \neq \emptyset$.

Lemma 17 For any **countable** $N \subseteq M$ and $C \subseteq \mathcal{C}$ there exist **countable** $M_0 \supseteq N$ and $\mathcal{C}_0 \supseteq C$ such that $(M_0, \mathcal{C}_0)$ is a core substructure. ■

We then fix a core substructure $(M_0, \mathcal{C}_0)$ with M_0 and $\mathcal{C}_0$ countable.

Remark 18 (Core-substructures of $\mathcal{K}_{\mathrm{ex}}$) Start with the prime field of the algebraically closed field K we can construct a core substructure.

Definition 19 **($\mathcal{C}_0$-predimension)** For any finite subset X of M, we define the $\mathcal{C}_0$-predimension

$$\delta(X) = \min\{\dim S : X \in S, S \subseteq_{an} U \subseteq_{op} M^n, S \text{ is } \mathcal{C}_0\text{-constructible}\}$$

and the dimension

$$\delta(X) = \min\{\delta(XY) : \text{finite } Y \subset M\}.$$

From now on we assume that $\dim M = 1$ and M is irreducible. Under this assumption we have that for any $y \in M$

$$0 \leq \delta(Xy) \leq \delta(X) + 1$$

since $Xy \in S \times M$ and $\dim(M) = 1$.

By the addition formula axiom (AF) we have that for any $F \subseteq_{an} U \subseteq_{op} M^k$ with positive dimension, there is $i \leq k$ such that $\dim \mathrm{pr}_i F > 0$.

Zilber proves a main proposition stating the relation between the original dimension "dim" and the dimension defined by the $\mathcal{C}_0$-predimension.

Proposition 20 Let S be an analytic subset of M^{n+k} with $S \subseteq_{an} U \subseteq_{op} M^{n+k}$. Consider a $\mathcal{C}_0$-constructible $P = \mathrm{pr}S$ for some projection pr. Then

$$\dim P = \max\{\delta(x) : x \in P\}$$

■

Now define the closure notion with predimension;

Definition 21 For any finite $X \subseteq M$,

$$\mathrm{cl}_{\mathcal{C}_0}(X) = \{y \in M : \partial(Xy) = \partial(X)\}$$

Proposition 20 plays a major role to show that $\mathrm{cl}(A)$ is countable for any finite A. It follows that the operator "cl" defines a predimension on M.

4.3.2 Quasi-minimal excellent class associated with an analytic Zariski structure

First define a class $\mathcal{A}_0(M)$ of structures associated with M;

$$\mathcal{A}_0(M) = \{\text{countable } \mathcal{C}_0^{\exists}\text{-structures } N : N \simeq N' \subseteq M, \mathrm{cl}(N') = N'\}$$

With this class, Zilber then defines another class $\mathcal{A}(M)$ satisfying the following properties;

1. $\mathrm{cl}_{\mathcal{C}_0}$ with respect to H is defined,
2. $\mathcal{A}_0(H) \subseteq \mathcal{A}_0(M)$ as classes with embeddings,
3. for every finite $X \subseteq H$ there is $N \in \mathcal{A}_0(H)$ such that $X \subseteq N$.

We want this class $\mathcal{A}(M)$ to be

1. a class of analytic Zariski structures, and
2. uncountable categorical.

These two objectives are achieved by the following theorems and the proposition.

Theorem 22 (Thm 2.13, [Z08]) (i) Every $L_{\infty\omega}(\mathcal{C}_0)$-type realized in M is equivalent to a projective type.

(ii) There are only countably many $L_{\infty\omega}(\mathcal{C}_0)$-types realized in M

(iii) $(M, \mathcal{C}_0^{\exists})$ is quasi minimal ω-homogeneous over countable submodels.

■

Definition 23 For $H_1, H_2 \in \mathcal{A}(M)$ with $H_{\subseteq}H_2$, we define $H_1 \preceq H_2$ if for every finite $X \subseteq H_1$, $\mathrm{cl}_{H_1}(X) = \mathrm{cl}_{H_2}(X)$.

Theorem 24 (Uncountable Categoricity, Thm 2.15, [Z08]) Given an analytic Zariski structure M and a countable core substructure $(M_0, \mathcal{C}_0)$, assume that $\mathcal{A}_0(M)$ is excellent. Then the class $\mathcal{A}(M)$ contains a structure of any infinite cardinality and is categorical in uncountable cardinals. ■

Under the same assumption as above theorem, if also the language of M is *essentially countable* i.e., there exists a countable $\mathcal{C}_{\text{base}} \subseteq \mathcal{C}$ such that every $S \in \mathcal{C}$ is of the form $S = P(a, M)$ for some $P \in \mathcal{C}_{\text{base}}$ and $a \in M^l$, and assuming further that $\mathcal{C}_{\text{base}} \subseteq \mathcal{C}_0$, then we have

Proposition 25 (Prop. 2.16, [Z08]) Any uncountable $H \in \mathcal{A}(M)$ is an analytic Zariski structure in the language $\mathcal{C}_0$ with parameters in H. If M is presmooth, then so is each H. ■

5 $\mathcal{K}_{\text{ex}}$ as a class of analytic Zariski geometries

Recall that the class $\mathcal{K}_{\text{ex}}$ is uncountable categorical (Theorem 9). On the other hand, showing $\mathcal{K}_{\text{ex}}$ is a class of analytic Zarisky geometries is still an open problem. We still do not know how to introduce a topology on each member of $\mathcal{K}_{\text{ex}}$.

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