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Numerical Existence Proofs and Guaranteed Error Bounds for Solutions to Two-Point Boundary Value Problems

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Abstract — In this article, a numerical method is presented for verifying the existence and the uniqueness of solutions to two-point boundary value problems of second order ordinary differential equations. By solving the bilinear form of the problem, a weak solution is a zero point of a certain nonlinear map. The Fréchet differentiability of this nonlinear map is shown. Based on the Newton-Kantorovich theorem, a numerical existence and local uniqueness theorem is presented for a zero point of the nonlinear map. It is shown that taking into account all errors of numerical computations such as discretization errors and rounding errors, conditions of this theorem can be checked by numerical computations with result verification. Finally, an illustrative numerical result is presented for showing the usefulness of the method.

1 Introduction

Let $(0,1)$ be an open interval. This article is concerned with the two boundary value problem of the second order ordinary differential equation:

$$\begin{cases} -(pu')' = f(u) & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where $p(x)$ is a smooth function on $(0,1)$ with $p(x) \geq p_0 > 0$ for some p_0. Here, $f : H_0^1(0,1) \to L^2(0,1)$ is assumed to be Fréchet differentiable. For example, the following function

$$f(u) = -qu' - c_1 u + c_2 u^2 + c_3 u^3 + \ldots + c_N u^N + g$$

with $N \in \mathbb{N}$, $q(x), c_i(x) \in L^\infty(0,1)$, $(i = 1, \ldots, N)$ and $g(x) \in L^2(0,1)$ satisfies this condition. We shall propose a numerical verification method for proving the existence of solutions to problem (1).

Studies on this type of computer assisted proofs for the existence of solutions to two point boundary value problems have been started by pioneering works of Kantorovich [1] and Urabe [2]. The works of McCarthy and Tapia [3] and of Kedem [4] have followed. In 1988, M. T. Nakao [5] has presented a method of a computer assisted proof for the existence of solutions to elliptic problems including the problem (1). This method has shown to be quite useful to generate tight numerical inclusion of solutions [6]. Nakao's method can be considered as an interval extension of the finite element method based on some fixed-point theorem. In 1991, Plum [7] has also presented another

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method of proving the existence and uniqueness of solutions for the problem (1). In his method, the norm of the inverse of linearlized operator is bounded by an eigenvalue enclosing technique based on the homotopy method. In these two decades, both Nakao's method and Plum's method have been demonstrated to be quite useful for the computer assisted existence proof of solutions of various boundary value problems of differential equations.

This article presents another method of a computer assisted proof procedure for the existence of solutions to the problem (1). In the verification theory, a weak formulation is led from the original problem. A weak solution is defined as a zero point of a certain nonlinear map from $H_0^1(0,1)$ into $H^{-1}(0,1)$ in this formulation. Then the Fréchet differentiability of this nonlinear map is shown. Based on the Newton-Kantorovich theorem [1], a numerical existence and local uniqueness theorem is derived for a zero point of this nonlinear map. This method is based on the theorem of estimating operator norm of inverse. This theorem makes it possible to obtain a numerical existence and local uniqueness theorem. It is also shown that taking into account all errors of numerical computations such as discretization errors and rounding errors, conditions of this theorem applied to this nonlinear map can be checked by numerical computations with result verification. One of features in this method is that verification conditions can be derived by a pure analytic way.

2 Verification Theory

In this section, we shall present a numerical method for verifying the existence and the uniqueness of solutions to two-point boundary value problems of the second order ordinary differential equation (1).

2.1 Preliminary

Throughout this article, let $L^2(0,1)$ denote the functional space of Lebesgue-measurable square-integrable functions with L^2-inner product and L^2-norm

$$(u,v)=\int_0^1 u(x)v(x)dx \quad \text{and} \quad \|u\|_{L^2}=\sqrt{(u,u)}, \quad (u,v\in L^2(0,1)),$$

respectively. Let $H^m(0,1)$ denote L^2-Sobolev space of order m with the inner product

$$\langle u,v\rangle_m=(u,v)+(u',u')+\cdots+(u^{(m)},u^{(m)})$$

and the norm [8]

$$\|u\|_{H^m}=\sqrt{\langle u,u\rangle_m}=\sqrt{\|u\|_{L^2}^2+\|u'\|_{L^2}^2+\cdots+\|u^{(m)}\|_{L^2}^2}.$$

Here, both $'$ and $\frac{d}{dx}$ denote the differentiation with respect to x and $u^{(m)}$ is the m-th derivative of u with respect to x. Let further

$$H_0^1(0,1)=\{u\in H^1 : u(0)=u(1)=0\}$$

with the inner product (u',v') and the norm $\|u\|_{H_0^1}=\|u'\|_{L^2}$. Let $H^{-1}(0,1)$ be the topological dual space of $H_0^1(0,1)$, *i.e.* the space of linear continuous functionals on $H_0^1(0,1)$. Let $T\in H^{-1}(0,1)$ and $u\in H_0^1(0,1)$. We denote $Tu\in\mathbb{R}$ as $<T,u>$. The norm of $T\in H^{-1}(0,1)$ is defined as

$$\|T\|_{H^{-1}}=\sup_{u\in H_0^1(0,1)\backslash\{0\}}\frac{|<T,u>|}{\|u\|_{H_0^1}}.$$

Let $L^\infty(0,1)$ denote the space of functions that are essentially bounded on $[0,1]$ with the norm

$$\|u\|_\infty = \operatorname*{ess\,sup}_{0\le x\le 1} |u(x)|.$$

Let X and Y be Banach spaces. The set of bounded linear operators is denoted by $\mathcal{L}(X,Y)$ with the operator norm

$$\|\mathcal{T}\|_{\mathcal{L}(X,Y)} = \sup_{u\in X\setminus\{0\}} \frac{\|\mathcal{T}u\|_Y}{\|u\|_X}, \quad (\mathcal{T}\in\mathcal{L}(X,Y)).$$

Here, $\|\cdot\|_X$ is the norm of X and $\|\cdot\|_Y$ is the norm of Y. The Sobolev embedding theorem states [6, 8] that

(1) for $(k > l)$ the embedding $H^k(0,1) \hookrightarrow H^l(0,1)$ is compact and continuous,

(2) the embedding $H_0^1(0,1) \hookrightarrow C^0(0,1)$ is compact and continuous,

(3) and $H_0^1(0,1) \subset L^p(0,1)$ for $p \ge 2$ with

$$\|v\|_{L^p} \le C_{e,p}\|v\|_{H_0^1}, \quad \left(v \in H_0^1(0,1) \quad \text{ex. } C_{e,p} = \left(\frac{2}{p+2}\right)^{\frac{1}{p}}\right). \tag{2}$$

2.2 Weak Formulation and its Fréchet Differentiability

Let us be concerned with the two-point boundary value problem of the form

$$\begin{cases} -(pu')' = f(u) & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{3}$$

In this part, we shall present a numerical verification method of proving the existence of weak solutions for Eq. (3). For $u, v \in H_0^1(0,1)$ let us define a continuous bilinear form $a(u,v)$ as

$$a(u,v) = (pu', v').$$

If we fix $u \in H_0^1(0,1)$, then $a(u,\cdot) \in H^{-1}(0,1)$. Thus, we can define an operator $\mathcal{A} : H_0^1(0,1) \to H^{-1}(0,1)$ by

$$<\mathcal{A}u, v>= a(u,v),$$

which can be seen as a weak form of the differential operator $-\frac{d}{dx}(p\frac{d}{dx})$. It is noted that the bilinear form a is coercive, *i.e.*, $a(u,u) \ge p_0\|u\|_{H_0^1}^2$. Then, for $v \in H_0^1(0,1)$ Lax-Milgram's theorem states the existence of a unique solution for the following equation:

$$a(u,v) =< T, v >, \quad (T \in H^{-1}(0,1)). \tag{4}$$

If we denote the operator which maps T to the solution u of Eq. (4) by $\mathcal{K} : H^{-1}(0,1) \to H_0^1(0,1)$, then this theorem also declares that $\mathcal{K}$ becomes an isomorphism between $H^{-1}(0,1)$ and $H_0^1(0,1)$. It is easy to see that

$$\mathcal{AK} = \mathcal{I}_{H^{-1}} \quad \text{and} \quad \mathcal{KA} = \mathcal{I}_{H_0^1}.$$

Here, $\mathcal{I}_{H^{-1}}$ and $\mathcal{I}_{H_0^1}$ are identity operators on $H^{-1}(0,1)$ and $H_0^1(0,1)$, respectively. In the rest of this article, we denote the identity operator by $\mathcal{I}$ omitting the subscript. Thus, we see $\mathcal{K} : H^{-1}(0,1) \to H_0^1(0,1)$ is the inverse of $\mathcal{A} : H_0^1(0,1) \to H^{-1}(0,1)$, *i.e.* $\mathcal{K} = \mathcal{A}^{-1}$.

Similarly, for $u, v \in H_0^1(0,1)$ we can define an operator $\mathcal{N} : H_0^1(0,1) \to H^{-1}(0,1)$ by

$$<\mathcal{N}u, v>= N(u,v) = (f(u), v).$$

Then, a weak form of Eq. (3) can be written as

$$\mathcal{A}u = \mathcal{N}u. \tag{5}$$

In the following, we will discuss how to verify the existence and the uniqueness of the solution of Eq. (5), the weak solution of the problem (3). Here, we note that the bilinear form $a(u,v)$ is an inner product on $H_0^1(0,1)$ and there exist positive constants C_a and c_a satisfying

$$c_a\|u\|_{H_0^1} \le \|u\|_a \le C_a\|u\|_{H_0^1} \quad \text{for} \quad u \in H_0^1(0,1), \tag{6}$$

where $\|u\|_a = \sqrt{a(u,u)}$. In fact, we can choose $c_a = \sqrt{p_0}$ and $C_a = \sqrt{\|p\|_{L^\infty}}$. We define the operator $\mathcal{F} : H_0^1(0,1) \to H^{-1}(0,1)$ by $\mathcal{F}u = (\mathcal{A} - \mathcal{N})u$. Then, Eq.(5) can be written as

$$\mathcal{F}u = 0. \tag{7}$$

Definitely, the weak solution of (3) is defined as a zero point of this nonlinear map $\mathcal{F}$.

Next, we now show that $\mathcal{F} : H_0^1(0,1) \to H^{-1}(0,1)$ is Fréchet differentiable. For a fixed $u, \hat{u} \in H_0^1(0,1)$ we can define $N'(\hat{u})(u,v)$ for $v \in H_0^1(0,1)$. It is clear that $N'(\hat{u})(u,\cdot) \in H^{-1}(0,1)$. Thus, we can define an operator $\mathcal{N}'(\hat{u}) : H_0^1(0,1) \to H^{-1}(0,1)$ by

$$< \mathcal{N}'(\hat{u})u, v >= N'(\hat{u})(u,v) = (f'(\hat{u})u, v).$$

Here, $f'(\hat{u}) : H_0^1(0,1) \to L^2(0,1)$ is the Fréchet derivative of $f : H_0^1(0,1) \to L^2(0,1)$ at $\hat{u}$. We now show that for a given $u \in H_0^1(0,1)$ the Fréchet derivative $\mathcal{F}'(u) : H_0^1(0,1) \to H^{-1}(0,1)$ of $\mathcal{F} : H_0^1(0,1) \to H^{-1}(0,1)$ is given as

$$\mathcal{F}'(u)v = (\mathcal{A} - \mathcal{N}'(u))v.$$

In fact, for $u, v \in H_0^1(0,1)$, we have

$$\begin{aligned}
\|\mathcal{F}(u+v) - \mathcal{F}(u) - (\mathcal{A} - \mathcal{N}'(u))v\|_{H^{-1}} &= \sup_{w \in H_0^1(0,1)\setminus\{0\}} \frac{|< \mathcal{N}(u+v) - \mathcal{N}u - \mathcal{N}'(u)v, w >|}{\|w\|_{H_0^1}} \\
&= \sup_{w \in H_0^1(0,1)\setminus\{0\}} \frac{|(f(u+v) - f(u) - f'(u)v, w)|}{\|w\|_{H_0^1}} \\
&\le \|\mu(u,v)\|_{L^2}.
\end{aligned}$$

Here, $\mu(u,v) = f(u+v) - f(u) - f'(u)v$. From the Fréchet differentiability of $f : H_0^1(0,1) \to L^2(0,1)$, we have

$$\frac{\|\mu(u,v)\|_{L^2}}{\|v\|_{H_0^1}} \to 0, \quad (\|v\|_{H_0^1} \to 0).$$

This shows the Fréchet differentiability of $\mathcal{F} : H_0^1(0,1) \to H^{-1}(0,1)$ at $u \in H_0^1(0,1)$ and

$$\mathcal{F}'(u) = \mathcal{A} - \mathcal{N}'(u).$$

Now, we define the natural embedding operator $i_{L^2 \hookrightarrow H^{-1}} : L^2(0,1) \to H^{-1}(0,1)$ by

$$i_{L^2 \hookrightarrow H^{-1}} w = T_w, \quad T_w(v) = (w,v) \quad \text{for} \quad v \in H_0^1(0,1).$$

Since $i_{L^2 \hookrightarrow H^{-1}} : L^2(0,1) \to H^{-1}(0,1)$ is compact and $f'(\hat{u}) : H_0^1(0,1) \to L^2(0,1)$ is continuous, the composite operator

$$\mathcal{N}'(\hat{u}) = i_{L^2 \hookrightarrow H^{-1}} \circ f'(\hat{u}) : H_0^1(0,1) \to H^{-1}(0,1)$$

is compact.

Now, we assume that an approximate solution $\hat{u} \in H_0^1(0,1)$ is given for Eq.(7). In order to prove the existence and the uniqueness of solution of Eq.(7) in the neighborhood of $\hat{u}$, the following Newton-Kantorovich Theorem [1] is applicable.

Theorem 1 (Newton-Kantorovich Theorem). *Let $\hat{u} \in H_0^1$. Let $\mathcal{F} : H_0^1(0,1) \to H^{-1}(0,1)$ be Fréchet differentiable at $\hat{u}$. Assume that the Fréchet derivative $\mathcal{F}'(\hat{u})$ is nonsingular and satisfies*

$$\|\mathcal{F}'(\hat{u})^{-1}\mathcal{F}\hat{u}\|_{H_0^1} \le \alpha,$$

for a certain positive α. Then, let $\mathcal{F} : H_0^1(0,1) \to H^{-1}(0,1)$ be Fréchet differentiable on $B(\hat{u}, 2\alpha) = \{v \in H_0^1(0,1) : \|v - \hat{u}\|_{H_0^1} \le 2\alpha\} \subset H_0^1(0,1)$ and assume that for a certain positive ω and for any $v, w \in B(\hat{u}, 2\alpha)$, the following holds:

$$\|\mathcal{F}'(\hat{u})^{-1}(\mathcal{F}'(v) - \mathcal{F}'(w))\|_{\mathcal{L}(H_0^1, H_0^1)} \le \omega\|v - w\|_{H_0^1}.$$

If

$$\alpha\omega \le \frac{1}{2},$$

then there is a solution $u^* \in H_0^1(0,1)$ of $\mathcal{F}u = 0$ satisfying

$$\|u^* - \hat{u}\|_{H_0^1} \le \rho := \frac{1 - \sqrt{1 - 2\alpha\omega}}{\omega}.$$

Furthermore, the solution u^* is unique in $B(\hat{u}, \rho)$.

This form of Newton-Kantorovich Theorem is called an affine invariant form. Verification constants α and ω have an invariance. Namely, we consider α which is defined by

$$\|\mathcal{F}'(\hat{u})^{-1}\mathcal{F}\hat{u}\|_{H_0^1} \le \alpha.$$

If we define $\mathcal{G} : H_0^1(0,1) \to H_0^1(0,1)$ by

$$\mathcal{G} = \mathcal{A}^{-1}\mathcal{F} = \mathcal{I} - \mathcal{A}^{-1}\mathcal{N},$$

then

$$\|\mathcal{G}'(\hat{u})^{-1}\mathcal{G}\hat{u}\|_{H_0^1} = \|\mathcal{F}'(\hat{u})^{-1}\mathcal{A}\mathcal{A}^{-1}\mathcal{F}\hat{u}\|_{H_0^1} = \|\mathcal{F}'(\hat{u})^{-1}\mathcal{F}\hat{u}\|_{H_0^1}.$$

This invariance also holds for ω. Thus,

$$\mathcal{G}u = 0 \tag{8}$$

is equivalent to Eq. (7) if one tries to prove the existence and uniqueness of solution of Eq. (8) in the neighborhood of $\hat{u}$ by the Newton Kantorovich theorem. Eq. (8) has been proposed by Nakao [5].

2.3 Finite Element Approximation

Let X_n denote a finite-dimensional space spanned by linearly independent H_0^1-conforming finite element basis functions $S_h = \{\phi_1, \phi_2, \cdots, \phi_n\}$ depending on the mesh size h, $(0 < h < 1)$:

$$X_n = \mathrm{span}\{\phi_1, \phi_2, ..., \phi_n\} \subset H_0^1(0,1).$$

The Ritz-projection $\mathcal{P}_n : H_0^1(0,1) \to X_n$ is defined by

$$(p(x)(u' - (\mathcal{P}_n u)'), v') = 0, \quad \forall\, v \in X_n. \tag{9}$$

Since $\mathcal{P}_n$ is the orthogonal projection with respect to the bilinear form $a(\cdot,\cdot)$, $\|\mathcal{P}_n u\|_a \le \|u\|_a$ holds. Now, let us consider a finite dimensional approximation of Eq. (7) of the following form:

$$\mathcal{P}_n\mathcal{A}^{-1}\mathcal{F}\mathcal{P}_n u = \mathcal{P}_n\mathcal{A}^{-1}(\mathcal{A} - \mathcal{N})\mathcal{P}_n u = \mathcal{P}_n(u - \mathcal{A}^{-1}\mathcal{N}\mathcal{P}_n u) = 0.$$

Let $u_h \in X_n$ be a solution of

$$\mathcal{P}_n(u_h - \mathcal{A}^{-1}\mathcal{N}\mathcal{P}_n u_h) = 0. \tag{10}$$

From Eq. (10), we have

$$(p(x)(u_h - \mathcal{P}_n\mathcal{A}^{-1}\mathcal{N}u_h)', \phi_i') = 0, \quad (i = 1, 2, \cdots, n). \tag{11}$$

The left hand side of Eq.(11) can be rewritten as

$$\begin{aligned} (p(x)u_h' - p(x)(\mathcal{P}_n\mathcal{A}^{-1}\mathcal{N}u_h)', \phi_i') &= (p(x)u_h' - p(x)(\mathcal{A}^{-1}\mathcal{N}u_h)', \phi_i') \\ &= (p(x)u_h', \phi_i') - (f(u_h), \phi_i). \end{aligned}$$

Thus, it turns out that Eq. (10) becomes

$$(p(x)u_h', \phi_h') = (f(u_h), \phi_h), \quad (\forall \phi_h \in S_h),$$

which is nothing but the finite element approximation [9] of the nonlinear equation (7).

2.4 Norm Estimation of Inverse Operator

Let $\hat{u} \in H_0^1(0,1)$. For the estimation of $\|(\mathcal{I} - \mathcal{A}^{-1}\mathcal{N}'(\hat{u}))^{-1}\|_{\mathcal{L}(H_0^1,H_0^1)}$, we present the following theorem, which is a modification of the main theorem in [10, 11] presented by one of authors (S. Oishi):

Theorem 2. *Let $\hat{u} \in H_0^1(0,1)$. Let further $\mathcal{N}'(\hat{u}) : H_0^1(0,1) \to H^{-1}(0,1)$ be a linear compact operator. Let X_n be a finite dimensional subspace of $H_0^1(0,1)$ spanned by the finite element bases $S_h = \{\phi_1, \phi_2, \cdots, \phi_n\}$. Let $\mathcal{P}_n : H_0^1(0,1) \to X_n$ be the Ritz-projection and $\mathcal{T} = \mathcal{A}^{-1}\mathcal{N}'(\hat{u})$. We assume that $\mathcal{P}_n\mathcal{T} : H_0^1(0,1) \to H_0^1(0,1)$ is bounded and satisfies*

$$\|\mathcal{P}_n\mathcal{T}\|_{\mathcal{L}(H_0^1,H_0^1)} \le K,$$

the difference between $\mathcal{T}$ and $\mathcal{P}_n\mathcal{T}$ is bounded and enjoys

$$\|\mathcal{T} - \mathcal{P}_n\mathcal{T}\|_{\mathcal{L}(H_0^1,H_0^1)} \le L$$

and the finite dimensional operator $\mathcal{P}_n(\mathcal{I} - \mathcal{T})|_{X_n} : X_n \to X_n$ is invertible with

$$\|(\mathcal{P}_n(\mathcal{I} - \mathcal{T})|_{X_n})^{-1}\|_{\mathcal{L}(H_0^1,H_0^1)} \le M.$$

Here, $\mathcal{P}_n(\mathcal{I} - \mathcal{T})|_{X_n} : X_n \to X_n$ is the restriction of the operator $\mathcal{P}_n(\mathcal{I} - \mathcal{T}) : H_0^1(0,1) \to X_n$ on X_n. If $(1 + MK)L < 1$, then $\mathcal{I} - \mathcal{T} : H_0^1(0,1) \to H_0^1(0,1)$ is invertible and enjoys

$$\|(\mathcal{I} - \mathcal{T})^{-1}\|_{\mathcal{L}(H_0^1,H_0^1)} \le \frac{1 + MK}{1 - (1 + MK)L} := C_1.$$

□

Proof. Since

$$u = (\mathcal{I} - \mathcal{T})u + (\mathcal{T} - \mathcal{P}_n\mathcal{T})u + \mathcal{P}_n\mathcal{T}u,$$

we have

$$\begin{aligned} \|u\|_{H_0^1} &\le \|(\mathcal{I} - \mathcal{T})u\|_{H_0^1} + \|(\mathcal{T} - \mathcal{P}_n\mathcal{T})\|_{\mathcal{L}(H_0^1,H_0^1)}\|u\|_{H_0^1} + \|\mathcal{P}_n\mathcal{T}u\|_{H_0^1} \\ &\le \|(\mathcal{I} - \mathcal{T})u\|_{H_0^1} + L\|u\|_{H_0^1} + \|\mathcal{P}_n\mathcal{T}u\|_{H_0^1}. \end{aligned} \tag{12}$$

From

$$\begin{aligned}\mathcal{P}_n(\mathcal{I}-\mathcal{T})\mathcal{P}_n\mathcal{T}u &= \mathcal{P}_n(\mathcal{I}-\mathcal{T})(\mathcal{P}_n\mathcal{T}-\mathcal{T})u+\mathcal{P}_n(\mathcal{I}-\mathcal{T})\mathcal{T}u \\ &= \mathcal{P}_n\mathcal{T}(\mathcal{T}-\mathcal{P}_n\mathcal{T})u+\mathcal{P}_n\mathcal{T}(\mathcal{I}-\mathcal{T})u\end{aligned}$$

and the invertibility of $\mathcal{P}_n(\mathcal{I}-\mathcal{T})|_{X_n}: X_n \to X_n$ with

$$\|(\mathcal{P}_n(\mathcal{I}-\mathcal{T})|_{X_n})^{-1}\|_{\mathcal{L}(H_0^1,H_0^1)} \leq M,$$

we have

$$\begin{aligned}\|\mathcal{P}_n\mathcal{T}u\|_{H_0^1} &\leq M\|\mathcal{P}_n\mathcal{T}\|_{\mathcal{L}(H_0^1,H_0^1)}\|(\mathcal{T}-\mathcal{P}_n\mathcal{T})\|_{\mathcal{L}(H_0^1,H_0^1)}\|u\|_{H_0^1}+M\|\mathcal{P}_n\mathcal{T}\|_{\mathcal{L}(H_0^1,H_0^1)}\|(\mathcal{I}-\mathcal{T})u\|_{H_0^1} \\ &\leq MKL\|u\|_{H_0^1}+MK\|(\mathcal{I}-\mathcal{T})u\|_{H_0^1}. \qquad (13)\end{aligned}$$

Substituting the inequality (13) into the inequality (12), we have

$$\|u\|_{H_0^1} \leq (1+MK)\|(\mathcal{I}-\mathcal{T})u\|_{H_0^1}+(1+MK)L\|u\|_{H_0^1}.$$

Thus, if $(1+MK)L<1$, then we obtain

$$\|u\|_{H_0^1} \leq \frac{1+MK}{1-(1+MK)L}\|(\mathcal{I}-\mathcal{T})u\|_{H_0^1}. \qquad (14)$$

From the inequality (14), if $(\mathcal{I}-\mathcal{T})u=0$, $u=0$ follows. This implies the operator $(\mathcal{I}-\mathcal{T}): H_0^1(0,1) \to H_0^1(0,1)$ is injective. Since the the operator $(\mathcal{I}-\mathcal{T}): H_0^1(0,1) \to H_0^1(0,1)$ is of Fredholm type with the index 0, it is also surjective. Thus, $\mathcal{I}-\mathcal{T}: H_0^1(0,1) \to H_0^1(0,1)$ is invertible and enjoys

$$\|(\mathcal{I}-\mathcal{T})^{-1}\|_{\mathcal{L}(H_0^1,H_0^1)} \leq \frac{1+MK}{1-(1+MK)L}.$$

□

2.5 Estimating Constants K and L

Let $\hat{u}, v \in H_0^1(0,1)$. By (6), we have

$$\|\mathcal{P}_n\mathcal{T}(\hat{u})v\|_{H_0^1}^2 \leq \frac{1}{c_a^2}(p(x)(\mathcal{P}_n\mathcal{T}(\hat{u})v)', (\mathcal{P}_n\mathcal{T}(\hat{u})v)').$$

Here, $\mathcal{T}(\hat{u})=\mathcal{A}^{-1}\mathcal{N}'(\hat{u})$. From the definition of the Ritz-projection (9), it follows

$$\begin{aligned}(p(x)(\mathcal{P}_n\mathcal{T}(\hat{u})v)', (\mathcal{P}_n\mathcal{T}(\hat{u})v)') &= (p(x)(\mathcal{T}(\hat{u})v)', (\mathcal{P}_n\mathcal{T}(\hat{u})v)') \\ &= (f'(\hat{u})v, \mathcal{P}_n\mathcal{T}(\hat{u})v).\end{aligned}$$

We note here that

$$\begin{aligned}(f'(\hat{u})v, \mathcal{P}_n\mathcal{T}(\hat{u})v) &\leq \|f'(\hat{u})v\|_{L^2}\|\mathcal{P}_n\mathcal{T}(\hat{u})v\|_{L^2} \\ &\leq \|f'(\hat{u})\|_{\mathcal{L}(H_0^1,L^2)}\|v\|_{H_0^1}C_{e,2}\|\mathcal{P}_n\mathcal{T}(\hat{u})v\|_{H_0^1},\end{aligned}$$

where $C_{e,2}$ is an embedding constant defined by (2). Thus, it turns out that

$$\|\mathcal{P}_n\mathcal{T}(\hat{u})v\|_{H_0^1} \leq \frac{C_{e,2}}{c_a^2}\|f'(\hat{u})\|_{\mathcal{L}(H_0^1,L^2)}\|v\|_{H_0^1},$$

which implies

$$\|\mathcal{P}_n\mathcal{T}(\hat{u})\|_{\mathcal{L}(H_0^1,H_0^1)} \le \frac{C_{e,2}}{c_a^2}\|f'(\hat{u})\|_{\mathcal{L}(H_0^1,L^2)}.$$

Consequently, one can put K as

$$K = \frac{C_{e,2}}{c_a^2}\|f'(\hat{u})\|_{\mathcal{L}(H_0^1,L^2)}.$$

Now, we derive the constant L. For $w \in L^2(0,1)$ we define $T_w \in H^{-1}(0,1)$ by

$$T_w(v) = (w,v) \quad \text{for} \quad v \in H_0^1(0,1).$$

We assume for $w \in L^2(0,1)$

$$\|(\mathcal{A}^{-1} - \mathcal{P}_n\mathcal{A}^{-1})T_w\|_{H_0^1} \le C_0(h)\|w\|_{L^2} \tag{15}$$

holds. In case of $p(x) = 1$, one can take $C_0(h) = \frac{h}{\pi}$ for one-dimensional piecewise linear hat functions. From Eq. (15), for $v \in H_0^1(0,1)$ we have

$$\begin{aligned}\|(\mathcal{A}^{-1} - \mathcal{P}_n\mathcal{A}^{-1})\mathcal{N}'(\hat{u})v\|_{H_0^1} &= \|(\mathcal{A}^{-1} - \mathcal{P}_n\mathcal{A}^{-1})T_{f'(\hat{u})v}\|_{H_0^1} \\ &\le C_0(h)\|f'(\hat{u})v\|_{L^2} \\ &\le C_0(h)\|f'(\hat{u})\|_{\mathcal{L}(H_0^1,L^2)}\|v\|_{H_0^1},\end{aligned}$$

which implies

$$\|\mathcal{T}(\hat{u}) - \mathcal{P}_n\mathcal{T}(\hat{u})\|_{\mathcal{L}(H_0^1,H_0^1)} \le C_0(h)\|f'(\hat{u})\|_{\mathcal{L}(H_0^1,L^2)}.$$

Thus, as the constant L, one can put

$$L = C_0(h)\|f'(\hat{u})\|_{\mathcal{L}(H_0^1,L^2)}.$$

2.6 Method of Calculating M

Let $\hat{u} \in H_0^1(0,1)$. We shall show how to calculate the constant M defined by

$$\|(\mathcal{I} - \mathcal{P}_n\mathcal{T}(\hat{u}))|_{X_n}^{-1}\|_{\mathcal{L}(H_0^1,H_0^1)} \le M.$$

Let $\phi, \psi \in X_n$ be related by $(\mathcal{I} - \mathcal{P}_n\mathcal{T}(\hat{u}))|_{X_n}^{-1}\phi = \psi$. Since $\phi, \psi \in X_n$, we can put

$$\phi = \sum_{j=1}^n s_j\phi_j, \quad \psi = \sum_{j=1}^n t_j\phi_j.$$

From $(\mathcal{I} - \mathcal{P}_n\mathcal{T}(\hat{u}))\psi = \phi$, we have

$$(p(x)(\psi - \mathcal{P}_n\mathcal{A}^{-1}\mathcal{N}'(\hat{u})\psi)', \phi_i') = (p(x)\phi', \phi_i'), \quad (i = 1, 2, \cdots, n). \tag{16}$$

The left hand side of Eq.(16) can be rewritten as

$$\begin{aligned}\sum_{j=1}^n t_j(p(x)\phi_j' - p(x)(\mathcal{P}_n\mathcal{A}^{-1}\mathcal{N}'(\hat{u})\phi_j)', \phi_i') &= \sum_{j=1}^n t_j(p(x)\phi_j' - p(x)(\mathcal{A}^{-1}\mathcal{N}'(\hat{u})\phi_j)', \phi_i') \\ &= \sum_{j=1}^n t_j\left[(p(x)\phi_j', \phi_i') - (f'(\hat{u})\phi_j, \phi_i)\right]. \end{aligned} \tag{17}$$

The right hand side of Eq.(16) can be rewritten as

$$\sum_{j=1}^{n} s_j(p(x)\phi_j', \phi_i'). \tag{18}$$

Let D and G be $n \times n$ real matrices whose i-j elements are given by $(p(x)\phi_j', \phi_i')$ and $(p(x)\phi_j', \phi_i') - (f'(\hat{u})\phi_j, \phi_i)$, respectively. Then, from (17) and (18) it turns out that

$$G\tau = D\sigma,$$

where $\sigma = (s_1, s_2, \cdots, s_n)^t$ and $\tau = (t_1, t_2, \cdots, t_n)^t$. Here, the superscript ' t ' denotes the transpose. Since stiffness matrix D is symmetric positive definite, there exists a lower triangular matrix $\hat{L}$ forming the Cholesky decomposition, $D = \hat{L}\hat{L}^t$. We denote the Euclidean norm of σ as $\|\sigma\|_2 = \sqrt{s_1^2 + s_2^2 + \cdots + s_n^2}$. Then, we have

$$\|\phi\|_a^2 = \sigma^t D\sigma = \sigma^t \hat{L}\hat{L}^t\sigma = \|(\hat{L}^t\sigma)^t(\hat{L}^t\sigma)\|_2 = \|\hat{L}^t\sigma\|_2^2.$$

Thus, it turns out that $\|\phi\|_a = \|\hat{L}^t\sigma\|_2$. Similarly, we have $\|\psi\|_a = \|\hat{L}^t\tau\|_2$. The invertibility of G can be checked by the numerical computation with result verification. Here, assuming the existence of G^{-1}, we consider

$$\|\psi\|_a^2 = \tau^t D\tau = \tau^t DG^{-1}D\sigma = (\hat{L}^t\tau)^t(\hat{L}^tG^{-1}\hat{L})(\hat{L}^t\sigma). \tag{19}$$

Using Schwarz's inequality for n-dimensional vectors w, y, $w^ty \le \|w\|_2\|y\|_2$, from Eq. (19) we have

$$\|\psi\|_a^2 \le \|\hat{L}^t\tau\|_2\|(\hat{L}^tG^{-1}\hat{L})(\hat{L}^t\sigma)\|_2 \le \|\psi\|_a\|\hat{L}^tG^{-1}\hat{L}\|_2\|\phi\|_a.$$

Thus, it turns out that one can put

$$M = \frac{C_a}{c_a}\|\hat{L}^tG^{-1}\hat{L}\|_2. \tag{20}$$

We note that this kind of arguments can be found in Nakao, Hashimoto and Watanabe [12]. The spectral norm of (20) can be obtained by the method in [13], which is suggested by Prof. Rump at Hamburg Institute of Technology.

2.7 Method of Calculating Norm of Residual

Let $\hat{u} \in X_n \subset H_0^1(0,1)$ be an approximate solution of the problem (1). In this subsection, we shall show how to calculate the upper bound of the norm of the residual:

$$\begin{aligned}
\|\mathcal{G}\hat{u}\|_{H_0^1} &= \|\hat{u} - \mathcal{A}^{-1}\mathcal{N}\hat{u}\|_{H_0^1} \\
&= \left\|\hat{u} - \mathcal{P}_n\mathcal{A}^{-1}\mathcal{N}\hat{u} - \mathcal{A}^{-1}\mathcal{N}\hat{u} + \mathcal{P}_n\mathcal{A}^{-1}\mathcal{N}\hat{u}\right\|_{H_0^1} \\
&\le \|\hat{u} - \mathcal{P}_n\mathcal{A}^{-1}\mathcal{N}\hat{u}\|_{H_0^1} + \|(\mathcal{A}^{-1} - \mathcal{P}_n\mathcal{A}^{-1})\mathcal{N}\hat{u}\|_{H_0^1} \\
&\le \|\hat{u} - \mathcal{P}_n\mathcal{A}^{-1}\mathcal{N}\hat{u}\|_{H_0^1} + C_0(h)\|f(\hat{u})\|_{L^2} =: C_2.
\end{aligned}$$

We show now how to calculate $\|\hat{u} - \mathcal{P}_n\mathcal{A}^{-1}\mathcal{N}(\hat{u})\|_{H_0^1}$. Since $\hat{u} \in X_n$, one can put $\hat{u} = \sum_{j=1}^n \hat{u}_j\phi_j$. Let $\hat{u}^h = (\hat{u}_1, \hat{u}_2, \cdots, \hat{u}_n)$. From $\hat{u} - \mathcal{P}_n\mathcal{A}^{-1}\mathcal{N}(\hat{u}) \in X_n$, we put

$$\hat{u} - \mathcal{P}_n\mathcal{A}^{-1}\mathcal{N}\hat{u} = \sum_{j=1}^{n} r_j\phi_j$$

and $r^h = (r_1, r_2, \cdots, r_n)^t$. For ϕ_i, $(i = 1, ..., n)$, we have

$$(p(x)(\hat{u} - \mathcal{P}_n\mathcal{A}^{-1}\mathcal{N}(\hat{u}))', \phi_i') = \sum_{j=1}^{n} r_j(p(x)\phi_j', \phi_i'), \quad (i = 1, 2, \cdots, n). \tag{21}$$

The left hand side of Eq.(21) can be rewritten as

$$\sum_{j=1}^{n} \hat{u}_j(p(x)\phi_j', \phi_i') - (f(\hat{u}), \phi_i).$$

Put $f^h = (f_1, f_2, \cdots, f_n)^t$ with $f_i = (f(\hat{u}), \phi_i)$, $(i = 1, 2, \cdots, n)$. Then, Eq.(21) reduces to

$$Dr^h = D\hat{u}^h - f^h.$$

Thus, we have $r^h = D^{-1}(D\hat{u}^h - f^h)$, which implies

$$\|\hat{u} - \mathcal{P}_n\mathcal{A}^{-1}\mathcal{N}\hat{u}\|_{H_0^1} = \frac{1}{c_a}\sqrt{(r^h)^t Dr^h} \leq \frac{1}{c_a}\sqrt{\|D^{-1}\|_2}\|D\hat{u}^h - f^h\|_2.$$

2.8 Estimation of Lipschitz Constant

Finally, we estimate the Lipschitz constant of $\mathcal{T}(u)$ by assuming $f' : H_0^1(0,1) \to L^2(0,1)$ is Lipschitz continuous on $B(\hat{u}, 2\alpha)$. We note that for $u, v, w \in H_0^1(0,1)$ we have

$$\begin{aligned}
\|(\mathcal{T}(v) - \mathcal{T}(w))u\|_{H_0^1}^2 &\leq \frac{1}{c_a^2}\|\mathcal{A}^{-1}(\mathcal{N}'(v) - \mathcal{N}'(w))u\|_a^2 \\
&= \frac{1}{c_a^2}((f'(v) - f'(w))u, \mathcal{A}^{-1}(\mathcal{N}'(v) - \mathcal{N}'(w))u) \\
&\leq \frac{1}{c_a^2}\|(f'(v) - f'(w))u\|_{L^2}\|\mathcal{A}^{-1}(\mathcal{N}'(v) - \mathcal{N}'(w))u\|_{L^2}.
\end{aligned}$$

Thus, it follows that

$$\|(\mathcal{T}(v) - \mathcal{T}(w))u\|_{H_0^1} \leq \frac{C_{e,2}}{c_a^2}\|\left(f'(v) - f'(w)\right)u\|_{L^2}.$$

Here, if $f' : H_0^1(0,1) \to L^2(0,1)$ is Lipschitz continuous on $B(\hat{u}, 2\alpha)$, *i.e.*, there exists a positive constant C_L satisfying

$$\|f'(v) - f'(w)\|_{\mathcal{L}(H_0^1, L^2)} \leq C_L\|v - w\|_{H_0^1(0,1)}, \quad (v, w \in B(\hat{u}, 2\alpha)),$$

then we have

$$\|\mathcal{F}'(v) - \mathcal{F}'(w)\|_{\mathcal{L}(H_0^1, H_0^1)} \leq \frac{C_{e,2}}{c_a^2}C_L\|v - w\|_{H_0^1}, \quad (v, w \in B(\hat{u}, 2\alpha)).$$

2.9 Computer Assisted Existence Algorithm

In this subsection, we present an algorithm of verifying the existence and the uniqueness of solution of Eq.(8) in the neighborhood of $\hat{u}$ by the Newton Kantorovich theorem. The following is a computer assisted proof algorithm based on our verification method.

Algorithm 1 (TWO-POINT BOUNDARY VALUE PROBLEMS [4]). *(Existence and uniqueness test of solutions for two-point boundary value problems of nonlinear ordinary differential equations (1).)*

1. *Compute an approximate solution* $\hat{u}$ *of the problem (1) by any numerical method.*

2. *Compute rigorous upper bound of* $\|(\mathcal{I}-\mathcal{T})^{-1}\|_{\mathcal{L}(H_0^1,H_0^1)}$ *by the following steps:*

   2.1 *Compute* $\|\hat{u}\|_\infty$ *and calculate* K *and* L *by*

$$K = \frac{C_{e,2}}{p_0}\|f'(\hat{u})\|_{\mathcal{L}(H_0^1,L^2)} \quad \text{and} \quad L = C_0(h)\|f'(\hat{u})\|_{\mathcal{L}(H_0^1,L^2)},$$

   respectively.

   2.2 *Let* D *and* G *be* $n \times n$ *matrices whose i-j elements are given by*

$$(p(x)\phi_j', \phi_i') \text{ and } (p(x)\phi_j', \phi_i') - (f'(\hat{u})\phi_j, \phi_i),$$

   respectively. Let a lower triangular matrix $\hat{L}$ *be the Cholesky decomposition of* D, $D = \hat{L}\hat{L}^t$. *If* G *is invertible, then set*

$$M = \frac{C_a}{c_a}\|\hat{L}^t G^{-1}\hat{L}\|_2.$$

   When G *is not invertible, stop with failure.*

   2.3 *Check whether* $(1+MK)L < 1$ *holds or not. If this holds, then by Theorem 2 we have*

$$\|(\mathcal{I}-\mathcal{T})^{-1}\|_{\mathcal{L}(H_0^1,H_0^1)} \le \frac{1+MK}{1-(1+MK)L} =: C_1.$$

   Otherwise, stop with failure.

3. *Calculate the residual by the formula*

$$C_2 := \|\hat{u} - \mathcal{P}_n\mathcal{A}^{-1}\mathcal{N}(\hat{u})\|_{H_0^1} + C_0(h)\|f(\hat{u})\|_{L^2}.$$

   Set $\alpha = C_1C_2$.

4. *Calculate the Lipschitz constant* C_3 *by*

$$C_3 := \frac{C_{e,2}}{p_0}C_L.$$

   Set $\omega = C_1C_3$.

5. *Check the condition* $\alpha\omega \le \frac{1}{2}$. *If this condition is satisfied, there is a solution* $u^* \in H_0^1(0,1)$ *of* $\mathcal{F}u = 0$ *satisfying*

$$\|u^* - \hat{u}\|_{H_0^1} \le \rho := \frac{1-\sqrt{1-2\alpha\omega}}{\omega}.$$

   Furthermore, the solution u^* *is unique in* $B(\hat{u},\rho)$.

   Otherwise, stop with failure.

[4]Linear case (ex. $c_i = 0$, $(i = 2, ..., N)$) can be treated by more direct estimate of the error analysis: $\|u-\hat{u}\|_{H_0^1} \le \|(I-\mathcal{T})^{-1}\|_{\mathcal{L}(H_0^1,H_0^1)}\|g-(I-\mathcal{T})\hat{u}\|_{H_0^1} \le C_1C_2$. The constant C_3 is not needed.

3 Computational Results

As a numerical example, we consider the following quadratic nonlinear two-point boundary value problem [6]:

$$\begin{cases} -u'' = u^2 & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{22}$$

An approximate solution $\hat{u}$ is calculated by the finite element method with bases as one-dimensional piecewise linear hat functions. The proposal verification method is applicable to Eq. (22). Our computer assisted proof method yields

$$K = 2.391,\ L = 0.005,\ M = 1.852,\ C_1 = 5.568,\ C_2 = 0.059,\ C_3 = 0.226.$$

Then we have $\alpha = 0.333$ and $\omega = 1.254$ so that $\alpha\omega < 0.417$. Consequently, it follows that there exists an unique solution in the ball $B(\hat{u}, \rho)$ with the radius

$$\|u - \hat{u}\|_{H_0^1} \le \rho = 0.472.$$

Figure 1 shows the guaranteed inclusion of the exact solution of Eq. (22). It is proved that there exists a unique solution between two curves. Since $H_0^1(0,1) \hookrightarrow C^0(0,1)$, we can obtain the guaranteed error bound in maximum norm by Poincaré's inequality.

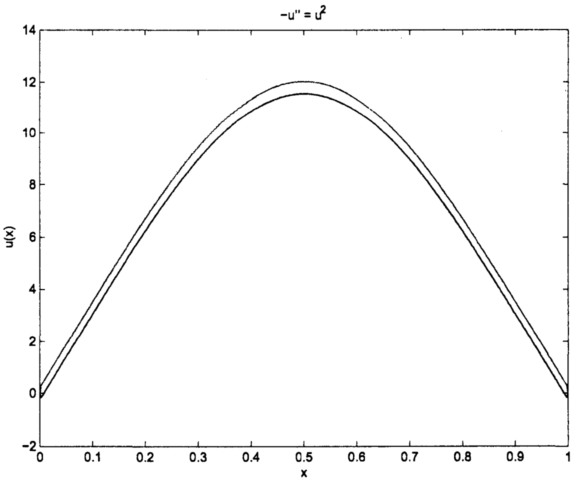

Figure 1: Guaranteed Inclusion of the Exact Solution (Mesh size $\frac{1}{512}$)

By increasing grid points, guaranteed error bounds are improved with $O(h)$. The guaranteed error and the ratio are presented in Table 1. All computations are carried out on Mac OS X, Intel Core2 Duo 1.86GHz by using MATLAB 2009a with a toolbox for verified computations, INTLAB[14].

Table 1: Verification Results for Problem (22)

Grid Points: 2^x	Guaranteed Error: ρ	Ratio: $O(h^\gamma)$
9	4.72×10^{-1}	-
10	1.85×10^{-1}	1.27
11	8.60×10^{-2}	1.08
12	4.17×10^{-2}	1.03
13	2.05×10^{-2}	1.01
14	1.02×10^{-2}	1.01
15	5.11×10^{-3}	1.00

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