数理解析研究所講究録
第 1719 巻 2010 年 142-154

DISCRETIZATION BY THE THEORY OF REPRODUCING KERNELS

Luís P. Castro, Dep. of Math. University of Aveiro
Saburou Saitoh, Dep. of Math. University of Aveiro
Yoshihiro Sawano, Dep. of Math. Kyoto University
and
Alberto M. Simões, Dep. of Math. University of Beira Interior

March 15, 2010

1 Introduction

The main purpose of this work is to analyse linear partial differential equations and corresponding inverse source problems via a reproducing kernel Hilbert spaces approach. The method here proposed is built in a very general framework. Detailed applications of the present method, e.g. to the most known basic linear partial differential equations (such that the heat equation, the wave equation and the Laplace equation), are expected to appear in forthcoming works by using the present approach as a starting point.

Assuming g to be any complex valued L_2 function, we shall first give very simple approximate solutions to the partial differential equation,

$$L(D)u = g \quad \text{on} \quad \mathbb{R}^n, \tag{1.1}$$

in the class of the functions of the s order Sobolev Hilbert space $H^s(s > \frac{n}{2})$ on the whole real space $\mathbb{R}^n$, with $n \geq 1$, $s \geq m \geq 1$ and $s > n/2$ and where m denotes the order of the nontrivial general linear partial differential operator $L(D)$ with complex constant coefficients on $\mathbb{R}^n$. That is, we are given a linear partial differential operator

$$L(D) = \sum_{|\alpha| \le m} a_\alpha \left(\frac{\partial}{\partial x} \right)^\alpha,$$

where the a_α's are complex numbers and there exists a multi-index α_0 of length m such that $a_\alpha \neq 0$. Anyway, many constant coefficient partial differential equations fall under the scope of our main results in case of being inhomogeneous linear partial differential equations with complex constant coefficients of all types on the whole space $\mathbb{R}^n$.

For simplicity, we write

$$L(D)e^{ix\cdot\xi} = \mathbf{L}(\xi)e^{ix\cdot\xi}$$

by using a complex polynomial $\mathbf{L}$.

Theorem 1. *Let $n \ge 1, s \ge m \ge 1$ and $s > n/2$.*

(i) For any complex valued function $g \in L_2$ and for any $\lambda > 0$,

$$\inf_{F \in H^s} \{\lambda \|F\|^2_{H^s} + \|g - L(D)F\|^2_{L_2}\} \tag{1.2}$$

is attained by a unique function $F^*_{\lambda,s,g}$.

(ii) Let us write

$$Q_{\lambda,s}(\eta) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\overline{\mathbf{L}(p)} e^{-ip\cdot\eta} dp}{\lambda(|p|^2+1)^s + |\mathbf{L}(p)|^2}. \tag{1.3}$$

Then, the extremal function $F^*_{\lambda,s,g}$ is represented by

$$F^*_{\lambda,s,g}(x) = \int_{\mathbb{R}^n} g(\xi) Q_{\lambda,s}(\xi - x) d\xi. \tag{1.4}$$

(iii) If g is expressed as $g = L(D)F$, for a function $F \in H^s$, then we have the favourable result: as $\lambda \to 0$

$$F^*_{\lambda,s,g} \to F, \tag{1.5}$$

uniformly.

As first examples that we envisage to consider within the basis of the present work, include the differential equations characterized by the following operators:

1. The $\overline{\partial}$-operator
$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right). \tag{1.6}$$

2. The heat operator
$$\partial_t u(x,t) - c^2 \Delta_x u(x,t) \quad \text{on} \quad \mathbb{R}^{n-1} \times \mathbb{R}, \tag{1.7}$$
where Δ_x denotes the Laplacian on $\mathbb{R}^{n-1}$.

3. The wave operator
$$\partial_t^2 u(x,t) - c^2 \Delta_x u(x,t) \quad \text{on} \quad \mathbb{R}^{n-1} \times \mathbb{R} \quad (c > 0). \tag{1.8}$$

Note that although there are not global solutions to (1.7), our main result is still applicable.

From concrete examples we see that the representations (1.4) are effectively computable by using computers and the approximate solutions (1.4) converge to the true solutions of (1.1) as in (1.5) (cf. [1] and [6]).

When the practical data g contain error or noises, we need error estimates for the approximate solution (1.4). As an improved version in [4], we can establish the following result.

Theorem 2. *For $g, g_\delta \in L^2(\mathbb{R}^n)$ we have*
$$\|F^*_{\lambda,s,g} - F^*_{\lambda,s,g_\delta}\|_{H^s} \leq \frac{1}{2\sqrt{\lambda}}\|g - g_\delta\|_{L_2}. \tag{1.9}$$

In this note, we shall first give the corresponding results for the general partial differential equation (1.1). Furthermore, we shall establish error estimates for our solutions because practical data contain error and noises. In the second part, we shall construct the solutions for discrete differential equations, following the recent general idea ([2]) and furthermore, we shall give their inversions. That is, we shall give the corresponding construction of the solutions of the inverse source problems.

2 Preliminaries on reproducing kernel Hilbert spaces

2.1 Sobolev reproducing kernel Hilbert spaces

We shall use the Sobolev reproducing kernel Hilbert spaces.

Let m be an integer such that $m > \frac{n}{2}$. Then, by H^m we shall denote the Sobolev Hilbert space whose norm is given by

$$\|F\|_{H^m} = \left[\sum_{\nu=0}^{m} {}_mC_\nu \left(\sum_{\alpha \in \mathbb{Z}_+^n,\, |\alpha| \le \nu} \frac{\nu!}{\alpha!} \int_{\mathbb{R}^n} \left| \frac{\partial^\alpha F(x)}{\partial x^\alpha} \right|^2 dx \right)\right]^{\frac{1}{2}}. \tag{2.1}$$

Observe that the corresponding inner product of this space is given by

$$\langle F, G \rangle_{H^m} = \sum_{\nu=0}^{m} {}_mC_\nu \left(\sum_{\alpha \in \mathbb{Z}_+^n,\, |\alpha| \le \nu} \frac{\nu!}{\alpha!} \int_{\mathbb{R}^n} \frac{\partial^\alpha F(x)}{\partial x^\alpha} \overline{\frac{\partial^\alpha G(x)}{\partial x^\alpha}}\, dx \right). \tag{2.2}$$

Here we have denoted

$$x = (x_1, x_2, ..., x_n), \quad \xi = (\xi_1, \xi_2, ..., \xi_n), \quad d\xi = d\xi_1 \cdot d\xi_2 \cdot \cdots \cdot d\xi_n \tag{2.3}$$

as usual. If we define the Fourier transform

$$(\mathcal{F}F)(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} F(x) e^{-ix\cdot\xi}\, dx, \tag{2.4}$$

where $x \cdot \xi = x_1\xi_1 + \cdots\cdots + x_n\xi_n$ is a standard inner product in $\mathbb{R}^n$, then we have, by virtue of the Fourier transform and the multinomial expansion

$$\begin{aligned}
\langle F, G \rangle_{H^m} &= \sum_{\nu=0}^{m} {}_mC_\nu \left(\sum_{\alpha \in \mathbb{Z}_+^n,\, |\alpha| \le \nu} \frac{\nu!}{\alpha!} \int_{\mathbb{R}^n} \xi^{2\alpha} (\mathcal{F}F)(\xi) \overline{(\mathcal{F}G)(\xi)}\, d\xi \right) \\
&= \sum_{\nu=0}^{m} {}_mC_\nu \int_{\mathbb{R}^n} |\xi|^{2\nu} (\mathcal{F}F)(\xi) \overline{(\mathcal{F}G)(\xi)}\, d\xi \\
&= \int_{\mathbb{R}^n} (1 + |\xi|^2)^m (\mathcal{F}F)(\xi) \overline{(\mathcal{F}G)(\xi)}\, d\xi.
\end{aligned}$$

If we set

$$K_m(x, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{i(x-y)\cdot\xi}}{(1 + |\xi|^2)^m}\, d\xi, \tag{2.5}$$

then we have

$$\langle F, K_m(\cdot, x) \rangle_{H^m} = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} (\mathcal{F}F)(\xi) e^{ix\cdot\xi}\, d\xi = F(x) \tag{2.6}$$

for all $F \in H^m$. Hence H^m is a reproducing kernel Hilbert space whose reproducing kernel is given by K_m.

We have an analogue to the above model case when a real number s satisfies $s > \frac{n}{2}$. If defining

$$K_s(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} e^{i(x-y)\cdot\xi}\, d\xi, \tag{2.7}$$

then it follows

$$H_{K_s} = H^s, \text{ that is, } F(x) = \langle F, K_s(\cdot, x)\rangle_{H^s},\ F \in H^s, \tag{2.8}$$

and the norm is given by

$$\|f\|_{H^s} := \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |(\mathcal{F}f)(\xi)|^2\, d\xi \right)^{\frac{1}{2}}. \tag{2.9}$$

Observe that

$$|F(x)| \le \|F\|_{H^s} \left(\int_{\mathbb{R}^n} \frac{d\xi}{(1+|\xi|^2)^{2s}} \right), \tag{2.10}$$

if $s > \frac{n}{2}$. So the convergence in H^s implies the uniform convergence. We refer e.g. to [10] for additional details on this. Here we content ourselves with mentioning that H^s is called the potential space.

2.2 Paley-Wiener reproducing kernel Hilbert spaces

We shall also use the Paley-Wiener reproducing kernel. See J. R. Higgins [3] and F. Stenger[11] for the details in connection with the sampling theory.

Let $h > 0$ be fixed. Let us consider the following integral transform:

$$g \in L_2((-\pi/h, +\pi/h)^n) \mapsto f(z) = \frac{1}{(2\pi)^n} \int_{(-\pi/h,+\pi/h)^n} g(t) e^{-iz\cdot t} dt. \tag{2.11}$$

Here, $\chi_h(t) = \prod_{\nu=1}^{n} \chi_{(-\pi/h,+\pi/h)}(t_\nu)$ stands for the characteristic function of $(-\pi/h, +\pi/h)^n$. In order to identify the image space by means of the theory of reproducing kernels, we consider the reproducing kernel

$$\begin{aligned} K_h(z,\overline{u}) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \chi_h(t) e^{-iz\cdot t} \overline{e^{-iu\cdot t}} dt \\ &= \prod_{\nu=1}^{n} \frac{1}{\pi(z_\nu - \overline{u}_\nu)} \sin \frac{\pi}{h}(z_\nu - \overline{u}_\nu),\ z, u \in \mathbb{C}^n. \end{aligned}$$

The image space of (2.11) is called the Paley-Wiener space $W\left(\frac{\pi}{h}\right)$ (:= W_h) comprised of all analytic functions of exponential type satisfying, for each ν, for some constant C_ν

$$|f(z_1, z_2, \ldots, z_\nu, z_{\nu+1}, \ldots, z_n)| \leq C_\nu \exp\left(\frac{\pi|z_\nu|}{h}\right), \; z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$$

and

$$\int_{\mathbb{R}^n} |f(x)|^2 dx < \infty.$$

Let δ denote the Kronecker's delta. For multi-indices $j = (j_1, j_2, \cdots, j_n)$ and $j' = (j'_1, j'_2, \cdots, j'_n)$, from the identity

$$K_h(jh, j'h) = \prod_{\nu=1}^{n} \frac{1}{h}\delta(j_\nu, j'_\nu) = \begin{cases} h^{-n}, & j = j', \\ 0, & \text{otherwise} \end{cases}$$

(for each ν), taking into account that $\delta(j_\nu, j'_\nu)$ is the reproducing kernel for the Hilbert space ℓ^2, and using the general theory (see [9]) of integral transforms and Parseval's identity, we have the isometric identities in connection to (2.11)

$$\frac{1}{(2\pi)^n} \int_{(-\pi/h,\pi/h)^n} |g(t)|^2 dt = h^n \sum_j |f(jh)|^2 = \int_{\mathbb{R}^n} |f(x)|^2 dx.$$

That is, the reproducing kernel Hilbert space H_{K_h} with $K_h(z, \overline{u})$ is characterized as a space comprising the Paley-Wiener space W_h and with the norms above in the senses of both discrete version and continuous version. Here we invoked the well-known result that $\{jh\}_j$ is a *uniqueness set* for the Paley-Wiener space W_h, in the sense that if $f(jh) = 0$ for all j then $f \equiv 0$. Therefore, the reproducing property of $K_h(z, \overline{u})$ states that

$$\begin{aligned} f(x) &= (f(\cdot), K_h(\cdot, x))_{H_{K_h}} \\ &= h^n \sum_j f(jh) K_h(jh, x) \\ &= \int_{\mathbb{R}^n} f(\xi) K_h(\xi, x) d\xi. \end{aligned}$$

This representation is the sampling theorem which represents the whole data $f(x)$ in terms of the discrete data $\{f(jh)\}_j$. Furthermore, we refer to [10] for a general theory about sampling theory and error estimates for finite points $\{hj\}_j$.

3 Tikhonov regularization

Let E be an arbitrary set, and let H_K be a reproducing kernel Hilbert space (RKHS) admitting the reproducing kernel $K(p,q)$ on E. For any Hilbert space $\mathcal{H}$ we consider a bounded linear operator L from H_K into $\mathcal{H}$. We are generally interesting in the best approximate problem

$$\inf_{f \in H_K} \|Lf - \mathbf{d}\|_{\mathcal{H}} \tag{3.1}$$

for any vector $\mathbf{d}$ in $\mathcal{H}$. However, this extremal problem is involved in the both senses of the existence of the extremal functions in (3.1) and their representations. See [10] for the details. So, we shall consider its Tikhonov regularization.

We set, for any fixed positive $\lambda > 0$,

$$K_L(\cdot, p; \lambda) = \frac{1}{L^*L + \lambda I} K(\cdot, p) = (L^*L + \lambda I)^{-1} K(\cdot, p),$$

where L^* denotes the adjoint operator of L. Then, by introducing the inner product

$$(f, g)_{H_K(L;\lambda)} = \lambda (f, g)_{H_K} + (Lf, Lg)_{\mathcal{H}}, \tag{3.2}$$

we shall construct the Hilbert space $H_K(L;\lambda)$ comprising functions of H_K. It is clear that this space admits a reproducing kernel.

Proposition 1.

(i) *The extremal function $f_{\mathbf{d},\lambda}(p)$ in the Tikhonov regularization*

$$\inf_{f \in H_K} \{\lambda \|f\|^2_{H_K} + \|\mathbf{d} - Lf\|^2_{\mathcal{H}}\} \tag{3.3}$$

exists uniquely and it is represented in terms of the kernel $K_L(p,q;\lambda)$ as follows:

$$f_{\mathbf{d},\lambda}(p) = (\mathbf{d}, LK_L(\cdot, p; \lambda))_{\mathcal{H}} \tag{3.4}$$

where the kernel $K_L(p,q;\lambda)$ is the reproducing kernel for the Hilbert space $H_K(L;\lambda)$.

(ii) *$K_L(p,q;\lambda)$ is determined as the unique solution $\widetilde{K}(p,q;\lambda)$ of the equation:*

$$\widetilde{K}(p,q;\lambda) + \frac{1}{\lambda}(L\widetilde{K}_q, LK_p)_{\mathcal{H}} = \frac{1}{\lambda} K(p,q) \tag{3.5}$$

with

$$\widetilde{K}_q = \widetilde{K}(\cdot, q; \lambda) \in H_K \quad for \quad q \in E, \tag{3.6}$$

and

$$K_p = K(\cdot, p) \in H_K \quad for \quad p \in E. \tag{3.7}$$

In (3.4), when $\mathbf{d}$ contains error or noises, we need its error estimate. For this, we can obtain the general result:

Proposition 2. *In* (3.4), *we have the estimate*

$$|f_{\mathbf{d},\lambda}(p)| \leq \frac{1}{2\sqrt{\lambda}} \sqrt{K(p,p)} \|\mathbf{d}\|_{\mathcal{H}}. \tag{3.8}$$

We would like to point out that in a corresponding estimate of $|f_{\mathbf{d},\lambda}(p)|$ in [4], the factor 2 is missing when compared with (3.8) above. That is, the estimate (3.8) is improved here. As the example of $\mathcal{H} = H_K$ and $L = I$ shows, non-trivial equality (3.8) is attained.

Proposition 3. *Suppose that H_K is a reproducing kernel Hilbert space with kernel K and that $\mathcal{H}$ is a Hilbert space. If we are given a surjective continuous linear transform $L : H_K \to \mathcal{H}$, then for $g \in \mathcal{H}$ we can find $f \in H_K$ so as to minimize $\|f\|_{H_K}$. Such f belongs to the range of L^*.*

4 Discrete differential equation

The representation (1.4) may not yield any solution of the discrete differential equation (1.1) for a given finite number of the data g and so we will be interested in such discrete differential equation, because we obtain only a finite number of values g as observation data.

4.1 Paley-Wiener type space

In the first half of this section, we work on H_{K_h}, the Paley-Wiener reproducing kernel Hilbert space described in Section 2. Recall that any element in H_{K_h} is smooth. For different points $\{x_j\}_{j=1}^N$, we shall consider the bounded linear operators from the RKHS H_{K_h} into $\mathbb{R}$:

$$H_{K_h} \ni F \mapsto (LF)(x_j) := (L(D)F)(x_j); \quad j = 1, 2, ..., N. \tag{4.1}$$

Needless to say, K_h being nice, the mapping $F \mapsto L(D)F$ is continuous. We define

$$\mathbf{L} : H_{K_h} \ni F \mapsto ((LF)(x_1), (LF)(x_2), ..., (LF)(x_N)) \in \mathbb{R}^N. \tag{4.2}$$

We shall use a standard orthonormal system $\{\mathbf{e}_j\}_{j=1}^N$ in the space $\mathbb{R}^N$. Then, we see

$$(LF)(x_j) = (\mathbf{L}F, \mathbf{e}_j)_{\mathbb{R}^N} = (F, \mathbf{L}^*\mathbf{e}_j)_{H_{K_h}}. \tag{4.3}$$

Let us set

$$A := \{a_{jj'}\}_{j,j'=1,2,\cdots,N} := \{(\mathbf{L}^*\mathbf{e}_{j'}, \mathbf{L}^*\mathbf{e}_j)_{H_{K_h}}\}_{j,j'=1,2,\cdots,N}. \tag{4.4}$$

Here note that $a_{jj'}$ are calculated as follows:

$$\begin{aligned}(\mathbf{L}^*\mathbf{e}_j)(x) &= ((\mathbf{L}^*\mathbf{e}_j)(\cdot), K_h(\cdot, x))_{H_{K_h}} \\ &= (\mathbf{e}_j, \mathbf{L}K_h(\cdot, x))_{\mathbb{R}^N} \\ &= L[K_h(\cdot, x)](x_j) \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \chi_h(\eta)\mathbf{L}(-\eta)e^{-i\eta\cdot(x-x_j)}d\eta.\end{aligned}$$

Hence, it follows that

$$a_{jj'} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \chi_h(\eta)|\mathbf{L}(-\eta)|^2 e^{-i\eta\cdot(x_j - x_{j'})}d\eta. \tag{4.5}$$

In particular, we conclude from this that the matrix $A = \{a_{jj'}\}_{j,j'=1,2,\cdots,N}$ is strictly positive definite.

Denote by H_A the image of L. Since the matrix $\{a_{jj'}\}_{j,j'=1,2,\cdots,N}$ is invertible, then, the norm in H_A is given by

$$||\mathbf{L}F||_{H_A}^2 = (\mathbf{L}F)^* A^* (\mathbf{L}F), \tag{4.6}$$

where $A^* = \overline{A^{-1}} = \{\widetilde{a}_{jj'}\}_{j,j'=1,2,\cdots,N}$ (see [10]. p. 250).

We thus obtain the desired result from Proposition 1.

Theorem 3. *For any given N values $\mathbf{d} = \{g(x_j)\}_{j=1}^N \in \mathbb{R}^n$, among the H_{K_h} functions F taking the values*

$$(LF)(x_j) = g(x_j) \quad j = 1, 2, ..., N, \tag{4.7}$$

the function $F^*_{\mathbf{d}}(x)$ with the minimum norm $\|F\|_{H_{K_h}}$ is uniquely determined and it is represented as follows:

$$F^*_{\mathbf{d}}(x) = \sum_{j,j'=1}^{N} g(x_j)\widetilde{a}_{jj'}(\mathbf{L}^*\mathbf{e}_{j'})(x). \tag{4.8}$$

Proof. The function given by (4.8) belongs to the range of L^* and agrees with g at x_j, $j = 1, 2, \cdots, N$. So, the result is an immediate consequence of Proposition 3. □

Theorem 3 can now be directly applied. In connection with this, we refer to [5] for a quite simple algorithm for inversion of any matrix (with several lines) using the theory of reproducing kernels and the Tikhonov regularization.

As we see from the isometrical relation between the data (4.1) and the inverse with the minimum norm $F^*_{\mathbf{d}}(x)$, the observation data and the corresponding inverse are rigid, and our inversion formula is given in the sense of well-posed problem.

4.2 Sobolev space

In the Sobolev space H^s of $s \geq m$ and $s > \frac{n}{2}$, we have the corresponding formulae:

$$\begin{aligned}(\mathbf{L}^*\mathbf{e}_j)(x) &= ((\mathbf{L}^*\mathbf{e}_j)(\cdot), K_s(\cdot, x))_{H_{K_s}} \\ &= (\mathbf{e}_j, \mathbf{L}K_s(\cdot, x))_{\mathbb{R}^N} \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{\overline{\mathbf{L}(\eta)}e^{-i\eta\cdot(x-x_j)}}{(1+|\eta|^2)^s}d\eta\end{aligned}$$

and

$$a_{jj'} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{|\mathbf{L}(\eta)|^2 e^{i\eta\cdot(x_j - x_{j'})}}{(1+|\eta|^2)^s}d\eta. \tag{4.9}$$

5 Inverse source problem

We shall consider the inversion problem for (4.1) and (4.2). That is, from a finite number of observation data $u(x_j), j = 1, 2, .., N$, we wish to determine the optimal source g in (1.1). For this purpose, we shall use the integral representation (1.4); that is, for $u(x_j)$ we shall use the approximation data $F^*_{\lambda,s,g}(x_j)$. So, we shall consider the bounded linear operators from the RKHS H_{K_h} into $\mathbb{R}$:

$$H_{K_h} \ni g \mapsto (Mg)(x_j) := F^*_{\lambda,s,g}(x_j); \quad j = 1, 2, ..., N. \tag{5.1}$$

We set

$$\mathbf{M} : H_{K_h} \ni g \mapsto ((Mg)(x_1), (Mg)(x_2), ..., (Mg)(x_N)) \in \mathbb{R}^N. \tag{5.2}$$

Then, we see

$$(Mg)(x_j) = (\mathbf{M}g, \mathbf{e}_j)_{\mathbb{R}^N} = (g, \mathbf{M}^*\mathbf{e}_j)_{H_{K_h}}. \tag{5.3}$$

Let us set

$$B := \{b_{jj'}\}_{j,j'=1,2,\cdots,N} := \{(\mathbf{M}^*\mathbf{e}_{j'}, \mathbf{M}^*\mathbf{e}_j)_{H_{K_h}}\}_{j,j'=1,2,\cdots,N}. \tag{5.4}$$

Here note that $b_{jj'}$ are calculated as follows:

$$\begin{aligned}(\mathbf{M}^*\mathbf{e}_j)(x) &= ((\mathbf{M}^*\mathbf{e}_j)(\cdot), K_h(\cdot, x))_{H_{K_h}} \\ &= (\mathbf{e}_j, \mathbf{M}K_h(\cdot, x))_{\mathbb{R}^N} \\ &= M[K_h(\cdot, x)](x_j) \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\chi_h(\eta)\mathbf{L}(-\eta)e^{-i\eta\cdot(x-x_j)}}{\lambda(|\eta|^2+1)^s + |\mathbf{L}(\eta)|^2} d\eta.\end{aligned}$$

Hence, it follows that

$$b_{jj'} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\chi_h(\eta)|\mathbf{L}(-\eta)|^2 e^{-i\eta\cdot(x_j-x_{j'})}}{(\lambda(|\eta|^2+1)^s + |\mathbf{L}(\eta)|^2)^2} d\eta. \tag{5.5}$$

In particular, we conclude from this that the matrix $B = \{b_{jj'}\}_{j,j'=1,2,\cdots,N}$ is strictly positive definite.

We thus obtain the desired result from Proposition 1.

Theorem 4. *For any given N values $\mathbf{d} = \{F^*_{\lambda,s,g}(x_j)\}_{j=1}^N \in \mathbb{R}^n$, among the H_{K_h} functions g taking the values*

$$F^*_{\lambda,s,g}(x_j) = (Mg)(x_j) \quad j = 1, 2, ..., N, \tag{5.6}$$

the function $g^*_{\mathbf{d}}(x)$ with the minimum norm $\|g\|_{H_{K_h}}$ is uniquely determined and it is represented as follows:

$$g^*_{\mathbf{d}}(x) = \sum_{j,j'=1}^{N} F^*_{\lambda,s,g}(x_j)\widetilde{b}_{jj'}(\mathbf{L}^*\mathbf{e}_{j'})(x). \tag{5.7}$$

In the Sobolev space of $s \geq m$ and $s > \frac{n}{2}$, we have the corresponding formulae:

$$
\begin{aligned}
(\mathbf{M}^*\mathbf{e}_j)(x) &= ((\mathbf{M}^*\mathbf{e}_j)(\cdot), K_s(\cdot, x))_{H_{K_s}} \\
&= (\mathbf{e}_j, \mathbf{M}K_s(\cdot, x))_{\mathbb{R}^N} \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\overline{\mathbf{L}(\eta)} e^{-i\eta\cdot(x-x_j)}}{(1+|\eta|^2)^s(\lambda(|\eta|^2+1)^s + |\mathbf{L}(\eta)|^2)} d\eta
\end{aligned}
$$

and

$$
b_{jj'} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{|\mathbf{L}(\eta)|^2 e^{i\eta\cdot(x_j - x_{j'})}}{(1+|\eta|^2)^s(\lambda(|\eta|^2+1)^s + |\mathbf{L}(\eta)|^2)^2} d\eta. \tag{5.8}
$$

Acknowledgements. This work was supported in part by *Center of Research and Development in Mathematics and Applications*, University of Aveiro, Portugal, through FCT - Portuguese Foundation for Science and Technology. The second author was supported in part by the Grant-in-Aid for the Scientific Research (C)(2)(No. 21540111) from the Japan Society for the Promotion Science, and the third author was supported by Grant-in-Aid for Young Scientists (B) (No.21740104) Japan Society for the Promotion of Science.

References

[1] M. Asaduzzaman, T. Matsuura, and S. Saitoh, *Constructions of approximate solutions for linear differential equations by reproducing kernels and inverse problems,* Advances in Analysis, Proceedings of the 4th International ISAAC Congress (2005), 335-344. (World Scientific).

[2] L. P. Castro, Q. Chen and S. Saitoh, *Source inversion of heat conduction from a finite number of observation data*, Applicable Analysis (to appear).

[3] J. R. Higgins, *A sampling principle associated with Saitoh's fundamental theory of linear transformations*, Analytic Extension Formulas and their Applications, (2001), Kluwer Academic Publishers, 73–86.

[4] H. Itou and S. Saitoh, *Analytical and numerical solutions of linear singular integral equations*, Int. J. Appl. Math. Stat., **12**(2007), 76–89.

[5] K. Iwamura, T. Matsuura and S. Saitoh, *A numerical construction of a natural inverse of any matrix by using the theory of reproducing kernels with the Tikhonov regularization*, (submitted).

[6] T. Matsuura and S. Saitoh, *Analytical and numerical solutions of linear ordinary differential equations with constant coefficients,* J. Anal. Appl., **3**(2005), 1-17.

[7] T. Matsuura and S. Saitoh, *Dirichlet's Principle Using Computers,* Applicable Analysis, **84**(2005), 989-1003.

[8] T. Matsuura and S. Saitoh, *Analytical and numerical inversion formulas in the Gaussian convolution by using the Paley-Wiener spaces,* Applicable Analysis, **85**(2006), 901-915.

[9] S. Saitoh, *Hilbert spaces induced by Hilbert space valued functions*, Proc. Amer. Math. Soc., **89** (1983), 74–78.

[10] S. Saitoh, *Integral Transforms, Reproducing Kernels and their Applications*, Pitman Research Notes in Mathematics Series, **369** (1997). Addison Wesley Longman, UK.

[11] F. Stenger, *Numerical Methods Based on Sinc and Analytic Functions*, Springer Series in Computational Mathematics, 20, 1993.

e-mail: castro@ua.pt; saburou.saitoh@gmail.com; yosihiro@math.kyoto-u.ac.jp; asimoes@mat.ubi.pt