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A COLLOCATION METHOD FOR SINGULAR INTEGRAL OPERATORS WITH REFLECTION

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Abstract

We will use a polynomial collocation method to compute the kernel dimension of singular integral operators with reflection and piecewise continuous functions as coefficients. The so-called k-splitting property of the operators is also discussed. An example is included to illustrate the proposed procedure.

Keywords: Polynomial collocation method, singular integral operator, reflection operator, kernel dimension.

1 Introduction

Let $L^2(\mathbb{T},\varpi)$ be the weighted Lebesgue space over $\mathbb{T}:=\{t\in\mathbb{C}\ :\ |t|=1\}$ equipped with the norm

$$\|f\|_{2,\varpi}:=\|\varpi f\|_2\,, \tag{1.1}$$

where $\|\cdot\|_2$ denotes the usual norm of the Hilbert space $L^2(\mathbb{T})$. We will assume that all weights $\varpi:\mathbb{T}\longrightarrow[0,+\infty]$ are such that $\varpi,\varpi^{-1}\in L^2(\mathbb{T})$, and

$$c_{\varpi}:=\sup_{t\in\mathbb{T}}\sup_{\varepsilon>0}\left(\frac{1}{\varepsilon}\int_{\mathbb{T}(t,\varepsilon)}\varpi(\tau)^2|d\tau|\right)^{1/2}\left(\frac{1}{\varepsilon}\int_{\mathbb{T}(t,\varepsilon)}\varpi(\tau)^{-2}|d\tau|\right)^{1/2}<\infty\,, \tag{1.2}$$

where

$$\mathbb{T}(t,\varepsilon):=\{\tau\in\mathbb{T}\ :\ |\tau-t|<\varepsilon\},\qquad \varepsilon>0.$$

The property (1.2) is the so-called *Hunt–Muckenhoupt–Wheeden condition*, and $A_2(\mathbb{T})$ is referred to as the set of *Hunt–Muckenhoupt–Wheeden weights.*

In the present work we deal with the singular integral operators

$$\mathcal{A}=a_0I_{\mathbb{T}}+b_0S_{\mathbb{T}}+a_1J+b_1S_{\mathbb{T}}J:L^2(\mathbb{T},w)\to L^2(\mathbb{T},w), \tag{1.3}$$

with essentially bounded piecewise continuous coefficients $a_0, b_0, a_1, b_1 \in PC(\mathbb{T})$, the identity operator $I_{\mathbb{T}}$, the Cauchy singular integral operator $S_{\mathbb{T}}$ defined almost everywhere by

$$(S_{\mathbb{T}}f)(t) = \frac{1}{\pi i}\,\text{p.v.}\int_{\mathbb{T}} \frac{f(\tau)}{\tau - t} d\tau, \quad t \in \mathbb{T},$$

the reflection operator

$$(J\varphi)(t) = \varphi(-t), \quad t \in \mathbb{T}, \tag{1.4}$$

and where the weighted Lebesgue space $L^2(\mathbb{T}, w)$ is considered for weights w belonging to $A_2^e(\mathbb{T}) := \{w \in A_2(\mathbb{T}) : \; w(-t) = w(t), \; t \in \mathbb{T}\}$.

We will apply a *collocation method* to the operator $\mathcal{A}$ which will help us to obtain information about the *k-splitting property* and the kernel dimension of the operators in consideration.

The paper is organized as follows: Section 2 is devoted to the collocation method, which will be used to compute de kernel dimension of the operators under consideration. The *approximation* and *projection methods*, as well as the notion of *singular values* and *stability* are considered in a general setting in subsection 2.1 and applied to our case in subsection 2.2. These previous results will be useful in Section 3 for obtaining an estimation of the operator $\mathcal{A}$ kernel dimension. A specific example where the singular values of some associated operators are computed is provided at the end of the paper.

2 A polynomial collocation method for singular integral operators

Under the assumption that the operator $\mathcal{A}$ given by (1.3) is a Fredholm operator (see [2] for corresponding criteria), we will study their kernel dimension by means of a *polynomial collocation method for singular integral operators* proposed by A. Rogozhin and B. Silbermann in [8].

2.1 General framework

2.1.1 Approximation numbers.

Let F be a finite dimensional Banach space with $\dim F = m$. The k-th approximation number ($k \in \{0, 1, \ldots, m\}$) of an operator $A \in \mathcal{L}(F)$ is defined as

$$s_k(A) = \text{dist}(A, \mathcal{F}_{m-k}) := \inf\{\|A - F\| \; : \; F \in \mathcal{F}_{m-k}\},$$

where $\mathcal{F}_{n-k}$ denotes the collection of all operators (or matrices from $\mathbb{C}^{n\times n}$) having the dimension of the range equal to at most $n - k$. It is clear that

$$0 \leq s_1(A) \leq \cdots \leq s_m(A) = \|A\|_{\mathcal{L}(F)}.$$

Notice that the approximation numbers can be also defined as the singular values of a square matrix $A_n \in \mathbb{C}^{nN \times nN}$ which are the square roots of the spectral points of $A_n^* A_n$, where A_n^* denotes the adjoint matrix of A_n.

Definition 1 *A sequence (A_n) of matrices $nN \times nN$ is said to have the k-splitting property if there is an integer $k \geq 0$ such that*

$$\lim_{n\to\infty} s_k(A_n) = 0 \quad \textit{and} \quad \liminf_{n\to\infty} s_{k+1}(A_n) > 0.$$

The number k is called the splitting number. Alternatively, we say the singular values Λ_n (computed via $A_n^* A_n$) of a sequence (A_n) of $k(n) \times l(n)$ matrices A_n have the splitting property if there exist a sequence $c_n \to 0$ ($c_n \geq 0$) and a number $d > 0$ such that

$$\Lambda_n \subset [0, c_n] \cup [d; \infty) \quad \textit{for all } n,$$

and the singular values of A_n are said to meet the k-splitting property if, in addition, for all sufficiently large n exactly k singular values of A_n lie in $[0, c_n]$.

2.1.2 Approximation method.

For the sake of self-contained global presentation we will describe here the approximation method in the scope of Banach spaces. Afterwards, we will show the natural adaptation to our cases. More information about this method can be found, for instance, in [3, 7, 8].

Let X be a Banach space. Given a bounded linear operator A on X, $A \in \mathcal{L}(X)$, and an element f of X, consider the operator equation

$$A\varphi = f. \tag{2.1}$$

To obtain approximate solutions of this equation, we consider approximate closed subspaces X_n in which the approximate solutions φ_n of (2.1) will be sought. In practice, the X_n spaces usually have finite dimension but we will not require this assumption. We will assume that X_n are ranges of certain projection operators $L_n : X \longrightarrow X_n$ so that these projections converge strongly to the identity operator: $s - \lim_{n\to\infty} L_n = I$. This strong convergence implies that $\cup_{n=1}^{\infty} X_n$ is dense in X.

Having fixed subspaces X_n, we choose convenient linear operators $A_n : X_n \longrightarrow X_n$ and consider in the place of (2.1) the equations

$$A_n\varphi_n = L_n f, \quad n = 1, 2, \dots, \tag{2.2}$$

with their solutions sought in $X_n = \operatorname{Im} L_n$.

A sequence (A_n) of operators $A_n \in \mathcal{L}(\operatorname{Im} L_n)$ is an *approximation method* for $A \in \mathcal{L}(X)$ if $A_n L_n$ converges strongly to A as $n \to \infty$.

Note that even if (A_n) is an approximation method for A, we do not yet know anything about the solvability of the equations (2.2), and about the relations between (eventual) solutions φ_n of (2.2) and the (possible) solution φ of (2.1).

The approximation method (A_n) for A is *applicable* if there exists a number n_0 such that the equations (2.2) possess unique solutions φ_n for every $n \geq n_0$ and every right-hand side $f \in X$, and if these solutions converge in the norm of X to a solution of (2.1). An equivalent characterization of applicable approximation methods is the notion of *stability*, where a sequence (A_n) of operators $A_n \in \mathcal{L}(\operatorname{Im} L_n)$ is called *stable* if there exists a number n_0 such that the operators A_n are invertible for every $n \geq n_0$ and if the norms of their inverses are uniformly bounded:

$$\sup_{n \geq n_0} \|A_n^{-1} L_n\| < \infty.$$

These notions are connected by the Polski's Theorem.

Theorem 1 (Polski; see [3, Theorem 1.4]) *Let (L_n) be a sequence of projections which converges strongly to the identity operator, and let (A_n) with $A_n \in \mathcal{L}(\operatorname{Im} L_n)$ be an approximation method for the operator $A \in \mathcal{L}(X)$. This method is applicable if and only if the operator A is invertible and the sequence (A_n) is stable.*

2.1.3 Projection methods and the algebraization of stability.

Let A be a bounded linear operator on X and (L_n) a sequence of projections converging strongly to the identity $I \in \mathcal{L}(X)$. The idea of any projection method for the approximate solution of (2.1) is to choose a further sequence (R_n) of projections which also converges strongly to the identity and which satisfy $\operatorname{Im} R_n = \operatorname{Im} L_n$. Thus, we choose $A_n = R_n A L_n : \operatorname{Im} L_n \longrightarrow \operatorname{Im} L_n$ as the approximate operators of A. In fact, Lemma 1.5 in [3] proves that $(R_n A L_n)$ is indeed an approximate method for A.

Let X be an infinite dimensional Banach space and let (X_n) be a sequence of finite dimensional subspaces of X. Moreover, we assume that there is a sequence (L_n) of projections from X onto X_n with strong limit $I \in X$ as $n \to \infty$. Let $\mathcal{F}$ refer to the set of all sequences $(A_n)_{n=0}^{\infty}$ of operators $A_n \in \mathcal{L}(\operatorname{Im} L_n)$ which are uniformly bounded: $\sup\{\|A_n L_n\| : n \geq 0\} < \infty$. The "algebraization" of $\mathcal{F}$ is given by the natural operations

$$\lambda_1(A_n) + \lambda_2(B_n) := (\lambda_1 A_n + \lambda_2 B_n), \quad (A_n)(B_n) := (A_n B_n) \tag{2.3}$$

and

$$\|(A_n)\|_{\mathcal{F}} := \sup\{\|A_n L_n\| : n \geq 0\}$$

which make $\mathcal{F}$ to be an initial Banach algebra with identity $(I_{|\operatorname{Im} L_n})$. The set $\mathcal{G}$ of all sequences (G_n) in $\mathcal{F}$ with $\lim_{n\to\infty} \|G_n L_n\| = 0$ is a closed two sided ideal in $\mathcal{F}$. The Kozak's Theorem (Theorem 1.5 in [3]) establish that a sequence $(A_n) \in \mathcal{F}$ is stable if and only if its coset $(A_n) + \mathcal{G}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{G}$.

If instead of a Banach space X we consider a Hilbert space $\mathcal{H}$ and L_n to be the orthogonal projections P_n from $\mathcal{H}$ onto $\mathcal{H}_n$, then $(A_n)^* = (A_n^*)$ defines an involution in $\mathcal{F}$ which makes $\mathcal{F}$ a C^*-algebra. Note that in this case the approximation numbers of an operator $A_n \in \mathcal{L}(\mathcal{H}_n)$ are just the singular values of A_n.

Let further T be a (possible infinite) index set and suppose that, for every $t \in T$, we are given an infinite dimensional Hilbert space $\mathcal{H}^t$ with identity operator I^t as well as a sequence (E_n^t) of partial isometries $E_n^t : \mathcal{H}^t \longrightarrow \mathcal{H}$ such that the initial projections P_n^t of E_n^t converge strongly to I^t as $n \to \infty$, the range projection of E_n^t is P_n and the separation condition

$$(E_n^s)^* E_n^t \longrightarrow 0 \quad \text{weakly as } n \to \infty \tag{2.4}$$

holds for every $s, t \in T$ with $s \neq t$. Recall that an operator $E : \mathcal{H}' \longrightarrow \mathcal{H}''$ is a partial isometry if $EE^*E = E$ and that E^*E and EE^* are orthogonal projections (which are called the initial and the range projections of E, respectively). The restriction of E to $\operatorname{Im}(E^*E)$ is an isometry from $\operatorname{Im}(E^*E)$ onto $\operatorname{Im}(EE^*) = \operatorname{Im} E$. We write E_{-n}^t instead of $(E_n^t)^*$, and set $\mathcal{H}_n := \operatorname{Im} P_n$ and $\mathcal{H}_n^t := \operatorname{Im} P_n^t$.

Let $\mathcal{F}^T$ stand for the set of all sequences $(A_n) \in \mathcal{F}$ for which the strong limits

$$s - \lim_{n\to\infty} E_{-n}^t A_n E_n^t \quad \text{and} \quad s - \lim_{n\to\infty} (E_{-n}^t A_n E_n^t)^*$$

exist for every $t \in T$, and define mappings $W^t : \mathcal{F}^T \longrightarrow \mathcal{L}(\mathcal{H}^t)$ by

$$W^t(A_n) := s - \lim_{n\to\infty} E_{-n}^t A_n E_n^t.$$

The algebra $\mathcal{F}^T$ is a C^*-subalgebra of $\mathcal{F}$ which contains the identity, and W^t are $*$-homomorphisms. Moreover, $\mathcal{F}^T$ is a *standard* algebra. This means that any sequence $(A_n) \in \mathcal{F}^T$ is stable if and only if all the operators $W^t(A_n)$ are invertible.

The separation condition (2.4) ensures that, for every $t \in T$ and every compact operator $K^t \in \mathcal{K}(\mathcal{H}^t)$, the sequence $(E_n^t K^t E_{-n}^t)$ belongs to the algebra $\mathcal{F}^T$, and for all $s \in T$

$$W^s(E_n^t K^t E_{-n}^t) = \begin{cases} K^t & \text{if } s = t \\ 0 & \text{if } s \neq t. \end{cases} \tag{2.5}$$

Conversely, the above identity implies the separation condition (2.4). Moreover, the ideal $\mathcal{G}$ belongs to $\mathcal{F}^T$. So we can introduce the smallest closed ideal $\mathcal{J}^T$ of $\mathcal{F}^T$ which contains all sequences $(E_n^t K^t E_{-n}^t)$ with $t \in T$ and $K^t \in \mathcal{K}(\mathcal{H}^t)$, as well as all sequences $(G_n) \in \mathcal{G}$.

Corresponding to the ideal $\mathcal{J}^T$, we introduce a class of Fredholm sequences by calling a sequence $(A_n) \in \mathcal{F}^T$ Fredholm if the coset $(A_n)+\mathcal{J}^T$ is invertible in the quotient algebra $\mathcal{F}^T/\mathcal{J}^T$. It is also known (see [3]) that if $(A_n) \in \mathcal{F}^T$ is Fredholm, then all operators $W^t(A_n)$ are Fredholm on $\mathcal{H}^t$, and the number of the non-invertible operators among the $W^t(A_n)$ is finite.

The main result concerning standard algebras reads as follows:

Theorem 2 (see [3]) *Let (A_n) be a sequence from the standard C^*-algebra $\mathcal{F}^T$.*

(i) *If the coset $(A_n)+\mathcal{J}^T$ is invertible in the quotient algebra $\mathcal{F}^T/\mathcal{J}^T$, then all operators $W^t(A_n)$ are Fredholm on $\mathcal{H}^t$, the number of the non-invertible operators among the $W^t(A_n)$ is finite, and the singular values of A_n have the k-splitting property with*

$$k(A_n) = \sum_{t\in T} \dim\ker W^t(A_n).$$

(ii) *If $W^t(A_n)$ is not Fredholm for at least one $t \in T$, then for every integer $k \geq 0$*

$$s_k(A_n) \longrightarrow 0, \quad as \quad n \longrightarrow \infty.$$

2.2 The collocation method for singular integral operators on $[L^2(\mathbb{T}, w)]^2$

In this part we will consider pure (matrix) singular integral operators defined on weighted Lebesgue spaces $[L^2(\mathbb{T}, w)]^2$, where the weight w belongs to $A_2(\mathbb{T})$.

In addition, let us consider the following singular integral equation on $[L^2(\mathbb{T}, w)]^2$:

$$(aI_{\mathbb{T}} + bS_{\mathbb{T}})u = f. \tag{2.6}$$

In view to obtain an approximate solution of (2.6) by the collocation method, we seek to polynomials u_n by solving the linear $(2n + 1) \times (2n + 1)$-system

$$a(z_j)u_n(z_j) + b(z_j)(S_{\mathbb{T}})u_n(z_j) = f(z_j), \quad j \in \{-n, \dots, n\},$$

which can be equivalently written in the form

$$L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n u_n = L_n f$$

and our goal is to examine the stability of the sequence $(L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n)$.

The algebraization of the stability runs as follows in this case. We start by considering the Fourier projection $P_n \in \mathcal{L}([L^2(\mathbb{T}, w)]^2)$ that in terms of the Fourier coefficients of a function $\psi \in [L^2(\mathbb{T}, w)]^2$ acts componentwise according to the rule

$$\psi = \sum_{k \in \mathbb{Z}} \psi_k t^k \longmapsto \sum_{k=-n}^{n} \psi_k t^k, \qquad n \in \mathbb{N}.$$

In addition, we take the Lagrange interpolation operator L_n (which is bounded in $[L^2(\mathbb{T}, w)]^2$, see for instance [1]) associated to the points

$$t_j = \exp\left(\frac{2\pi i j}{2n + 1}\right), \quad j = 0, 1, \dots, 2n.$$

That is, L_n assigns to a function ψ its Lagrange interpolation polynomial $L_n\psi \in \operatorname{Im} P_n$, uniquely determined, on each component, by the conditions $(L_n\psi)(t_j) = \psi(t_j)$, $j = 0, 1, \dots, 2n$. One can show that $\|P_n\psi - \psi\|_{2,w} \longrightarrow 0$ as $n \longrightarrow \infty$ for every $\psi \in [L^2(\mathbb{T}, w)]^2$ and in [5] it was proved (for the scalar case) that $\|L_n\psi - \psi\|_{2,w} \longrightarrow 0$, $n \longrightarrow \infty$.

For $r \in \mathbb{Z}_+$ given, we construct

$$A_{n,r} := L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n(P_n - W_n P_{r-1} W_n), \quad n \in \mathbb{Z}_+, \tag{2.7}$$

where the operator $W_n \in \mathcal{L}([L^2(\mathbb{T}, w)]^2)$ acts by the rule

$$W_n\psi = \sum_{k=0}^{n} \psi_{n-k}t^k + \sum_{k=-n}^{-1} \psi_{-n-k-1}t^k.$$

Note that if $r = 0$, then we get a polynomial collocation method A_n for the solution of the singular integral equation (2.6).

First, note that the operators W_n and P_n are related as follows:

$$W_n^2 = P_n, \quad W_nP_n = P_nW_n = W_n. \tag{2.8}$$

On the other hand, in [3, 4, 6] it was shown that:

$$L_naI_{\mathbb{T}} = L_naL_n, \quad S_{\mathbb{T}}P_n = P_nS_{\mathbb{T}}P_n, \quad W_nL_naW_n = L_n\widetilde{a}P_n \tag{2.9}$$

$$(L_naP_n)^* = L_n\overline{a}P_n, \quad (P_nS_{\mathbb{T}}P_n)^* = P_nS_{\mathbb{T}}P_n \tag{2.10}$$

where for $a \in L^\infty(\mathbb{T})$,

$$\widetilde{a}(t) = a\left(\frac{1}{t}\right), \quad t \in \mathbb{T}.$$

We denote by T_2 the index set $\{1, 2\}$ and by $\mathcal{F}^{T_2}$ the C^*-algebra of all operator sequences (A_n), with $A_n \in \mathcal{L}(\mathrm{Im}\, P_n)$, for which there exist operators ($*$-homomorphisms) $W^1(A_n)$, $W^2(A_n) \in \mathcal{L}([L^2(\mathbb{T}, w)]^2)$ such that

$$s - \lim_{n\to\infty} P_nA_nP_n = W^1(A_n) \quad \text{and} \quad s - \lim_{n\to\infty} W_nA_nW_n = W^2(A_n)$$

$$s - \lim_{n\to\infty} (P_nA_nP_n)^* = W^1(A_n)^* \quad \text{and} \quad s - \lim_{n\to\infty} (W_nA_nW_n)^* = W^2(A_n)^*.$$

Furthermore, let us introduce the subsets $\mathcal{J}^1$ and $\mathcal{J}^2$ of the C^*-algebra $\mathcal{F}^{T_2}$:

$$\begin{aligned} \mathcal{J}^1 &= \{(P_nKP_n) + (G_n) \ : \ K \in \mathcal{K}([L^2(\mathbb{T}, w)]^2), \ \|G_n\| \to \infty\} \\ \mathcal{J}^2 &= \{(W_nLW_n) + (G_n) \ : \ L \in \mathcal{K}([L^2(\mathbb{T}, w)]^2), \ \|G_n\| \to \infty\}. \end{aligned}$$

Again, $\mathcal{J}^{T_2}$ is the smallest closed two-sided ideal of $\mathcal{F}^{T_2}$ which contains all sequences (J_n) such that J_n belongs to one of the ideals $\mathcal{J}^t$, $t = 1, 2$.

Theorem 3 *Let $a, b \in [PC(\mathbb{T})]^{2\times 2}$ and consider the operators*

$$A_{n,r} := L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n(P_n - W_nP_{r-1}W_n), \ n \in \mathbb{Z}_+.$$

(1) *The sequence $(A_{n,r})$ belongs to the C^*-algebra $\mathcal{F}^{T_2}$. In particular*

$$W^1(A_{n,r}) = aI_{\mathbb{T}} + bS_{\mathbb{T}}, \quad \textit{and} \quad W^2(A_{n,r}) = (\widetilde{a}I_{\mathbb{T}} + \widetilde{b}S_{\mathbb{T}})Q_{r-1}$$

where $Q_{r-1} = I - P_{r-1}$.

(2) *The coset* $(A_{n,r})+\mathcal{J}^{T_2}$ *is invertible in* $\mathcal{F}^{T_2}/\mathcal{J}^{T_2}$ *if and only if the operator* $W^1(A_{n,r}) = aI_{\mathbb{T}} + bS_{\mathbb{T}}$ *is Fredholm.*

(3) *If the operators* $W^1(A_{n,r})$ *and* $W^2(A_{n,r})$ *are Fredholm on* $[L^2(\mathbb{T},w)]^2$*, then the approximation numbers of* $A_{n,r}$ *have the k-splitting property with*

$$k(A_{n,r}) = \dim\ker(aI_{\mathbb{T}} + bS_{\mathbb{T}}) + \dim\ker((\widetilde{a}I_{\mathbb{T}} + \widetilde{b}S_{\mathbb{T}})Q_{r-1}).$$

(4) *Otherwise,* $s_l(A_{n,r}) \longrightarrow 0$ *for each* $l \in \mathbb{N}$.

Proof. We are going to compute $W^1(A_{n,r})$ and $W^2(A_{n,r})$. Having this goal in mind, we will use the relations (2.8) and (2.9). First note that for each $r \in \mathbb{N}$ the sequence $(W_nP_{r-1}W_n)$ belongs to $\mathcal{J}^2$. So, from (2.5) we have that $W^1(P_n - W_nP_{r-1}W_n) = I$ and $W^2(P_n - W_nP_{r-1}W_n) = I - P_{r-1}$. Since W^t, $t \in T_2$, are $*$-homomorphisms, then it only remains to compute

$$\begin{aligned} W^1(L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n) &= s-\lim_{n\to\infty} L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_nP_n \\ &= \lim_{n\to\infty} L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n \\ &= aI_{\mathbb{T}} + bS_{\mathbb{T}} \end{aligned}$$

and

$$\begin{aligned} W^2(L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n) &= s-\lim_{n\to\infty} W_n(L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n)W_n \\ &= \lim_{n\to\infty} W_n(L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n) \\ &= \lim_{n\to\infty} L_n(\widetilde{a}I_{\mathbb{T}} + \widetilde{b}S_{\mathbb{T}})P_n \\ &= \widetilde{a}I_{\mathbb{T}} + \widetilde{b}S_{\mathbb{T}}. \end{aligned}$$

Therefore, $W^1(A_{n,r}) = aI_{\mathbb{T}} + bS_{\mathbb{T}}$ and $W^2(A_{n,r}) = (\widetilde{a}I_{\mathbb{T}} + \widetilde{b}S_{\mathbb{T}})Q_{r-1}$. Similarly, using the above mentioned properties (2.8) and (2.9), as well as (2.10), we are able to compute $W^1(A_{n,r})^*$ and $W^2(A_{n,r})^*$, which proves proposition (1) above.

On the other hand, from the previous part we have that $W^1(A_{n,r}) = aI_{\mathbb{T}} + bS_{\mathbb{T}}$ and $W^2(A_{n,r}) = (\widetilde{a}I_{\mathbb{T}} + \widetilde{b}S_{\mathbb{T}})Q_{r-1}$. Then, propositions (2), (3) and (4) follow from Theorem 2. □

3 On the kernel dimension of the operator $\mathcal{A}$

Now, we are in condition to compute the kernel dimension of the operator $\mathcal{A}$ given in (1.3).

Theorem 4 *If the singular integral operator $\mathcal{A}$ is Fredholm, then the singular values of the operators $A_{n,r}$ defined in (2.7) have the k-splitting property with*

$$k = k(A_{n,r}) = \dim\ker(\mathcal{A}) + \dim\ker(\widetilde{u}_{\mathbb{T}} I_{\mathbb{T}} + \widetilde{v}_{\mathbb{T}} S_{\mathbb{T}}) Q_{r-1}$$

where $Q_{r-1} := I - P_{r-1}$.

Proof. From [2, Theorem 2.2] we know that the operator $\mathcal{A}$ is equivalent to a matrix singular integral operator of the form

$$\mathcal{D}_{\mathbb{T}} = u_{\mathbb{T}} I_{\mathbb{T}} + v_{\mathbb{T}} S_{\mathbb{T}} \in \mathcal{L}([L^2(\mathbb{T}, w)]^2), \tag{3.1}$$

with coefficients given by

$$u_{\mathbb{T}}(t) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ t^{-1/2} & -t^{-1/2} \end{pmatrix} u_1(t^{1/2}) \begin{pmatrix} 1 & t^{1/2} \\ 1 & -t^{1/2} \end{pmatrix} \tag{3.2}$$

and

$$v_{\mathbb{T}}(t) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ t^{-1/2} & -t^{-1/2} \end{pmatrix} v_1(t^{1/2}) \begin{pmatrix} 1 & t^{1/2} \\ 1 & -t^{1/2} \end{pmatrix}, \tag{3.3}$$

where

$$u_1(t) = \begin{pmatrix} r_{\mathbb{T}_+} a_0(t) & r_{\mathbb{T}_+} a_1(t) \\ r_{\mathbb{T}_+} a_0(-t) & r_{\mathbb{T}_+} a_1(-t) \end{pmatrix},$$

and

$$v_1(t) = \begin{pmatrix} r_{\mathbb{T}_+} b_0(t) & r_{\mathbb{T}_+} b_1(t) \\ r_{\mathbb{T}_+} b_0(-t) & r_{\mathbb{T}_+} b_1(-t) \end{pmatrix}.$$

The conclusion is now obtained from proposition (3) in Theorem 3, taking into account that $W^1(A_{n,r}) = \mathcal{D}_{\mathbb{T}}$, and the fact that two equivalent after extension operators have the same kernel dimension. □

Lemma 3.7 in [7] implies that if r is large enough then the kernel dimension of the operator $\widetilde{u}_{\mathbb{T}} I_{\mathbb{T}} + \widetilde{v}_{\mathbb{T}} S_{\mathbb{T}}$ is equal to the rank of the projection P_{r-1}, that is $2(2r-1)$. Observe that if r is replaced by $r+1$ and the number of singular values increases exactly by 2, then a correct r is found. I.e., $k(A_{n,r+1}) = k(A_{n,r}) = 4$ (see [9] for a more detailed explanation). Moreover, we would like to know the number $\dim\ker(\mathcal{A})$ provided that we would be able to compute $\Lambda_n \cap [0, c_n]$ where Λ_n is the set of the singular values of $(A_{n,r})$.

3.1 Order of convergence of $s_k(A_{n,k})$

In order to analyse $\dim\ker(\mathcal{A})$, we have to identify the number of singular values of $A_{n,r}$ tending to zero. This suggests us to investigate the convergence speed of $s_k(A_{n,k})$ to zero. To this end, by using the operator equivalence relation given in Theorem 2.2 of [2] and Theorem 3, we can adapt the results of Section 4 in [8], as follows:

Corollary 1 *Let $a_0, a_1, b_0, b_1 \in PC(\mathbb{T})$. If the singular integral operator $\mathcal{A}$ is Fredholm, then*

$$s_k(A_{n,r}) \le C \max(\|A_{n,r}\varphi_1\|, \dots, \|A_{n,r}\varphi_l\|, \|W_n A_{n,r} W_n \psi_1\|, \dots, \|W_n A_{n,r} W_n \psi_m\|)$$

with $k = \dim\ker(\mathcal{A}) + \dim\ker(\widetilde{u}_{\mathbb{T}} I_{\mathbb{T}} + \widetilde{v}_{\mathbb{T}} S_{\mathbb{T}})Q_{r-1}$, where the constant C does not depend on n, and $\{\varphi_i\}_{i=1}^{l}$ and $\{\psi_i\}_{i=1}^{m}$ are some orthonormal bases of $\ker(u_{\mathbb{T}} I_{\mathbb{T}} + v_{\mathbb{T}} S_{\mathbb{T}})$ and $\ker(\widetilde{u}_{\mathbb{T}} I_{\mathbb{T}} + \widetilde{v}_{\mathbb{T}} S_{\mathbb{T}})Q_{r-1}$, respectively.

Thus, we have to estimate the norms $\|A_{n,r}\varphi\|$ and $\|W_n A_{n,r} W_n \varphi\|$, where is taken $\varphi \in \ker(u_{\mathbb{T}} I_{\mathbb{T}} + v_{\mathbb{T}} S_{\mathbb{T}})$, $\psi \in \ker(\widetilde{u}_{\mathbb{T}} I_{\mathbb{T}} + \widetilde{v}_{\mathbb{T}} S_{\mathbb{T}})Q_{r-1}$, and $\|\varphi\| = \|\psi\| = 1$. Such estimates are provided in [8]. Here, for the sake of the presentation completeness, we are going to include them. First, we will deal with smooth coefficients $u_{\mathbb{T}}$ and $v_{\mathbb{T}}$. By $C(\mathbb{T}) \subset PC(\mathbb{T})$ we denote the algebra of all continuous functions on $\mathbb{T}$, by $\mathcal{H}^s(\mathbb{T}) \subset C(\mathbb{T})$ the Hölder-Zygmund space and by $\mathcal{R}(\mathbb{T}) \subset C(\mathbb{T})$ the algebra of all rational functions on $\mathbb{T}$. For each continuous function $f \in [C(\mathbb{T})]^{2\times 2}$, we put

$$E_n(f) := \inf_{p \in [\mathcal{R}^n(\mathbb{T})]^{2\times 2}} \|f - p\|_\infty, \quad n \in \mathbb{Z}_+,$$

where $[\mathcal{R}^n(\mathbb{T})]^{2\times 2}$ is the set of all matrix trigonometric polynomials p on $\mathbb{T}$ of the form $p(t) = \sum_{k=-n}^{n} p_k t^k$, with $p_n \in \mathbb{C}^{2\times 2}$. Recall that for any $f \in [C(\mathbb{T})]^{2\times 2}$ and $n \in \mathbb{Z}_+$, there is a polynomial $p_n(f) \in [\mathcal{R}^n(\mathbb{T})]^{2\times 2}$ such that $E_n(f) = \|f - p_n(f)\|_\infty$.

In what follows, by $[\alpha n]$ we denote the integer part of αn (with $n \in \mathbb{Z}_+$).

Lemma 1 *Let $a_0, a_1, b_0, b_1 \in PC(\mathbb{T})$ and let $\alpha \in (0,1)$. If the singular integral operator $\mathcal{A}$ is Fredholm, then*

$$\begin{aligned} s_k(A_{n,r}) \ \le\ & C \max(E_{[\alpha n]}(u_{\mathbb{T}}), E_{[\alpha n]}(v_{\mathbb{T}}), \|Q_{n-[\alpha n]}\varphi_1\|, \dots, \|Q_{n-[\alpha n]}\varphi_l\|, \\ & \|Q_{n-[\alpha n]}\psi_1\|, \dots, \|Q_{n-[\alpha n]}\psi_m\|) \end{aligned}$$

for $\alpha \in (0,1)$ with $k = \dim\ker(\mathcal{A}) + \dim\ker(\widetilde{u}_{\mathbb{T}} I_{\mathbb{T}} + \widetilde{v}_{\mathbb{T}} S_{\mathbb{T}})Q_{r-1}$, where the constant C does not depend on n, and $\{\varphi_i\}_{i=1}^{l}$ and $\{\psi_i\}_{i=1}^{m}$ are some orthonormal bases of $\ker(u_{\mathbb{T}} I_{\mathbb{T}} + v_{\mathbb{T}} S_{\mathbb{T}})$ and $\ker(\widetilde{u}_{\mathbb{T}} I_{\mathbb{T}} + \widetilde{v}_{\mathbb{T}} S_{\mathbb{T}})Q_{r-1}$, respectively.

The last inequality can be used in order to estimate the convergence speed for a_0, a_1, b_0 and b_1 smooth functions.

Proposition 1 *Let $a_0, a_1, b_0, b_1 \in C(\mathbb{T})$ and let the singular integral operator*

$$\mathcal{A} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 J + b_1 S_{\mathbb{T}} J,$$

be Fredholm. If the functions $u_{\mathbb{T}}, v_{\mathbb{T}}$ given by (3.2) and (3.3) belong to $[\mathcal{H}^s(\mathbb{T})]^{2\times 2}$ for some $s > 0$, then

$$s_k(A_{n,r}) = O(n^{-s}), \quad \text{as } n \to \infty. \tag{3.4}$$

On the other hand, if the functions a_0, a_1, b_0 and b_1 belong to $\mathcal{R}(\mathbb{T})$, then there is a $\rho > 0$ such that

$$s_k(A_{n,r}) = O(e^{-\rho n}), \quad \text{as } n \to \infty. \tag{3.5}$$

For more general cases where non-smooth conditions are imposed over the coefficients a_0, a_1, b_0 and b_1, similar estimates to (3.4) and (3.5) can be also obtained. For this situation, the equivalence relation between the operator $\mathcal{A}$ and the Toeplitz operator $\mathcal{T}_\psi$, with $\psi = (u_\mathbb{T} - v_\mathbb{T})^{-1}(u_\mathbb{T} + v_\mathbb{T})$ (see [2, Corollay 2.1]), allows us to use the results of Section 2 in [8], and in particular Theorem 2.2, which gives the estimates (3.4) and (3.5) for corresponding truncated Toeplitz matrices $A_{n,r} := \mathcal{T}_{n,r}(\psi)$.

Example 3.1 In view of illustrating the applicability of Theorem 4, we will present here an example within the smooth coefficients case. Let us consider the operator $\mathcal{A}$ as in (1.3) with reflection operator J defined in (1.4) and coefficients given by

$$
\begin{aligned}
a_0(t) &= \frac{1}{2}[t^{2(s-1)} + t^{-2} + t^{-2s}], \\
a_1(t) &= \frac{1}{2}[-t^{2(s-1)} - t^{-2} + t^{-2s}], \\
b_0(t) &= \frac{t^{-2s}}{2t^{2\kappa_1}+1}\left(\frac{1}{2}(2t^{2\kappa_1} - 1) + \frac{2t^{2\kappa_1} + 3t^{2(-\kappa_2-\alpha-1/2)}}{3t^{-2\kappa_2}+1}\right) \\
&\quad + \frac{1}{2}\frac{3t^{-2\kappa_2}-1}{3t^{-2\kappa_2}+1}(t^{2(s-1)} + t^{-2}), \\
b_1(t) &= \frac{t^{-2s}}{2t^{2\kappa_1}+1}\left(\frac{1}{2}(2t^{2\kappa_1} - 1) - \frac{2t^{2\kappa_1} + 3t^{2(-\kappa_2-\alpha-1/2)}}{3t^{-2\kappa_2}+1}\right) \\
&\quad - \frac{1}{2}\frac{3t^{-2\kappa_2}-1}{3t^{-2\kappa_2}+1}(t^{2(s-1)} + t^{-2}),
\end{aligned}
$$

with $\kappa_1, \kappa_2, s \in 2\mathbb{Z}$ and $\alpha = (4k-1)/2$, $k \in \mathbb{Z}$. From the theory exposed above, $\mathcal{A}$ is equivalent to the operator $\mathcal{D}_\mathbb{T}$ with coefficients $u_\mathbb{T}$ and $v_\mathbb{T}$ given by

$$
u_\mathbb{T}(t) = \begin{pmatrix} t^{-s} & 0 \\ 0 & t^{s-1} + t^{-1} \end{pmatrix} \quad \text{and} \quad v_\mathbb{T}(t) = \begin{pmatrix} t^{-s}\frac{2t^{\kappa_1}-1}{2t^{\kappa_1}+1} & \frac{t^{-s}(4t^{\kappa_1+1/2}+6t^{-\kappa_2-\alpha})}{(2t^{\kappa_1}+1)(3t^{-\kappa_2}+1)} \\ 0 & \frac{3t^{-\kappa_2}-1}{3t^{-\kappa_2}+1}(t^{s-1} + t^{-1}) \end{pmatrix}.
$$

To perform our computations, in a similar manner as in [8, 10], instead of the operators $A_{n,r}$ defined in (2.7) we are going to consider the following operators which have the same singular values as $A_{n,r}$:

$$
B_{n,r} := F_{2n+1}A_{n,r}F_{2n+1}^{-1} = (u_\mathbb{T}(t_j)\delta_{j,k})_{j,k=0}^{2n} + (v_\mathbb{T}(t_j)\delta_{j,k})_{j,k=0}^{2n}F_{2n+1}Q_{n,r}F_{2n+1}^{-1}
$$

where $\delta_{j,k}$ is the Kronecker symbol and F_{2n+1} (with inverses F_{2n+1}^{-1}) are the $2(2n+1) \times 2(2n+1)$ matrices

$$
F_{2n+1} := \left(\frac{1}{\sqrt{2n+1}}e^{\frac{2\pi ij}{2n+1}}I_2\right)_{i,j=0}^{2n}, \quad F_{2n+1}^{-1} := \left(\frac{1}{\sqrt{2n+1}}e^{-\frac{2\pi ij}{2n+1}}I_2\right)_{i,j=0}^{2n}
$$

(with I_2 being the identity 2×2 matrix). Considering these matrices we rewrite $A_{n,r}$ with respect to the standard basis $\operatorname{Im} P_n$ as

$$A_{n,r} = F_{2n+1}^{-1}(u_{\mathbb{T}}(t_j)\delta_{j,k})_{j,k=0}^{2n} F_{2n+1} + F_{2n+1}^{-1}(v_{\mathbb{T}}(t_j)\delta_{j,k})_{j,k=0}^{2n} F_{2n+1} Q_{n,r};$$

here

$$Q_{n,r} = \operatorname{diag}(\underbrace{0I_2, \ldots, 0I_2}_{n+1}, \underbrace{I_2, \ldots, I_2}_{n-\max(0,r-1)}, \underbrace{0I_2, \ldots, 0I_2}_{\max(0,r-1)}).$$

On the other hand, from Corollary 2.1 in [2] we know that $\mathcal{A}$ is equivalent to the Toeplitz operator $\mathcal{T}_\psi$ with

$$\psi(t) = (u_{\mathbb{T}}(t) - v_{\mathbb{T}}(t))^{-1}(u_{\mathbb{T}}(t) + v_{\mathbb{T}}(t)) = \begin{pmatrix} 2t^{\kappa_1} & 2t^{\kappa_1+1/2} + 3t^{-\kappa_2-\alpha} \\ 0 & 3t^{-\kappa_2} \end{pmatrix},$$

where in the case $\alpha > 0$, we have that ψ admits a (right) Wiener-Hopf factorization

$$\psi(t) = \begin{pmatrix} 2 & t^{-\alpha} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{\kappa_1} & 0 \\ 0 & t^{-\kappa_2} \end{pmatrix} \begin{pmatrix} 1 & t^{1/2} \\ 0 & 3 \end{pmatrix}.$$

This implies, from the well-known Simonenko's Theorem, that

$$\dim \ker \mathcal{T}_\psi = \sum_{j \in \{\kappa_1, -\kappa_2\}} \max(0, -j).$$

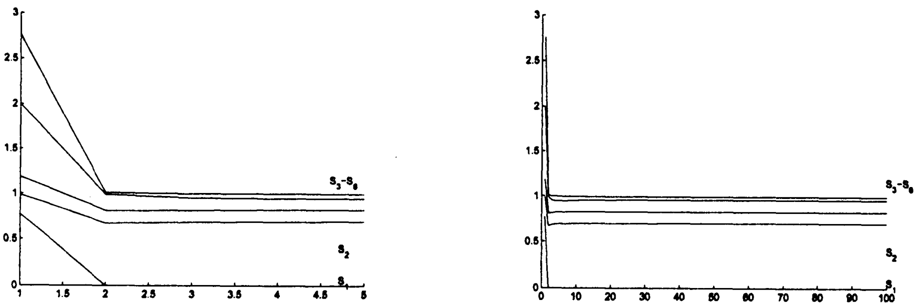

Figure 1: The behavior of the first 6 singular values of $A_{n,0}$ ($n = 5$ and $n = 100$).

Notice that for $\kappa_1, \kappa_2 \geq 0$, $\widetilde{\psi}(t) = \psi\left(\frac{1}{t}\right)$ also admits a right Wiener-Hopf factorization

$$\widetilde{\psi}(t) = \begin{pmatrix} 2t^{-\kappa_1} & 2t^{-\kappa_1-1/2} + 3t^{\kappa_2+\alpha} \\ 0 & 3t^{\kappa_2} \end{pmatrix} = \begin{pmatrix} 2 & \frac{2}{3}t^h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-\kappa_1} & 0 \\ 0 & t^{\kappa_2} \end{pmatrix} \begin{pmatrix} 1 & \frac{3}{2}t^g \\ 0 & 3 \end{pmatrix}$$

with $g = \kappa_1 + \kappa_2 + \alpha$ and $h = -\kappa_1 - \kappa_2 - 1/2$. Therefore, $\dim\ker(\widetilde{u}_{\mathbb{T}} I_{\mathbb{T}} + \widetilde{v}_{\mathbb{T}} S_{\mathbb{T}}) = \dim\ker \mathcal{T}_{\widetilde{\psi}} = \kappa_1$. Thus, these facts give us the value of $k(A_{n,r})$ in Theorem 4, which is $k = \kappa_1 + \kappa_2$. For the case $\kappa_1 = 2$, $\kappa_2 = 0$ and $\alpha = 7/2$, Figure 1 illustrates that in fact $A_{n,r}$ has the 2-splitting property.

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