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THE FORMAL DEGREE OF DISCRETE SERIES REPRESENTATIONS OF GL_N
(GL_N の離散系列表現の形式的次数について)

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Introduction

Let $\boldsymbol{G}$ be a connected reductive group defined over a non-Archimedean local field F, and $G = \boldsymbol{G}(F)$. It is important to determine the formal degrees of the discrete series representations of G for the explicit Plancherel measure for G. In Aubert and Plymen [1], the explicit Plancherel measure for $G = GL_N(D)$ is derived via the work of Silberger and Zink [13], where D is a division algebra over F.

For $GL_N(F)$, there are many works, e.g., [16], [8], [4], and [13] of computing formal degrees of discrete series representations. Indeed, the general formal degree formula of the discrete series representations are given in [4], and the explicit values of the formal degrees are computed in [8] in the tame case, and in [13] in the general case.

In this article, we improve the method of [13] by using the results of [4], and compute the values of the formal degrees of the discrete series representations of $GL_N(F)$ (Teorem 1.4), which are expressed in terms of critical exponents (see 1.1 below). These expressions are implicit in the formula of [13, Theorem 1.1]. Thus, our formulas are not essentially new. But our improved method remains valid for some other classical groups. In fact, we obtained analogous results for a symplectic group $Sp_N(F)$ and for a unramified unitary group $U(V, h)$, where N is an even integer ≥ 4 and h is a non-degenerate Hermitian form of an N-dimensional F-vector space.

The contents of this article are summarized as follows: In Section 1, we give the improvement of the method of Silberger and Zink [13] for $GL_N(F)$, and in Sectin 2, we present results (Theorem 2.7) obtained using the recent works on Hecke algebras of self-dual simple types of Kariyama and Miyauchi [11] (cf. [10]) for the unramified unitary group $U(V, h)$.

1. An improvement on the method of Silberger-Zink

1.1. **Preliminaries.** Let F be a non-Archimedean local field. Let $\mathfrak{o}_F$ be the ring of integers of F, $\mathfrak{p}_F$ its maximal ideal, and $k_F = \mathfrak{o}_F/\mathfrak{p}_F$ the residue field. We denote by $q = |k_F|$ the cardinality of k_F.

Let N be an integer ≥ 2, and V an N-dimensional vector space over F. We set $A = \mathrm{End}_F(V)$ and denote by $G = A^{\times}$ the multiplicative group of A. By an appropriate F-basis of V, we identify $A = M_N(F)$ and $G = GL_N(F)$.

We use the notations of Bushnell-Kutzko [4]. Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in A with Jacobson radical $\mathfrak{P} = \mathrm{rad}(\mathfrak{A})$. We define a subgroup $\mathcal{K}_{\mathfrak{A}}$ of G by $\mathcal{K}_{\mathfrak{A}} = \{g \in G | g\mathfrak{A}g^{-1} = \mathfrak{A}\}$. For an element β in A, the integer $k_0(\beta, \mathfrak{A})$ is defined in [4, (1.4.5), (1.4.6)].

Following [4, (1.5)], a stratum in A is a 4-tuple $[\mathfrak{A}, n, r, \beta]$, where $\mathfrak{A}$ is as above, n, r are integers such that $n > r$, and $\beta \in A$ with $\beta \in \mathfrak{P}^{-n}$.

Definition 1.1. ([4, (1.5.5)]) A stratum $[\mathfrak{A}, n, r, \beta]$ is called *pure*, if the following conditions are satisfied:

(1) the algebra $E = F[\beta]$ is a field,
(2) $E^\times \subset \mathcal{K}_{\mathfrak{A}}$,
(3) $\beta \in \mathfrak{P}^{-n} \backslash \mathfrak{P}^{1-n}$.

It is called *simple* if, in addition,

(4) $r < -k_0(\beta, \mathfrak{A})$.

Thus, for a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A, the integer $k_0(\beta, \mathfrak{A})$ satisfies $k_0(\beta, \mathfrak{A}) = -\min\{r \in \mathbb{Z} : [\mathfrak{A}, n, r, \beta]$ in not simple$\}$ of [15, (3.6)], and it is called the *critical exponent*.

1.2. **Simple types.** Hereafter, we assume that the hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A is always principal, that is, there exists an element z in A such that $\mathfrak{P} = z\mathfrak{A} = \mathfrak{A}z$.

Let $e_0 = e(\mathfrak{A}|\mathfrak{o}_F)$ be the $\mathfrak{o}_F$-period of $\mathfrak{A}$, that is, $\mathfrak{P}^{e_0} = \mathfrak{p}_F\mathfrak{A}$, and set

$$f_0 = N/e_0.$$

Then each element x of $\mathfrak{A}$ has the block form $x = (x_{ij})_{1 \le i,j \le e_0}$ with $x_{ij} \in M_{f_0}(\mathfrak{o}_F)$ if $i \le j$, and $x_{ij} \in M_{f_0}(\mathfrak{p}_F)$ otherwise.

Let B be the A-centralizer of β, and $\mathfrak{B} = \mathfrak{A} \cap B$. Then it follows from Definition 1.1(2) that $\mathfrak{B}$ is a hereditary $\mathfrak{o}_E$-order in B. Let $e_1 = e(\mathfrak{B}|\mathfrak{o}_E)$ be the $\mathfrak{o}_E$-period of $\mathfrak{B}$, defined as is e_0 above, where $\mathfrak{o}_E$ is the maximal ideal of E.

Associated with a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A, the following three compact open subgroups

$$H^1 = H^1(\beta, \mathfrak{A}) \subset J^1 = J^1(\beta, \mathfrak{A}) \subset J = J(\beta, \mathfrak{A})$$

of G are defined in [4, (3.1)]. Via these groups, a *simple type* in G, denoted by (J, λ), is constructed as follows: Take a simple character θ of H^1 (see [4, (3.2)] for the definition). Then it is known that there exists a unique irreducible representation $\eta = \eta(\theta)$ of J^1 containing θ. We obtain an extension, κ, of η to J, which is called a *β-extension*.

Write $G_E = B^\times$. Then G_E is isomorphic to $GL_{N/[E:F]}(E)$. Set

$$f_1 = N/([E : F]e_1).$$

For $U(\mathfrak{B}) = \mathfrak{B}^\times \supset U^1(\mathfrak{B}) = 1 + \mathrm{rad}(\mathfrak{B})$, it follows from [4, (3.1.15)] that $J = U(\mathfrak{B})J^1$ and that

$$J/J^1 \simeq U(\mathfrak{B})/U^1(\mathfrak{B}) \simeq GL_{f_1}(k_E)^{e_1},$$

where k_E denotes the residue field of E. Then J/J^1 is isomorphic to a Levi subgroup of $G_{N/[E:F]}(k_E)$. Let σ_0 be an irreducible cuspidal representation of $GL_{f_1}(k_E)$, and σ the inflation of the representation $\sigma_0^{\otimes e_1} = \sigma_0 \otimes \cdots \otimes \sigma_0$ (e_1-times) of J/J^1 to J. Now the simple type (J, λ) in G is defined by

$$\lambda = \kappa \otimes \sigma,$$

in [4, (5.5.10)(a)].

In particular, a simple type (J, λ) in G of *level zero* is defined in [4, (5.5.10)(b)]. This is a special case of [4, (5.5.10)(a)], by setting $E = F$, $\mathfrak{B} = \mathfrak{A}$, $J^t = U^t(\mathfrak{A})(t = 0, 1)$, and θ, η, κ all trivial. Thus, J/J^1 is isomorphic to $GL_{N/e_1}(k_F)^{e_1}$ for $e_1 = e(\mathfrak{A}|\mathfrak{o}_F)$, and λ is the inflation of a representation $\sigma_0^{\otimes e_1}$, where we set $f_1 = N/e_1$ and σ_0 is an irreducible cuspidal representation of $GL_{f_1}(k_F)$, as above.

A simple type (J, λ) in G is called *maximal*, if $e_1 = e(\mathfrak{A}|\mathfrak{o}_E) = 1$.

1.3. **Discrete series representations of $G = GL_N(F)$.** Let e_1 be a positive integer dividing N, and ρ an irreducible supercuspidal representation of $G' = GL_{N/e_1}(F)$. Then there exists a maximal simple type (J_0, λ_0) in G' containing λ_0 by [4, (8.4.1)].

Let M be a Levi subgroup of $G = GL_N(F)$ that is isomorphic to $G'^{e_1} = G' \times \cdots \times G'$ (e_1-times), and $P = MN$ a parabolic subgroup of G with Levi factor M and with unipotent radical N. Then $\rho^{\otimes e_1}$ is an irreducible supercuspidal representation of $M \simeq G'^{e_1}$. Set $J_M = J_0^{e_1}$, and $\lambda_M = \lambda_0^{\otimes e_1}$. Then (J_M, λ_M) is a $[M, \rho^{\otimes e_1}]_M$-type in the sense of [5, (8.1)], and by [6, Proposition 1.4], there exists an irreducible representation λ_P of a compact open subgroup J_P of G associated with the parabolic subgroup P such that (J_P, λ_P) is a G-cover of (J_M, λ_M). Thus, (J_P, λ_P) is a $[M, \rho^{\otimes e_1}]_G$-type. The pair (J_P, λ_P) is derived, as in [4, (7.2.17)], from a simple type (J, λ) in G associated with a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A as in Section 1.2, and satisfies $\mathrm{Ind}_{J_P}^{J} \lambda_P \simeq \lambda$.

By [17], the induced representation

$$\mathrm{Ind}_P^G(|\det|^{(1-e_1)/2}\rho \otimes \cdots \otimes |\det|^{(e_1-1)/2}\rho)$$

contains a unique irreducible discrete series representation, say $(\pi, \mathcal{V})$, of G. Hence, by [4, (7.3.14)], $(\pi, \mathcal{V})$ contains (J, λ) and so (J_P, λ_P).

1.4. **A formal degree formula.** Let $(\pi, \mathcal{V})$ be the irreducible discrete series representation of G in the previous section that contains a simple type (J, λ) in G associated with a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A with $E = F[\beta]$ as in Section 1.2. Let B be the A-centralizer of β, and $\mathfrak{B} = B \cap \mathfrak{A}$.

Let $e_1 = e(\mathfrak{B}|\mathfrak{o}_E)$ and $f_1 = N/([E : F]e_1)$ be as in Section 1.2. Let K/E be an unramified extension of degree f_1, and set $C^\times = GL_{e_1}(K)$. Let $\mathcal{I}$ be an Iwahori subgroup of $C^\times$, and $\mathbf{1}_{\mathcal{I}}$ the trivial representation of $\mathcal{I}$. Then the Hecke algebras $\mathcal{H}(G, \lambda)$ and $\mathcal{H}(C^\times, \mathbf{1}_{\mathcal{I}})$ for (J, λ) and

$(\mathcal{I}, 1_{\mathcal{I}})$, respectively, are obtained (cf. [4, Section 4]). It follows from [4, (5.5.16), (5.6.6)] that there exists a support-preserving isomorphism

$$\Psi : \mathcal{H}(G, \lambda) \simeq \mathcal{H}(C^{\times}, 1_{\mathcal{I}})$$

(cf. [4, (7.6.18)]). Via this isomorphism Ψ, the equivalence class of the irreducible discrete series representation $(\pi, \mathcal{V})$ of G corresponds to that of an unramified twist $\phi \cdot \mathrm{St}_{C^{\times}}$ of the Steinberg representation $\mathrm{St}_{C^{\times}}$ of $C^{\times}$ (cf. [13, p.13]). In the notation of [4, (7.7.1)], we have

$$\pi \simeq \mathfrak{Ad}_{\Psi}(\phi \cdot \mathrm{St}_{C^{\times}})$$

where $\mathfrak{Ad}_{\Psi}$ is defined as in [4, (7.6.21)]. Denote by St_G the Steinberg representation of G, and by $\deg(\mathrm{St}_G, dx)$ the formal degree of St_G relative to a Haar measure dx on G. Since the formal degrees of $\mathrm{St}_{C^{\times}}$ and $\phi \cdot \mathrm{St}_{C^{\times}}$ are the same, by [4, (7.7.11)], we obtain the following result.

Theorem 1.2. (*cf.* [13, Proposition 3.6]) *Let notations and assumptions be as above. Then for arbitrary measures dx on G and dy on $C^{\times}$,*

(1.1)

$$\frac{\dim(\lambda)}{e(E|F)} \mathrm{vol}(K^{\times}\mathcal{I}/K^{\times}, dy) \deg(\mathrm{St}_{C^{\times}}, dy) = \mathrm{vol}(F^{\times}J/F^{\times}, dx) \deg(\pi, dx),$$

where $e(E|F)$ denotes the index of ramification of E/F.

1.5. **A deformation of the formal degree formula.** There exists a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}_m$ in A such that $U(\mathfrak{A}_m) = \mathfrak{A}_m^{\times}$ is an Iwahori subgroup of G that is contained in the parahoric subgroup $U(\mathfrak{A})$. Now, we first normalize dx on G so that $\mathrm{vol}(F^{\times}U(\mathfrak{A}_m)/F^{\times}, dx) = 1$. Then

$$\begin{aligned} \mathrm{vol}(F^{\times}U(\mathfrak{A})/F^{\times}, dx) &= (U(\mathfrak{A}) : U(\mathfrak{A}_m))\mathrm{vol}(F^{\times}U(\mathfrak{A}_m)/F^{\times}, dx) \\ &= (U(\mathfrak{A}) : U(\mathfrak{A}_m)), \end{aligned}$$

and by [3, Proposition 5.3], the formal degree of the Steinberg representation St_G of G is given by

$$\deg(\mathrm{St}_G, dx) = \frac{1}{N} \widetilde{W}_{A_{N-1}}(q^{-1})^{-1},$$

where $\widetilde{W}_{A_{N-1}}(t)$ is the Poincaré series of type A_{N-1} (see [12]). Thus, we obtain Macdonald's formula:

$$\mathrm{vol}(F^{\times}U(\mathfrak{A})/F^{\times}, dx) \deg(\mathrm{St}_G, dx) = \frac{1}{N}(U(\mathfrak{A}) : U(\mathfrak{A}_m))\widetilde{W}_{A_{N-1}}(q^{-1})^{-1}$$

which is the same as that of [13, 3.7]. This formula holds for any Haar measure dx on G. We again normalize dx on G so that

$$\deg(\mathrm{St}_G, dx) = 1.$$

Then

$$\mathrm{vol}(F^{\times}U(\mathfrak{A})/F^{\times}, dx) = \frac{1}{N}(U(\mathfrak{A}) : U(\mathfrak{A}_m))\widetilde{W}_{A_{N-1}}(q^{-1})^{-1}$$

On the other hand, for $C^\times = GL_{e_1}(K)$, set

$$f = f(K|F) = [k_K : k_F],$$

where k_K is the residue field of K. Then, from Macdonald's formula for $\mathrm{St}_{C^\times}$, we also obtain

$$\mathrm{vol}(K^\times \mathcal{I}/K^\times, dy)\deg(\mathrm{St}_{C^\times}, dy) = \frac{1}{e_1}\widetilde{W}_{A_{e_1-1}}(q^{-f})^{-1}.$$

Hence, since $N/e_1 = [K : F] = fe(E|F)$ and

$$\mathrm{vol}(F^\times J/F^\times, dx) = \mathrm{vol}(F^\times U(\mathfrak{A})/F^\times, dx)(U(\mathfrak{A}) : J)^{-1},$$

we obtain

(1.2)

$$\frac{\deg(\mathrm{St}_{C^\times}, dy)}{e(E|F)}\frac{\mathrm{vol}(K^\times \mathcal{I}/K^\times, dy)}{\mathrm{vol}(F^\times J/F^\times, dx)} = f\frac{\widetilde{W}_{A_{N-1}}(q^{-1})}{\widetilde{W}_{A_{e_1-1}}(q^{-f})}\frac{(U(\mathfrak{A}) : J)}{(U(\mathfrak{A}) : U(\mathfrak{A}_m))}.$$

Since $J = U(\mathfrak{B})J^1$ and $J^1 \subset U^1(\mathfrak{A})$, we moreover obtain

$$(1.3) \qquad (U(\mathfrak{A}) : J) = \frac{(U(\mathfrak{A}) : U^1(\mathfrak{A}))}{(U(\mathfrak{B}) : U^1(\mathfrak{B}))}(U^1(\mathfrak{A}) : J^1).$$

Lemma 1.3. *Let notations and assumptions be as above. Then the formal degree* $\deg(\pi, dx)$ *is equal to*

$$\left(f\frac{\widetilde{W}_{A_{N-1}}(q^{-1})}{\widetilde{W}_{A_{e_1-1}}(q^{-f})}\right)(U(\mathfrak{A}_m) : U^1(\mathfrak{A}))\left(\frac{\dim(\sigma)}{(U(\mathfrak{B}) : U^1(\mathfrak{B}))}\right) \times \Big((U^1(\mathfrak{A}) : J^1)\dim(\eta)\Big).$$

Proof. By definition, $\dim(\lambda) = \dim(\eta)\dim(\sigma)$. Thus, the lemma follows from Eqs. (1.1) to (1.3).

1.6. **Calculation of the factors of** $\deg(\pi, dx)$**.** By the definition in [12],

$$(1.4) \qquad f\frac{\widetilde{W}_{A_{N-1}}(q^{-1})}{\widetilde{W}_{A_{e_1-1}}(q^{-f})} = f\frac{q^N - 1}{q^{N/e} - 1}\frac{(q^f - 1)^{e_1}}{(q-1)^N}$$

where we set $e = e(E|F)$.

Since $U(\mathfrak{A})/U^1(\mathfrak{A}) \simeq GL_{f_0}(k_F)^{e_0}$ in Section 1.2, the quotient group $U(\mathfrak{A}_m)/U^1(\mathfrak{A})$ is isomorphic to the product of e_0-copies of a Borel subgroup $\overline{B}_0$ of $GL_{f_0}(k_F)$. Hence,

$$(1.5) \qquad (U(\mathfrak{A}_m) : U^1(\mathfrak{A})) = |\overline{B}_0|^{e_0} = \{(q-1)^{f_0}q^{\frac{1}{2}f_0(f_0-1)}\}^{e_0}.$$

By the definition of σ_0,

$$\dim(\sigma) = (\dim(\sigma_0))^{e_1} = \prod_{i=1}^{f_1-1}(q^{f(E|F)i} - 1)^{e_1},$$

and $(U(\mathfrak{B}) : U^1(\mathfrak{B})) = |GL_{f_1}(k_E)|^{e_1}$. Hence,

$$\frac{\dim(\sigma)}{(U(\mathfrak{B}) : U^1(\mathfrak{B}))} = \frac{q^{-\frac{1}{2}f(E|F)e_1 f_1(f_1-1)}}{(q^f-1)^{e_1}}. \tag{1.6}$$

1.7. **Calculation of the factor** $(U^1(\mathfrak{A}) : J^1)\dim(\eta)$**.** Since by definition $U^1(\mathfrak{A}) = 1+\mathfrak{P} \supset J^1 = 1+\mathfrak{J}^1 \supset H^1 = 1+\mathfrak{H}^1$, it follows from [4, (5.1.1)] that the last factor $(U^1(\mathfrak{A}) : J^1)\dim(\eta)$ is equal to

$$\begin{aligned}(\mathfrak{P} : \mathfrak{J}^1)\sqrt{(J^1 : H^1)} &= (\mathfrak{P} : \mathfrak{J}^1)(\mathfrak{J}^1 : \mathfrak{H}^1)^{1/2} \\ &= (\mathfrak{P} : \mathfrak{J}^1)^{1/2}(\mathfrak{P} : \mathfrak{H}^1)^{1/2}.\end{aligned}$$

This amounts to $\dim \pi_\beta^1$ in [13, 7.1]. We compute this similarly to as done in [13].

For the simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A, $[\mathfrak{A}, n, r, \beta]$ with $r = -k_0(\beta, \mathfrak{A})$ is a pure stratum, and it determines a family $[\mathfrak{A}, n, r_i, \gamma_i], 0 \le i \le s$ of simple strata that satisfy the conditions [4, (2.4.2)]. This family is called a *defining sequence* for the pure stratum $[\mathfrak{A}, n, r, \beta]$. Hence we get a family of pairs $(r_i, \gamma_i), 0 \le i \le s$, such that

(1) $[\mathfrak{A}, n, r_i, \gamma_i]$ is simple, $0 \le i \le s$;
(2) $[\mathfrak{A}, n, r_0, \gamma_0] \sim [\mathfrak{A}, n, r, \beta]$;
(3) $0 < r = r_0 < r_1 < \cdots < r_s < r_{s+1} = n$;
(4) $r_{i+1} = -k_0(\gamma_i, \mathfrak{A})$, and for $0 \le i \le s-1$,

$$[\mathfrak{A}, n, r_{i+1}, \gamma_{i+1}] \sim [\mathfrak{A}, n, r_{i+1}, \gamma_i],$$

(5) $-k_0(\gamma_s, \mathfrak{A}) = n$ or ∞.

These conditions include all the conditions in [4, (2.4.2)] except condition (vi).

Denote by A_{γ_i} the A-centralizer of γ_i. Then by using the family $(r_i, \gamma_i), 0 \le i \le s$, we define $\mathfrak{J} = \mathfrak{J}(\beta) = \mathfrak{J}(\beta, \mathfrak{A})$ and $\mathfrak{H} = \mathfrak{H}(\beta) = \mathfrak{H}(\beta, \mathfrak{A})$ inductively as follows:

$$\begin{aligned}\mathfrak{J}(\gamma_i) &= \mathfrak{A} \cap A_{\gamma_i} + \mathfrak{J}(\gamma_{i+1}) \cap \mathfrak{P}^{[(r_{i+1}+1)/2]}, \\ \mathfrak{H}(\gamma_i) &= \mathfrak{A} \cap A_{\gamma_i} + \mathfrak{H}(\gamma_{i+1}) \cap \mathfrak{P}^{[r_{i+1}/2]+1},\end{aligned}$$

for $-1 \le i \le s$, where we set $\gamma_{-1} = \beta$ and $\mathfrak{J}(\gamma_{s+1}) = \mathfrak{H}(\gamma_{s+1}) = \mathfrak{A}$, (see [4, (3.1.7), (3.1.8)]).

For a real number r, set

$$\overline{r} = [r/2]+1, \ \underline{r} = [(r+1)/2],$$

where $[x]$ denotes the greatest integer $\le x$, for a real number x. From a filtration

$$\mathfrak{P} = \mathfrak{P} + \mathfrak{J}^1 \supset \mathfrak{P}^{\underline{r_1}} + \mathfrak{J}^1 \supset \mathfrak{P}^{\underline{r_2}} + \mathfrak{J}^1 \supset \cdots \supset \mathfrak{P}^{\underline{r_s}} + \mathfrak{J}^1 \supset \mathfrak{P}^{\underline{n}} + \mathfrak{J}^1 = \mathfrak{J}^1$$

in $\mathfrak{P} \supset \mathfrak{J}^1$, we get

$$\begin{aligned}(\mathfrak{P} : \mathfrak{J}^1) &= \prod_{i=-1}^{s} (\mathfrak{P}^{\underline{r_i}} + \mathfrak{J}^1 : \mathfrak{P}^{\underline{r_{i+1}}} + \mathfrak{J}^1) \\ &= \prod_{i=-1}^{s} \frac{(\mathfrak{P}^{\underline{r_i}} : \mathfrak{P}^{\underline{r_{i+1}}})}{(\mathfrak{P}^{\underline{r_i}} \cap \mathfrak{J} : \mathfrak{P}^{\underline{r_{i+1}}} \cap \mathfrak{J})} \\ &= \prod_{i=-1}^{s} \frac{(\mathfrak{P}^{\underline{r_i}} : \mathfrak{P}^{\underline{r_{i+1}}})}{(\mathfrak{P}^{\underline{r_i}} \cap A_{\gamma_i} : \mathfrak{P}^{\underline{r_{i+1}}} \cap A_{\gamma_i})},\end{aligned}$$

where we set $r_{-1} = 1$. The last line follows from [4, (3.1.8), (3.1.10)].

For $-1 \le i \le s$, set

$$d_i = \underline{r_{i+1}} - \underline{r_i}.$$

Since $\mathfrak{P}$ is principal,

$$(\mathfrak{P}^{\underline{r_i}} : \mathfrak{P}^{\underline{r_{i+1}}}) = (\mathfrak{A} : \mathfrak{P})^{d_i} = |\mathrm{M}_{f_0}(k_F)|^{e_0 d_i} = q^{N f_0 d_i}.$$

Set $f_i' = f(F[\gamma_i]|F)$, $e_i' = e(F[\gamma_i]|F)$, and

$$m_i = (N/[F[\gamma_i] : F])/(e_0/e_i') = f_0/f_i'.$$

Then

$$(\mathfrak{P}^{\underline{r_i}} \cap A_{\gamma_i} : \mathfrak{P}^{\underline{r_{i+1}}} \cap A_{\gamma_i}) = |\mathrm{M}_{m_i}(k_{F[\gamma_i]})|^{(e_0/e_i')d_i} = q^{N f_0 d_i/[F[\gamma_i]:F]}.$$

Hence,

$$(\mathfrak{P} : \mathfrak{J}^1) = q^{\mu};\ \mu = \sum_{i=-1}^{s} N f_0 (1 - [F[\gamma_i] : F]^{-1}) d_i.$$

Similarly, setting $d_i' = \overline{r_{i+1}} - \overline{r_i}$, we get

$$(\mathfrak{P} : \mathfrak{H}^1) = q^{\mu'};\ \mu' = \sum_{i=-1}^{s} N f_0 (1 - [F[\gamma_i] : F]^{-1}) d_i'.$$

Finally, we obtain $(\mathfrak{P} : \mathfrak{J}^1)^{1/2} (\mathfrak{P} : \mathfrak{H}^1)^{1/2} = q^{\nu/2}$ with

(1.7) $$\nu = \sum_{i=-1}^{s} N f_0 (1 - [F[\gamma_i] : F]^{-1})(r_{i+1} - r_i).$$

1.8. **Main Theorem for $GL_N(F)$.** We are now ready to determine the explicit formal degree formula for an irreducible discrete series representation $(\pi, \mathcal{V})$ of G as in Section 1.3.

Theorem 1.4. ([13, Theorem 1.1]) *Let $(\pi, \mathcal{V})$ be an irreducible discrete series representation of G that contains a simple type (J, λ) in G associated with a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A as in Section 1.3. Let*

dx be a Haar measure on G such that $\deg(\mathrm{St}_G, dx) = 1$. *For a family* (r_i, γ_i), $0 \le i \le s$, *as in Section 1.7, set*

$$\Delta = \frac{1}{e(\mathfrak{A}|\mathfrak{o}_F)} \sum_{i=-1}^{s} (1 - [F[\gamma_i] : F]^{-1})(r_{i+1} - r_i)$$

where we reset $(r_{-1}, \gamma_{-1}) = (0, \beta)$ *and* $r_{s+1} = n$. *Then this positive rational number* Δ *does not depend on the choice of defining sequence, and*

$$\deg(\pi, dx) = f \frac{q^N - 1}{q^{N/e} - 1} q^{\frac{1}{2}[N^2\Delta - N(1-1/e)]},$$

where $f = f(K|F)$ *and* $e = e(K|F) = e(E|F)$.

Proof. Denote by $\deg(\pi, dx)_{p'}$ the p-prime part of $\deg(\pi, dx)/f$. From Lemma 1.3 and from Eqs. (1.4) to (1.6), it follows immediately that

$$f \cdot \deg(\pi, dx)_{p'} = f \frac{q^N - 1}{q^{N/e} - 1}.$$

Since the q-power of (1.5) $\times$ (1.6) is equal to

$$\begin{aligned} \frac{1}{2} e_0 f_0 (f_0 - 1) \quad &- \quad \frac{1}{2} f(E|F) e_1 f_1 (f_1 - 1) \\ &= \quad \frac{1}{2} N(\frac{1}{e} - 1) + \frac{1}{2} N f_0 (1 - \frac{1}{[E:F]}), \end{aligned}$$

the sum of this value and the value ν of (1.7) is reduced to the q-power of the right-hand side of the formula in the assertion.

It follows directly from [4, (2.1.4)] that Δ does not depend on the choice of defining sequence. The proof is complete.

In Theorem 1.4, if $e_1 = e(\mathfrak{B}|\mathfrak{o}_E) = 1$, the irreducible discrete series representation $(\pi, \mathcal{V})$ of G is supercuspidal. Thus, Theorem 1.4 contains the formal degree formula for an irreducible supercuspidal representation of G containing a maximal simple type (J, λ) as follows:

Corollary 1.5. *Let* $(\pi, \mathcal{V})$ *be an irreducible supercuspidal representation of* G *containing a maximal simple type* (J, λ) *in* G, *and* $\{(r_i, \gamma_i) : 0 \le i \le s\}$ *a family as in Theorem 1.3. Then*

$$\deg(\pi, dx) = f \frac{q^N - 1}{q^{N/e} - 1} q^{\frac{1}{2}[N^2\Delta - N(1-1/e)]},$$

for a Haar measure dx *on* G *with* $\deg(\mathrm{St}_G, dx) = 1$, *where*

$$\Delta = \frac{1}{e} \sum_{i=-1}^{s} (1 - [F[\gamma_i] : F]^{-1})(r_{i+1} - r_i).$$

Remarks 1.6. (i) It is shown by [4, (6.2.2)] that an irreducible supercuspidal representation $(\pi, \mathcal{V})$ of G containing a maximal simple type

(J,λ) in G of positive level is equivalent to c-$\mathrm{Ind}_{E^\times J}^G \Lambda$ for an extension Λ of λ. From this fact, we obtain

$$\deg(\pi, dx) = \frac{\dim(\lambda)}{\mathrm{vol}(E^\times J/F^\times, dx)}$$

for any Haar measure dx on G, (cf. [7, 5.9]), independently from Theorem 1.2, and can similarly show Corollary 1.5.

(ii) In Theorem 1.4, the level zero case is implicit. In this case, $E = F, \mathfrak{B} = \mathfrak{A}, J^t = U^t(\mathfrak{A})(t = 0,1)$ and η is trivial, as in Section 1.2. Thus, in the formula for $\deg(\pi, dx)$ of Lemma 1.3, we have $(U^1(\mathfrak{A}) : J^1)\dim(\eta) = 1$, and can get

$$\deg(\pi, dx) = N/e(\mathfrak{A}|\mathfrak{o}_F)$$

similarly to as done in Section 1.6, for such a Haar measure dx on G as above.

(iii) This formula follows also from Theorem 1.4, by setting $e = 1$, $\Delta = 0$ and $\lambda = \lambda_1$, since $E = F$, $J = U(\mathfrak{A})$ and so the defining sequence is empty. Hence, by Theorem 1.4, the formal degrees of all discrete series representations of $GL_N(F)$ are computed.

2. An application to unramified p-adic unitary groups

2.1. Unramified unitary groups. Let F be a non-Archimedean local field of odd residual characteristic, with a non-trivial galois involution $x \mapsto \overline{x}$ with fixed field F_0. Let N be an even integer ≥ 4, and V an N-dimensional F-vector space equipped with a non-degenerate F/F_0-Hermitian form h with anisotropic part (0). Let $G = U(V,h)$ be the unitary group of (V,h). Hereafter, we assume that F/F_0 is unramified.

Recently, in [9], we defined a *self-dual* simple type (J,λ) in G associated with a certain skew simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A , and, in [11], we proved that the Hecke algebra $\mathcal{H}(G,\lambda)$ is the affine Hecke algebra of type $\widetilde{C}_m$ for some positive integer $m \geq 2$, and determined the parameters of the Hecke algebra completely. Thanks to these results on G, we can apply the improved method of the previous section for $GL_N(F)$ to the unramified unitary group G, and we obtain analogous results for the group G. Here we present a part of these results without proofs.

2.2. Self-dual simple types. We also denote by $x \mapsto \overline{x}$ the adjoint (anti-)involution on $A = \mathrm{End}_F(V)$ induced by the Hermitian form h.

A simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A, defined in Section 1.1, is called *skew* if $\mathfrak{A}$ is defined by a self-dual strict $\mathfrak{o}_F$-lattice sequence Λ in V (cf. [14, 1.2]) and β is skew in A, i.e., $\overline{\beta} = -\beta$. Assume that $[\mathfrak{A}, n, 0, \beta]$ is a skew simple stratum in A with $E = F[\beta]$. Write $E_0 = \{x \in E : \overline{x} = x\}$. Then there exists a non-degenerate E/E_0-Hermitian form h_E on the E-vector space V such that, setting $L^\# = \{v \in V : h_E(v, L) \subset \mathfrak{p}_E\}$, for an $\mathfrak{o}_E$-lattice L in V, we have $L^\# = \{v \in V : h(v, L) \subset \mathfrak{p}_F\}$ (cf. [15, Section 2]).

Definition 2.1. Following [9], we say that a skew simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A is *good* if the following conditions are satisfied:

(1) E/E_0 is an unramified quadratic extension;
(2) $R = \dim_E(V)$ is even;
(3) there exists an $\mathfrak{o}_E$-lattice L in $\{\Lambda(n) : n \in \mathbb{Z}\}$ such that $L^{\#} = \varpi_E L$. where Λ is as above and ϖ_E is a uniformizer of E.

Hereafter, we assume that $[\mathfrak{A}, n, 0, \beta]$ is a good skew simple stratum in A with $E = F[\beta]$. Let B be the A-centralizer of β, $\mathfrak{B} = B \cap \mathfrak{A}$, $e_1 = e(\mathfrak{B}|\mathfrak{o}_E)$, and $f_1 = N/([E : F]e_1)$, as before.

Similarly, we have compact open subgroups $H^1(\beta, \mathfrak{A}) \subset J^1(\beta, \mathfrak{A}) \subset J(\beta, \mathfrak{A})$ of G. Denote simply by $H^1 \subset J^1 \subset J$ these subgroups. As in Section 1.2, we begin with a skew simple character θ of H^1, and there exists a unique irreducible representation η of J^1 containing θ. Moreover, we also have a β-extension κ of η to J by [15, 4.2]. Set

$$m = [e_1/2].$$

It follows from the conditions of Definition 2.1 that the quotient group J/J^1 is isomorphic to

$$U(\mathfrak{B})/U^1(\mathfrak{B}) \simeq \begin{cases} GL_{f_1}(k_E)^m & \text{if } e_1 \text{ is even,} \\ GL_{f_1}(k_E)^m \times U_{f_1}(k_E/k_{E_0}) & \text{if } e_1 \text{ is odd,} \end{cases}$$

where by Definition 2.1(2), f_1 is even and $U_{f_1}(k_E/k_{E_0})$ denotes the unitary group of a non-degenerate k_E/k_{E_0}-Hermitian form on an f_1-dimensional k_E-vector space. Let σ_0 and σ_1 be irreducible cuspidal representations of $GL_{f_1}(k_E)$ and $U_{f_1}(k_E/k_{E_0})$, respectively. Let σ be the inflation of $\sigma_0^{\otimes m}$ to J if e_1 is even, and that of $\sigma_0^{\otimes m} \otimes \sigma_1$ if e_1 is odd. A simple type (J, λ) in G (of positive level) is defined similarly by $\lambda = \kappa \otimes \sigma$. By [15, p.334], on each factor $GL_{f_1}(k_E)$ of $U(\mathfrak{B})/U^1(\mathfrak{B})$, a certain Weyl group element of $G_E = B \cap G$ induces an involution $g \mapsto \overline{g}$.

Definition 2.2. ([11, Definition 5.2]) A simple type (J, λ) in G is called *self-dual* if σ_0 is equivalent to the representation $g \mapsto \sigma_0(\overline{g})$.

2.3. **A formal degree formula for unramified** $U(V, h)$**.** Recently, by [10] and [11], we determined the structure of the Hecke algebra $\mathcal{H}(G, \lambda)$ for such a self-dual simple type (J, λ) above as follows:

Proposition 2.3. *Let $[\mathfrak{A}, n, 0, \beta]$ be a good skew simple stratum in A with $E = F[\beta]$, and (J, λ) a self-dual simple type in G associated with $[\mathfrak{A}, n, 0, \beta]$ in A. Let B be the A-centralizer of β, $e_1 = e(\mathfrak{B}|\mathfrak{o}_E)$, and $m = [e_1/2]$. Assume that $m \geq 2$. Then the Hecke algebra $\mathcal{H}(G, \lambda)$ for (J, λ) is an affine Hecke algebra of type $\widetilde{C}_m$ with parameter $(q_1, q_2, q_3) = (q^{-N/e_0}, q^{-N/2e_0}, q^{-N/2e_0})$, where $e_0 = e(\mathfrak{A}|\mathfrak{o}_F)$ as in Section 1.1.*

Let K/E_0 be a field extension such that $K \supset E$, $K \supset K_0 \supset E_0$, $e(K|E_0) = 1$, and $[K : E] = [K_0 : E_0] = f_1$. Let $C^\times$ be the unitary group of type C_m defined by a non-degenerate K/K_0-Hermitian form, and $\mathcal{I}$ an Iwahori subgroup of $C^\times$. Immediately, $|k_K| = |k_E|^{f_1} = q^{N/e_0}$, and so the Iwahori-Hecke algebra $\mathcal{H}(C^\times, \mathbf{1}_\mathcal{I})$ turns out to be also the affine Hecke algebra of type $\widetilde{C}_m$ with parameter $(q_1, q_2, q_3) = (q^{-N/e_0}, q^{-N/2e_0}, q^{-N/2e_0})$ provided that $m \geq 2$.

When $m = [e_1/2] = 1$, again by [11], $\mathcal{H}(G, \lambda)$ is isomorphic to the affine Hecke algebra with parameter $|k_{K_0}| = q^{N/2e_0}$ over the infinite dihedral group, and so is the Iwahori-Hecke algebra $\mathcal{H}(C^\times, \mathbf{1}_\mathcal{I})$ of unramified $C^\times = U(1,1)(K_0)$ relative to $\mathbf{1}_\mathcal{I}$ as well (cf. [2, 3.d]). Hence, we obtain the following:

Proposition 2.4. *Let notations and assumptions be as above. In particular, let $m \geq 1$. Then there exists a canonical isomorphism*

$$\Psi : \mathcal{H}(G, \lambda) \simeq \mathcal{H}(C^\times, \mathbf{1}_\mathcal{I}),$$

that is support-preserving.

Theorem 2.5. *Via the Hecke isomorphism Ψ in Proposition 2.4, the Steinberg representation $\mathrm{St}_{C^\times}$ of $C^\times$ corresponds to the equivalence class of an irreducible square-integrable representation, say $(\pi, \mathcal{V})$, of G that contains the self-dual simple type (J, λ) in G as above.*

Proof. This is the analogue of [5, (7.7.1)] for $GL_N(F)$. The method of proof remains valid for unramified G.

Corollary 2.6. *Let notations and assumptions be as in Theorem 2.5. Then*

$$\mathrm{vol}(J, dx)\ \frac{\deg(\pi, dx)}{\dim(\lambda)} = \mathrm{vol}(\mathcal{I}, dy)\deg(\mathrm{St}_{C^\times}, dy)$$

for any Haar measures dx on G and dy on $C^\times$.

2.4. **A formal degree formula for unramified G.** We normalize dx on G so that the formal degree of the Steinberg representation St_G of G is equal to 1 relative to dx. Then, as in Section 1.5, the formal degree $\deg(\pi, dx)$ is rewritten as

$$\left(\frac{\widetilde{W}_{C_{N/2}}(q^{-1}, q^{-1/2}, q^{-1/2})}{\widetilde{W}_{C_m}(q^{-N/e_0}, q^{-N/2e_0}, q^{-N/2e_0})}\right)(U(\mathfrak{A}_m) : U^1(\mathfrak{A})) \times\left(\frac{\dim(\sigma)}{(U(\mathfrak{B}) : U^1(\mathfrak{B}))}\right)\Big((U^1(\mathfrak{A}) : J^1)\dim(\eta)\Big),$$

where $U(\mathfrak{A}_m)$ is an Iwahori subgroup of G that is contained in $U(\mathfrak{A})$, and, for example, $\widetilde{W}_{C_m}(t_1, t_2, t_3)$ denotes the Poincaré series of type $\widetilde{C}_m$ (see [12, Section 3]). We note that this also holds in the case of $m = 1$. For, if we formally set $m = 1$ in the Poincaé series $\widetilde{W}_{C_m}(t_1, t_2, t_3)$, we have

$$\frac{(1-t_1)(1+t_2)(1+t_3)}{(1-t_1)(1-t_2t_3)} = \frac{(1+t_2)(1+t_3)}{1-t_2t_3}.$$

This is nothing but the Poincaré series for the infinite dihedral group.

Similarly, we can easily compute the factors except for the last $(U^1(\mathfrak{A}) : J^1)\dim(\eta)$ as in Section 1.6.

The calculation of this last factor is rather more laborious than that for GL_N in Section 1.7. We also have a defining sequence for the pure stratum $[\mathfrak{A}, n, r, \beta]$ in A with $r = -k_0(\beta, \mathfrak{A})$, and get a family $\{(r_i, \gamma_i) | 0 \leq i \leq s\}$, together with $(r_{-1}, \gamma_{-1}) = (1, \beta)$, where γ_i is skew and simple in A, by [14, Section 3], as in Section 1.7. Set $d_i = [(r_{i+1}+1)/2] - [(r_i+1)/2]$ and $d'_i = [r_{i+1}/2] - [r_i/2]$, for $-1 \leq i \leq s$, as before.

To present the main theorem, we need to define positive integers δ_i, for $-1 \leq i \leq s$, as follows: set $\delta_i = 0$ if $e_0/e(F[\gamma_i]|F)$ is even, and otherwise,

$$\begin{aligned}
\delta_i &= [d_i/2] + [(d'_i+1)/2] && \text{if } r_i \equiv 0 \bmod 4,\\
\delta_i &= [(d_i+1)/2] + [d'_i/2] && \text{if } r_i \equiv 1 \bmod 4,\\
\delta_i &= [(d_i+1)/2] + [(d'_i+1)/2] && \text{if } r_i \equiv 2 \bmod 4,\\
\delta_i &= [d_i/2] + [d'_i/2] && \text{if } r_i \equiv 3 \bmod 4.
\end{aligned}$$

Theorem 2.7. *Let $(\pi, \mathcal{V})$ be an irreducible square-integrable representation of G containing a self-dual simple type (J, λ) in G as in Theorem 2.5. Let $\{(r_i, \gamma_i) | 0 \leq i \leq s\}$ be a family defined as above. Let $e_0 = e(\mathfrak{A}|\mathfrak{o}_F)$, $e = e(E|F)$, and $e_i^0 = e(F[\gamma_i]|F[\gamma_i]_0)$, for $0 \leq i \leq s$, where each $F[\gamma_i]_0$ is the fixed field of $F[\gamma_i]$ under the involution induced by the adjoint one $x \mapsto \overline{x}$ on A in Section 2.2. Set*

$$\Delta = \frac{1}{e_0}\sum_{i=-1}^{s}\Big(1 - \frac{1}{[F[\gamma_i] : F]}\Big)(r_{i+1} - r_i),$$

$$\Delta' = \frac{1}{e_0}\sum_{i=-1}^{s}(1 - e_i^0)\delta_i$$

where each δ_i is the integer defined above and we reset $(r_{-1}, \gamma_{-1}) = (0, \beta)$ and $r_{s+1} = n$. Then these Δ and Δ' do not depend on the choice of defining sequence, and the formal degree $\deg(\pi, dx)$ *is given by*

(1) when $e_0/e = e(\mathfrak{B}|\mathfrak{o}_E)$ is even,

$$q^{\frac{1}{4}[N^2\Delta - N(1-1/e) + N\Delta']} \times \prod_{i=0}^{e_0/2e-1} \frac{q^{N(i+e_0/2e)/e_0} - 1}{(q^{N(i+1)/e_0} - 1)(q^{N(i+1/2)/e_0} + 1)^2} \times \prod_{i=0}^{N/2-1} \frac{(q^{i+1}-1)(q^{i+1}+1)^2}{q^{N/2+i-1} - 1},$$

(2) when $e_0/e = e(\mathfrak{B}|\mathfrak{o}_E)$ is odd,

$$\frac{q^{\frac{1}{4}[N^2\Delta - N(1-1/e)+N\Delta']}}{q^{N/2e_0}-1} \times \prod_{i=0}^{(e_0/e-3)/2} \frac{q^{N(i+(e_0/e-1)/2)/e_0}-1}{(q^{N(i+1)/e_0}-1)(q^{N(i+1/2)/e_0}+1)^2} \times \prod_{i=0}^{N/2-1} \frac{(q^{i+1}-1)(q^{i+1}+1)^2}{q^{N/2+i-1}-1}.$$

Remarks 2.8. (1) We also obtained analogous results for $Sp_N(F)$. This formal degree formula is more complicated than that in Theorem 2.7 for the unramified unitary group G.

(2) In [11], it is proved that Theorem 2.5 holds for a self-dual simple type (J, λ) in G associated with not only a skew simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A which is not good (see Definition 2.1), but also a skew (non-simple) semisimple stratum $[\Lambda, n, 0, \beta]$ in A, that is, β is a skew semisimple element of A (cf. [14]).

(3) We constructed a special skew semisimple stratum $[\Lambda, n, 0, \beta]$ in A, and computed the corresponding formal degree as well.

References

[1] A.-M. Aubert and R. Plymen, Plancherel measure for GL(n, F) and GL(m, D): Explicit formulas and Bernstein decomposition, J. Number Theory, **112** (2005), 26-66.

[2] L. Blasco and C. Blondel, Algèbres de Hecke et séries principales généralisées de $Sp_4(F)$, Proc. London Math. Soc. (3) **85** (2002), 659-685.

[3] A. Borel, Admissible representations of a semisimple group over a local field with vectors fixed under an Iwahori subgroup, Invent. Math. **35** (1976), 233-259.

[4] C. J. Bushnell and P. C. Kutzko, The admissible dual of GL(N) via compact open subgroups, Ann. Math. Stud. **129**, Princeton Univ. Press, 1993.

[5] C. J. Bushnell and P. C. Kutzko, Smooth representations of reductive p-adic groups: structure theory via types, Proc. London Math. Soc. (3) **77** (1998), 582-634.

[6] C. J. Bushnell and P. C. Kutzko, Semisimple types in GL_n, Compositio Math. **119** (1999), 53-97.

[7] H. Carayol, Représentations cuspidales du groupes linéaire, Ann. Scient. Ec. Norm. Sup. (4) **17** (1984), 191-225.

[8] L. Corwin, A. Moy and P. J. Sally Jr., Degrees and formal degrees for division algebras and GL_n over a p-adic field, Pacific J. Math. **141** (1990), 21-45.

[9] K. Kariyama, On types for unramified p-adic unitary groups, Canad. J. Math. Vol. **60** (2008), No. 3, 1067-1107.

[10] K. Kariyama, On the Hecke algebra of a type for unramified p-adic unitary groups, J. Algebra **323** (2010), 2624-2637.

[11] K. Kariyama and M. Miyauchi, Hecke algebras of self-dual simple types for p-adic classical groups, Preprint 2010 submitted.

[12] I. G. Macdonald, The Poincaré series of a Coxeter group, Math. Annalen **199** (1972), 161-174.

[13] A. J. Silberger and E.-W. Zink, The formal degree of discrete series representations of central simple algebras over p-adic fields, Max-Plank-Institut für Math. MPI 96-154 (1996).

[14] S. Stevens, Semisimple characters for p-adic classical groups, Duke Math. J. **127** no.1 (2005), 123-173.

[15] S. Stevens, The supercuspidal representations of p-adic classical groups, Invent. Math. **172** (2) (2008), 289-352.

[16] J.-W. Waldspurger, Algèbres de Hecke et induites de représentations cuspidales, pour $GL(N)$, J. reine angew. Math. **370** (1986), 127-191.

[17] A. Zelvinsky, Induced representations of reductive p-adic groups II. On irreducible representations of $GL(n)$, Ann. Sci. École Norm. Sup. **13** (1980), 165-210.

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