数理解析研究所講究録
第 1722 巻 2010 年 126-145

Spherical functions on $U(2n)/(U(n)\times U(n))$ and hermitian Siegel series

Yumiko Hironaka
Department of Mathematics,
Faculty of Education and Integrated Sciences, Waseda University
Nishi-Waseda, Tokyo, 169-8050, JAPAN

§0 Introduction

For each nondegenerate hermitian matrix T of size n with respect to an unramified quadratic extension k'/k of non-archimedian local fileds of characteristic 0, we consider the space X_T which is equivalent to $U(2n)/(U(n)\times U(n))$ over the algebraic closure of k and study spherical functions on X_T.

In §1, we construct X_T which is an homogeneous space of $G=U(H_n)$ with stabilizer isomorphic to $U(T)\times U(T)$ over k', and define the spherical function $\omega_T(x;z)$ on X_T $(x\in X_T,\ z\in\mathbb{C}^n)$, which means that $\omega_T(x;z)$ is K-invariant and a common eigenfunction for the action of the Hecke algebra $\mathcal{H}(G,K)$, K being the maximal compact subgroup of G. By a general theory, $\omega_T(x;z)$ is continued to a rational function on $q^{z_1},\ldots,q^{z_n}$, where q is the cardinality of the residue class field of k.

The Weyl group W of G acts on $z\in\mathbb{C}^n$ via rational characters of the Borel group of G, and we show functional equations with respect to W and locations of possible poles and zeros of $\omega_T(x;z)$ by giving an explicit rational function $G(z)$ of $q^{z_1},\ldots,q^{z_n}$ for which $G(z)\cdot\omega_T(x;z)$ is holomorphic in $z\in\mathbb{C}^n$ and W-invariant, in §2.

Using the functional equations, we give an explicit expression of $\omega_T(x;z)$ at many points in X_T in §3, define the spherical Fourier transform on the Schwartz space $\mathcal{S}(K\backslash X_T)$ and show the image is a free $\mathcal{H}(G,K)$-module of rank 2^{n-1} in §4. In §5, as an application, we consider hermitian Siegel series $b_\pi(T;t)$ and prove their functional equations by use of results in §2.

2010 Mathematics Subject Classification: Primary 11F85; secondly 11E95, 11F70, 22E50.
Key words and phrases: spherical functions, unitary groups, hermitian Siegel series.
E-mail: hironaka@waseda.jp
This research is partially supported by Grant-in-Aid for scientific Research (C):20540029.
A full version of this paper will appear elsewhere.

§1 Spaces $\mathfrak{X}_T$ and X_T, and spherical functions $\omega_T(x;s)$

Let k'/k be an unramified quadratic extension of p-adic fields with involution $*$, and for each $A = (a_{ij}) \in M_{mn}(k')$, we denote by A^* the matrix $(a_{ji}{}^*) \in M_{nm}(k')$. We fix a unit $\epsilon \in \mathcal{O}_k^\times$ such that $k' = k(\sqrt{\epsilon})$ and $\varepsilon - 1 \in 4\mathcal{O}_k^\times$ (cf. [Om], 63.3 and 63.4), and set

$$\xi = \frac{1+\sqrt{\epsilon}}{2}. \tag{1.1}$$

Then $\{1, \ \xi\}$ forms an $\mathcal{O}_k$-basis for $\mathcal{O}_{k'}$, and $\{\alpha \in \mathcal{O}_{k'} \mid \alpha^* = -\alpha\} = \sqrt{\epsilon}\mathcal{O}_k$. We fix a prime element π of k, and denote by $v_\pi(\ )$ the additive value on k, by $|\ |$ the normalized absolute value on $k^\times$ with $|\pi|^{-1} = q$ being the cardinality of the residue class field of k.

We set

$$\mathcal{H}_m = \{ A \in M_m(k') \mid A^* = A\}, \quad \mathcal{H}_m^{nd} = \mathcal{H}_m \cap GL_m(k').$$

For $A \in \mathcal{H}_m$ and $X \in M_{mn}(k')$, we write

$$A[X] = X^* \cdot A = X^*AX \in \mathcal{H}_n,$$

and define the unitary group

$$U(A) = \{g \in GL_m(k') \mid A[g] = A\}.$$

In particular we set

$$G = U(H_n) \quad \text{with } H_n = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}, \qquad U(m) = U(1_m).$$

For $T \in \mathcal{H}_n^{nd}$, we set

$$\begin{aligned} \mathfrak{X}_T &= \{x \in M_{2n,n}(k') \mid H_n[x] = T\} \ni x_T = \begin{pmatrix} \xi T \\ 1_n \end{pmatrix}, \\ X_T &= \mathfrak{X}_T / U(T). \end{aligned} \tag{1.2}$$

The group G acts on $\mathfrak{X}_T$, as well as on X_T, through left multiplication, which is transitive by Witt's theorem for hermitian matrices (cf. [Sch], Ch.7, §9).

Lemma 1.1 *The stabilizer G_0 of G at $x_T U(T) \in X_T$ is isomorphic to $U(T) \times U(T)$:*

$$U(T) \times U(T) \xrightarrow{\sim} G_0, \ (h_1, h_2) \longmapsto \widetilde{T}^{-1} \begin{pmatrix} h_1^{*-1} & 0 \\ 0 & h_2^{*-1} \end{pmatrix} \widetilde{T},$$

where

$$\widetilde{T} = \begin{pmatrix} 1_n & \xi^* T \\ 1_n & -\xi T \end{pmatrix} \in GL_{2n}(k').$$

We fix the Borel subgroup B of G as

$$B = \left\{ \begin{pmatrix} b & 0 \\ 0 & b^{*-1} \end{pmatrix} \begin{pmatrix} 1_n & a \\ 0 & 1_n \end{pmatrix} \;\middle|\; \begin{array}{l} b \text{ is upper triangular of size } n, \\ a + a^* = 0 \end{array} \right\}. \tag{1.3}$$

For each element $x \in \mathfrak{X}_T$, we denote by x_2 the lower half n by n block of x. We define relative B-invariants on $\mathfrak{X}_T$ by

$$f_{T,i}(x) = d_i(x_2 \cdot T^{-1}) = d_i(x_2 T^{-1} x_2^*), \quad 1 \le i \le n, \tag{1.4}$$

where $d_i(y)$ is the determinant of the upper left i by i block of a matrix y. It is easy to see, for $b \in B$,

$$f_{T,i}(bx) = \psi_i(b) f_{T,i}(x), \quad \psi_i(b) = \prod_{j=1}^{i} N(b_j)^{-1}, \tag{1.5}$$

where b_j is the j-th diagonal component of b and $N = N_{k'/k}$. Hence $f_{T,i}(x)$, $1 \le i \le n$ are relative B-invariants on $\mathfrak{X}_T$ associated with the rational characters ψ_i of B, and we may regard them as relative B-invariants on X_T, since $f_{T,i}(xh) = f_{T,i}(x)$ for any $h \in U(T)$. We set

$$\mathfrak{X}_T^{op} = \{ x \in X_T \mid f_{T,i}(x) \neq 0,\ 1 \le i \le n \}, \quad X_T^{op} = \mathfrak{X}_T^{op}/U(T). \tag{1.6}$$

Remark 1.2 Though we may realize above objects as the sets of k-rational points of algebraic sets defined over k and develop the arguments, we take down to earth way for simplicity of notations. We only note here that X_T^{op} (resp. $\mathfrak{X}_T^{op}$) becomes a Zariski open B-orbit in X_T (resp. $B \times U(T)$-orbit in $\mathfrak{X}_T^{op}$) over the algebraic closure of k.

We introduce a spherical function $\omega_T(x; s)$ on $\mathfrak{X}_T$ as well as on $X_T = \mathfrak{X}_T/U(T)$. For $x \in \mathfrak{X}_T$ and $s \in \mathbb{C}^n$, set

$$\omega_T(x; s) = \omega_T^{(n)}(x; s) = \int_K |f_T(kx)|^{s+\varepsilon}\, dk, \tag{1.7}$$

where $K = G \cap GL_{2n}(\mathcal{O}_{k'})$, dk is the normalized Haar measure on K and k runs over the set $\{ k \in K \mid kx \in \mathfrak{X}^{op} \}$,

$$\varepsilon = \varepsilon_0 + \left(\frac{\pi\sqrt{-1}}{\log q}, \ldots, \frac{\pi\sqrt{-1}}{\log q}\right), \quad \varepsilon_0 = (-1, \ldots, -1, -\frac{1}{2}) \in \mathbb{C}^n,$$

$$|f_T(x)|^s = \prod_{i=1}^{n} |f_{T,i}(x)|^{s_i}.$$

The right hand side of (1.7) is absolutely convergent if $\mathrm{Re}(s_i) \ge 1$, $1 \le i \le n-1$, and $\mathrm{Re}(s_n) \ge \frac{1}{2}$, and continued to a rational function of $q^{s_1}, \ldots, q^{s_n}$. We note here that

$$|\psi(p)|^{\varepsilon_0} \left(= \prod_{i=1}^{n} |\psi_i(p)|^{\varepsilon_{0,i}} \right) = \delta^{\frac{1}{2}}(p),$$

where δ is the modulus character on B (i.e., $d(pp') = \delta(p')^{-1}d(p)$).

We denote by $\mathcal{C}^\infty(K\backslash X_T)$ the space of left K-invariant functions on X_T, which can be identified with the space $\mathcal{C}^\infty(K\backslash \mathfrak{X}_T/U(T))$ of left K-invariant right $U(T)$-invariant functions on $\mathfrak{X}_T$. The function $\omega_T(x; z)$ can be regarded as a function in $\mathcal{C}^\infty(K\backslash X_T)$ and becomes a common eigenfunction for the action of the Hecke algebra $\mathcal{H}(G, K)$ (cf. [H2] §1, or [H4] §1). In detail, the Hecke algebra $\mathcal{H}(G, K)$ is the commutative $\mathbb{C}$-algebra consisting of compactly supported two-sided K-invariant functions on G, acting on $\mathcal{C}^\infty(K\backslash X_T)$ by the convolution product

$$(\phi * \Psi)(x) = \int_G \phi(g)\Psi(g^{-1}x)dg, \quad (\phi \in \mathcal{H}(G, K),\ \Psi \in \mathcal{C}^\infty(K\backslash X_T)), \tag{1.8}$$

and we see

$$(\phi * \omega_T(\ ; s))\,(x) = \lambda_s(\phi)\omega_T(x; s), \quad (\phi \in \mathcal{H}(G, K)) \tag{1.9}$$

where λ_s is the $\mathbb{C}$-algebra homomorphism defined by

$$\begin{aligned} \lambda_s : \mathcal{H}(G, K) &\longrightarrow \mathbb{C}(q^{s_1}, \ldots, q^{s_n}), \\ \phi &\longmapsto \int_B \phi(p)\, |\psi(p)|^{-s+\varepsilon}\, dp, \qquad \left(|\psi(p)|^{-s+\varepsilon} = |\psi(p)|^{-s+\varepsilon_0}\right), \end{aligned}$$

with dp being the left invariant measure on B normalized by $\int_{K\cap B} dp = 1$.

We introduce a new variable z which is related to s by

$$s_i = -z_i + z_{i+1} \quad (1 \le i \le n-1), \quad s_n = -z_n \tag{1.10}$$

and write $\omega_T(x; z) = \omega_T(x; s)$. The Weyl group W of G relative to the maximal k-split torus in B acts on rational characters of B as usual (i.e., $\sigma(\psi)(b) = \psi(n_\sigma^{-1} b n_\sigma)$ by taking a representative n_σ of σ), so W acts on $z \in \mathbb{C}^n$ and on $s \in \mathbb{C}^n$ as well. We will determine the functional equations of $\omega_T(x; s)$ with respect to this Weyl group action. The group W is isomorphic to $S_n \ltimes C_2^n$, S_n acts on z by permutation of indices, and W is generated by S_n and $\tau : (z_1, \ldots, z_n) \longmapsto (z_1, \ldots, z_{n-1}, -z_n)$. Keeping the relation (1.10), we also write $\lambda_z(\phi) = \lambda_s(\phi)$; then λ_z gives a $\mathbb{C}$-algebra isomorphism (Satake isomorphism)

$$\begin{aligned} \lambda_z \ :\ \mathcal{H}(G, K) &\xrightarrow{\ \sim\ } \mathbb{C}[q^{\pm 2z_1}, \ldots, q^{\pm 2z_n}]^W, \\ \phi &\longmapsto \int_B \phi(p) \prod_{i=1}^{n} |N(p_i)|^{-z_i}\, \delta^{\frac{1}{2}}(p) dp, \end{aligned} \tag{1.11}$$

where p_i is the i-th diagonal component of $p \in B$.

Proposition 1.3 *Set $\mathcal{U} = (\mathbb{Z}/2\mathbb{Z})^{n-1}$ and*

$$\widetilde{u} = (u_1 \frac{\pi\sqrt{-1}}{\log q}, \ldots, u_{n-1}\frac{\pi\sqrt{-1}}{\log q}, 0) \in \mathbb{C}^n, \qquad u = (u_1, \ldots . u_{n-1}) \in \mathcal{U}.$$

Then $\omega_T(x; z + \widetilde{u})$, $u \in \mathcal{U}$, are linearly independent for generic $z \in \mathbb{C}^n$ and correspond to the same eigenvalue through $\lambda_z : \mathcal{H}(G, K) \longrightarrow \mathbb{C}$.

Proof. The set $\mathfrak{X}_T^{op}$ is decomposed into the disjoint union of B-orbits as follows:

$$\mathfrak{X}_T^{op} = \bigsqcup_{u \in \mathcal{U}} \mathfrak{X}_{T,u},$$
$$\mathfrak{X}_{T,u} = \{x \in \mathfrak{X}_T^{op} \mid v_\pi(f_{T,i}(x)) \equiv u_1 + \cdots + u_i \pmod 2,\ 1 \le i \le n-1\}.$$

We consider finer spherical functions

$$\omega_{T,u}(x;s) = \int_K |f_T(kx)|_u^{s+\varepsilon}\, dk, \quad |f_T(y)|_u^{s+\varepsilon} = \begin{cases} |f_T(y)|^{s+\varepsilon} & \text{if } y \in \mathfrak{X}_{T,u} \\ 0 & \text{otherwise}, \end{cases} \tag{1.12}$$

then $\{\omega_{T,u}(x;s) \mid u \in \mathcal{U}\}$ are linearly independent for generic s associated with the same λ_s. For each character χ of $\mathcal{U}$, we have

$$\sum_{u \in \mathcal{U}} \chi(u)\omega_{T,u}(x;s) = \omega_T(x; s + \widetilde{v}),$$

for some $v \in \mathcal{U}$, and the result follows from this. ∎

We note here the relation between $\omega_T(x;s)$ and $\omega_{T'}(y;s)$ when T and T' are equivalent under the action of $GL_n(k')$, which is easy to see.

Proposition 1.4 *For $T \in \mathcal{H}_n^{nd}$ and $h \in GL_n(k')$, we set $T' = T[h]\ (= h^*Th)$. Then*

$$\mathfrak{X}_{T'} = (\mathfrak{X}_T)\, h, \quad X_{T'} = \mathfrak{X}_T h/U(T') \quad \text{and} \quad f_{T',i}(xh) = f_{T,i}(x) \quad (x \in \mathfrak{X}_T),$$

and

$$\omega_{T'}(xh;s) = \omega_T(x;s), \qquad (x \in \mathfrak{X}_T).$$

By use of some results on spherical functions on the space $\mathcal{H}_n^{nd}$ of hermitian forms, we obtain the following.

Theorem 1.5 *For any $T \in \mathcal{H}_n^{nd}$, the function*

$$\prod_{1 \le i < j \le n} \frac{q^{z_j} + q^{z_i}}{q^{z_j} - q^{z_i - 1}} \times \omega_T(x;z)$$

is holomorphic for any z in $\mathbb{C}^n$ and S_n-invariant. In particular it is an element in $\mathbb{C}[q^{\pm z_1}, \ldots, q^{\pm z_n}]^{S_n}$.

Outline of a proof. By the embedding

$$K_0 = GL_n(\mathcal{O}_{k'}) \longrightarrow K, \quad h \longmapsto \widetilde{h} = \begin{pmatrix} h^{*-1} & 0 \\ 0 & h \end{pmatrix},$$

we obtain

$$\omega_T(x;z) = \int_{K_0} dh \int_K \left| f_T(\widetilde{h}kx) \right|^{s+\varepsilon} dk = \int_K \zeta^{(n)}(D(kx);s)dk.$$

Here $D(kx) = (kx)_2 \cdot T^{-1} \in \mathcal{H}_n$, $\zeta^{(n)}(y;s)$ is a spherical function on $\mathcal{H}_n^{nd}$ defined by

$$\zeta^{(n)}(y;s) = \int_{K_0} \prod_{i=1}^{n} |d_i(h\cdot y)|^{s_i+\varepsilon_i}\, dh, \quad (h\cdot y = hyh^*),$$

and we have already known (cf. [H1], or [H3]) that

$$\prod_{1\le i<j\le n} \frac{q^{z_j}+q^{z_i}}{q^{z_j}-q^{z_i-1}} \times \zeta^{(n)}(y;s) \in \mathbb{C}[q^{\pm z_1},\dots,q^{\pm z_n}]^{S_n},$$

the result follows from this. ∎

Remark 1.6 For the transposition $\tau_i = (i\ i+1) \in W$, $1 \le i \le n-1$, the following functional equation holds by Theorem 1.5

$$\omega_T(x;z) = \frac{1-q^{z_i-z_{i+1}-1}}{q^{z_i-z_{i+1}}-q^{-1}} \times \omega_T(x;\tau_i(z)), \quad 1\le i\le n-1. \tag{1.13}$$

On the other hand, one may obtain (1.13) directly in the similar way to the case of τ in § 2, where the sufficient condition in [H4]-§3 for having a functional equation with respect to τ_i is satisfied and the Gamma factor in (1.13) is essentially the same to that of the zeta function of prehomogeneous vector space $(U \times GL_1(k'), (k')^2)$, where $U \cong U(2)$ or $U(\begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix})$. Then Theorem 1.5 follows from (1.13).

§2 Functional equations of $\omega_T(x;z)$

We calculate the functional equation for $\tau \in W$, and give the functional equations with respect to the whole W.

2.1.

Theorem 2.1 *For any $T \in \mathcal{H}_n^{nd}$, the spherical function satisfies*

$$\omega_T(x;z) = \omega_T(x;\tau(z)),$$

where $\tau(z_1,\dots,z_n) = (z_1,\dots,z_{n-1},-z_n)$.

For $n=1$, we have the following by a direct calculation, where we set $K_1 = U(H_1) \cap GL_2(\mathcal{O}_{k'})$.

Proposition 2.2 (i) *The set*

$$\left\{ x_e = \begin{pmatrix} \pi^e & \\ & \xi\pi^{t-e} \end{pmatrix} \;\middle|\; e \in \mathbb{Z},\ 2e \le t \right\}, \quad \left(\xi = \frac{1+\sqrt{\epsilon}}{2} \right)$$

forms a set of complete representatives of $K_1 \backslash \mathfrak{X}_T$ *for* $T = \pi^t$.
(ii) *For* $x_e \in \mathfrak{X}_T$ *with* $T = \pi^t$ *as in (i), one has*

$$\omega_T^{(1)}(x_e; s) = \frac{(-1)^t q^{e-\frac{1}{2}t}}{1+q^{-1}} \times \frac{q^{(t-2e+1)s}(1-q^{-2s-1}) - q^{-(t-2e+1)s}(1-q^{2s-1})}{q^s - q^{-s}}.$$

(iii) *For any* $T \in \mathcal{H}_1^{nd}$, $\omega_T^{(1)}(x;s)$ *is holomorphic for all* $s \in \mathbb{C}$ *and satisfies the functional equation*

$$\omega_T^{(1)}(x;s) = \omega_T^{(1)}(x;-s).$$

Until the end of this subsection we assume $n \ge 2$. The parabolic subgroup P attached to τ, in the sense of [Bo] §21.11, is given as follows:

$$P = B \cup B w_\tau B$$
$$= \left\{ \left(\begin{array}{cc|cc} q & & & \\ & a & & b \\ \hline & & q^{*-1} & \\ & c & & d \end{array} \right) \left(\begin{array}{cc|cc} 1_{n-1} & \alpha & & \\ & 1 & & \\ \hline & & 1_{n-1} & \\ & & -\alpha^* & 1 \end{array} \right) \left(\begin{array}{c|cc} 1_n & \gamma & \beta \\ & -\beta^* & 0 \\ \hline & 1_n & \end{array} \right) \in G \;\middle|\; \begin{array}{l} q \text{ is upper triangular in } GL_{n-1}(k'), \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1,1),\ \alpha,\beta \in M_{n-1,1}(k'), \\ \gamma \in M_{n-1}(k'),\ \gamma + \gamma^* = 0 \end{array} \right\}, \tag{2.1}$$

where each empty place in the above expression means zero-entry.

Since it suffices to show Theorem 2.1 for diagonal T's (cf. Proposition 1.4), we fix a diagonal $T \in \mathcal{H}_n^{nd}$ and write $f_i(x) = f_{T,i}(x)$ for simplicity of notations. We consider the following action of $\widetilde{P} = P \times GL_1$ on $\widetilde{\mathfrak{X}}_T = \mathfrak{X}_T \times V$ with $V = M_{21}(k')$:

$$(p,r) \star (x,v) = (px, \rho(p) v r^{-1}), \qquad (p,r) \in \widetilde{P},\ (x,v) \in \widetilde{\mathfrak{X}}_T,$$

where $\rho(p) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by the decomposition of $p \in P$ as in (2.1). We define

$$g(x,v) = \det\left[\left(\begin{array}{c|c} 1_{n-1} & \\ \hline & {}^t v \end{array} \right) \begin{pmatrix} x_2 \\ -y \end{pmatrix} \cdot T^{-1} \right], \quad (x,v) \in \widetilde{\mathfrak{X}}_T, \tag{2.2}$$

where x_2 is the lower half n by n block of x (the same before) and y is the n-th row of x. Then we have

Lemma 2.3 (i) $g(x,v)$ *is a relative* $\widetilde{P}$*-invariant on* $\widetilde{\mathfrak{X}}_T$ *associated with character* $\widetilde{\psi}$*:*

$$\widetilde{\psi}(p,r) = \psi_{n-1}(p) N(r)^{-1}, \quad (p,r) \in \widetilde{P} = P \times GL_1,$$

where ψ_{n-1} *is given in (1.5) and well-defined on* P, *and satisfies*

$$g(x, v_0) = f_n(x), \qquad v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in V$$

(ii) $g(x,v)$ *is expressed as*

$$g(x, v) = D(x)[v], \tag{2.3}$$

with some hermitian matrix

$$D(x) = \begin{pmatrix} a(x) & \beta(x) \\ \beta(x)^* & d(x) \end{pmatrix} \qquad (a(x), d(x) \in k,\ \beta(x) \in k'), \tag{2.4}$$

such that $\det(D(x)) = 0$ *and* $\mathrm{Tr}(\beta(x)) = -f_{n-1}(x)$, *where* Tr *is the trace* $\mathrm{Tr}_{k'/k}$.

For $A \in \mathcal{H}_2$ and $s \in \mathbb{C}$, we define

$$\zeta_{K_1}(A; s) = \int_{K_1} |d_1(h \cdot A)|^{s - \frac{1}{2}}\, dh,$$

where dh is the normalized Haar measure on K_1, which is absolutely convergent if $\mathrm{Re}(s) \geq \frac{1}{2}$ and continued to the whole $\mathbb{C}$. Then we obtain

Lemma 2.4 *Assume* $x \in \mathfrak{X}_T^{op}$ *and* $D(x)$ *is given by (2.3). Set* $m = \min\{v_\pi(a(x)),\ v_\pi(d(x))\}$ *and* $t = v_\pi(\beta(x)) - m$ *for the expression of* $D(x)$ *as in (2.4). Then* $t \geq 0$ *and*

$$\zeta_{K_1}(D(x); s) = \frac{q^{\frac{m}{2}}}{1 + q^{-1}} \cdot |f_{n-1}(x)|^s \cdot \frac{q^{(t+1)s}(1 - q^{-2s-1}) - q^{-(t+1)s}(1 - q^{2s-1})}{q^s - q^{-s}}.$$

In particular, one has the functional equation

$$\zeta_{K_1}(D(x); s) = |f_{n-1}(x)|^{2s} \cdot \zeta_{K_1}(D(x); -s). \tag{2.5}$$

We give a sketch of a proof of Theorem 2.1. By the embedding

$$K_1 \longrightarrow K = K_n, \quad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \widetilde{h} = \left(\begin{array}{cc|cc} 1_{n-1} & & & \\ & a & & b \\ \hline & & 1_{n-1} & \\ & c & & d \end{array}\right),$$

we have

$$\begin{aligned} \omega_T(x; s) &= \int_{K_1} dh \int_K |f(kx)|^{s+\varepsilon}\, dk \\ &= \int_K \chi_\pi\Big(\prod_{i<n} f_i(kx)\Big) \prod_{i<n} |f_i(kx)|^{s_i - 1}\, \zeta_{K_1}\Big(D(kx); s_n + \tfrac{\pi\sqrt{-1}}{\log q}\Big) dk. \end{aligned}$$

By Lemma 2.4, we see

$$\omega_T(x; s) = \omega_T(x; s_1, \dots, s_{n-2}, s_{n-1} + 2s_n, -s_n),$$

and, in variable z, we have

$$\omega_T(x; z) = \omega_T(x; \tau(z)), \qquad \tau(z) = (z_1, \dots, z_{n-1}, -z_n).$$

∎

2.2. In order to describe functional equations of $\omega_T(x; z)$ with respect to W, we prepare some notations. We denote by Σ the set of roots of G with respect to the k-split torus of G contained in B and by Σ^+ the set of positive roots with respect to B. We may understand Σ as a subset in $\mathbb{Z}^n$, and set

$$\Sigma^+ = \Sigma_s^+ \cup \Sigma_\ell^+, \quad \Sigma_s^+ = \{e_i - e_j,\ e_i + e_j \mid 1 \le i < j \le n\}, \ \Sigma_\ell^+ = \{2e_i \mid 1 \le i \le n\},$$

where e_i is the i-th unit vector in $\mathbb{Z}^n$, $1 \le i \le n$. The set

$$\Delta = \{\tau_i = (i\ i+1) \in S_n \mid 1 \le i \le n-1\} \cup \{\tau\},$$

is associated with the set of simple roots and generates W. For each $\sigma \in W$, we set

$$\Sigma_s^+(\sigma) = \left\{\alpha \in \Sigma_s^+ \mid -\sigma(\alpha) \in \Sigma^+\right\}.$$

The pairing on $\mathbb{Z}^n \times \mathbb{C}^n$

$$\langle t,\, z\rangle = \sum_{i=1}^n t_i z_i, \qquad (t \in \mathbb{Z}^n,\ z \in \mathbb{C}^n),$$

is W-invariant on $\Sigma \times \mathbb{C}^n$, i.e.,

$$\langle \alpha,\, z\rangle = \langle \sigma(\alpha),\, \sigma(z)\rangle, \qquad (\alpha \in \Sigma,\ z \in \mathbb{C}^n,\ \sigma \in W). \tag{2.6}$$

Theorem 2.5 *For $T \in \mathcal{H}_n^{nd}$ and $\sigma \in W$, the spherical function $\omega_T(x; z)$ satisfies the following functional equation*

$$\omega_T(x; z) = \Gamma_\sigma(z) \cdot \omega_T(x; \sigma(z)), \tag{2.7}$$

where

$$\Gamma_\sigma(z) = \prod_{\alpha \in \Sigma_s^+(\sigma)} \frac{1 - q^{\langle \alpha, z\rangle - 1}}{q^{\langle \alpha, z\rangle} - q^{-1}}.$$

In particular, the Gamma factor $\Gamma_\sigma(z)$ does not depend on x nor T.

Outline of a proof. We determine $\Gamma_\sigma(z)$ by the equation (2.7), which is a rational function of $q^{z_1},\ldots,q^{z_n}$. We set for $\alpha \in \Sigma$ and $z \in \mathbb{C}^n$

$$f_\alpha(\langle\alpha,\, z\rangle) = \begin{cases} 1 & \text{if } \alpha = \pm 2e_i,\ (1 \le i \le n) \\ \dfrac{1 - q^{\langle\alpha,\, z\rangle - 1}}{q^{\langle\alpha,\, z\rangle} - q^{-1}} & \text{otherwise} \end{cases}$$

We see $\Gamma_\sigma(z)$ for $\sigma \in \Delta$ by (1.13) and Theorem 2.1. For general $\sigma \in W$, we obtain the result by cocycle relations of $\Gamma_\sigma(z)$ and W-invariancy of the inner product (2.6). ∎

We will use the following explicit value $\Gamma_\rho(z)$ in §5.

Corollary 2.6 *Set $\rho \in W$ by*

$$\rho(z_1,\ldots,z_n) = (-z_n, -z_{n-1},\ldots,-z_1).$$

Then

$$\Gamma_\rho(z) = \prod_{1\le i<j\le n} \frac{1-q^{z_i+z_j-1}}{q^{z_i+z_j}-q^{-1}}.$$

Remark 2.7 The above ρ gives the functional equation of the hermitian Siegel series (cf. §5), and it is interesting that such ρ corresponds to the unique automorphism of the extended Dynkin diagram of the root system of type (C_n), which was pointed out by Y. Komori.

By Theorem 1.5 and Theorem 2.5, we obtain the following theorem.

Theorem 2.8 *Set*

$$G(z) = \prod_{\alpha\in\Sigma_s^+} \frac{1+q^{\langle\alpha,\, z\rangle}}{1-q^{\langle\alpha,\, z\rangle-1}}.$$

Then, for any $T \in \mathcal{H}_n^{nd}$, the function $G(z)\cdot\omega_T(x;z)$ is holomorphic for all z in $\mathbb{C}^n$ and W-invariant. In particular it is an element in $\mathbb{C}[q^{\pm z_1},\ldots,q^{\pm z_n}]^W$.

§3 Explicit formula for $\omega_T(x; z)$

3.1. Set

$$\Lambda_n^+ = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}, \tag{3.1}$$

and, for each $\lambda \in \Lambda_n^+$,

$$\pi^\lambda = Diag(\pi^{\lambda_1}, \ldots, \pi^{\lambda_n}) \in \mathcal{H}_n^{nd}, \qquad x_\lambda = \begin{pmatrix} \xi\pi^\lambda \\ 1_n \end{pmatrix} \in \mathfrak{X}_{\pi^\lambda},$$
$$\omega_\lambda(x; z) = \omega_T(x; z) \quad \text{for } T = \pi^\lambda. \tag{3.2}$$

Theorem 3.1 *For $\lambda \in \Lambda_n^+$, one has the following explicit expression:*

$$\omega_\lambda(x_\lambda; z) = \frac{(-1)^{\sum_i \lambda_i(n-i+1)} q^{-\sum_i \lambda_i(n-i+\frac{1}{2})}(1-q^{-2})^n}{\prod_{i=1}^{2n}(1-(-q^{-1})^i)} \times \frac{1}{G(z)} \times \sum_{\sigma \in W} q^{-\langle \lambda, \sigma(z) \rangle} H(\sigma(z)),$$

where $G(z)$ is given in Theorem 2.8 and

$$H(z) = \prod_{\alpha \in \Sigma_s^+} \frac{1+q^{\langle \alpha, z \rangle - 1}}{1-q^{\langle \alpha, z \rangle}} \prod_{\alpha \in \Sigma_\ell^+} \frac{1-q^{\langle \alpha, z \rangle - 1}}{1-q^{\langle \alpha, z \rangle}}.$$

Remark 3.2 By Theorem 2.8, the main part

$$H_\lambda(z) = \sum_{\sigma \in W} \sigma\left(q^{-\langle \lambda, z \rangle} H(z)\right) = \sum_{\sigma \in W} q^{-\langle \lambda, \sigma(z) \rangle} H(\sigma(z))$$

of $\omega_\lambda(x_\lambda; z)$ belongs to $\mathbb{C}[q^{\pm z_1}, \ldots, q^{\pm z_n}]^W$. Further we see in a standard way that the set $\{H_\lambda(z) \mid \lambda \in \Lambda_n^+\}$ forms its $\mathbb{C}$-basis. On the other hand, $H_\lambda(z)$ is a special case of P_λ (up to a scalar factor) introduced by Macdonald [Mac] §10 in a generous context of orthogonal polynomials associated with root systems.

We will prove the above theorem by using a general expression formula (Theorem 2.6 in [H4] , or in [H2]) of spherical functions on homogeneous spaces, which is based on functional equations of finer spherical functions and some data depending only on the group G. We explain about the proof in the next subsection.

By Theorem 3.1 and Proposition 1.4, we may have the explicit formula of $\omega_T(x; s)$ at many points. For $T \in \mathcal{H}_n^{nd}$ and $\lambda \in \Lambda_n^+$, it is known that T and π^λ belong to the same $GL_n(k')$-orbit in $\mathcal{H}_n^{nd}$ if and only if

$$v_\pi(\det(T)) \equiv |\lambda| \pmod 2,$$

where $|\lambda| = \sum_{i=1}^n \lambda_i$. Thus we obtain

Theorem 3.3 *Let $T \in \mathcal{H}_n^{nd}$ and $\lambda \in \Lambda_n^+$ and assume that $v_\pi(\det(T)) \equiv |\lambda| \pmod 2$. Taking $h_\lambda \in GL_n(k')$ for which $\pi^\lambda[h_\lambda] = T$, one has $x_\lambda h_\lambda \in \mathfrak{X}_T$ and*

$$\begin{aligned} \omega_T(x_\lambda h_\lambda; z) &= \omega_\lambda(x_\lambda; z) \\ &= \frac{(-1)^{\sum_i \lambda_i(n-i+1)} q^{-\sum_i \lambda_i(n-i+\frac{1}{2})}(1-q^{-2})^n}{\prod_{i=1}^{2n}(1-(-q^{-1})^i)} \cdot \frac{1}{G(z)} \cdot \sum_{\sigma \in W} \sigma\left(q^{-\langle \lambda, z\rangle} H(z)\right). \end{aligned}$$

Further, each such a λ gives a different K-orbit

$$Kx_\lambda h_\lambda U(T) \quad in \quad K\backslash X_T \left(= K\backslash \mathfrak{X}_T/U(T)\right).$$

3.2. In order to apply Theorem 2.6 in [H4], we need to check the assumptions there. Let $\mathbb{G}$ be a connected reductive linear algebraic group and $\mathbb{X}$ be an affine algebraic variety which is $\mathbb{G}$-homogeneous, where everything is assumed to be defined over a p-adic field k. For an algebraic set, we use the same ordinary letter to indicate the set of k-rational points. Let K be a maximal compact open subgroup of G, and $\mathbb{B}$ a minimal parabolic subgroup of $\mathbb{G}$ defined over k satisfying $G = KB = BK$. We denote by $\mathfrak{X}(\mathbb{B})$ the group of rational character of $\mathbb{B}$ defined over k and by $\mathfrak{X}_0(\mathbb{B})$ the subgroup consisting of those characters associated with some relative $\mathbb{B}$-invariant on $\mathbb{X}$ defined over k. In these situation, the assumptions are the following:

($A1$) $\mathbb{X}$ has only a finite number of $\mathbb{B}$-orbits.

($A2$) A basic set of relative $\mathbb{B}$-invariants on $\mathbb{X}$ defined over k can be taken by regular functions on $\mathbb{X}$.

($A3$) For $y \in \mathbb{X}$ not contained in the open orbit, there exists some ψ in $\mathfrak{X}_0(\mathbb{B})$ whose restriction to the identity component of the stabilizer $\mathbb{H}_y$ of $\mathbb{G}$ at y is not trivial.

($A4$) The rank of $\mathfrak{X}_0(\mathbb{B})$ coincides with that of $\mathfrak{X}(\mathbb{B})$.

In the present situation, as is noted in Remark 1.2, we may understand $\mathbb{G} = U(H_n)$ defined over k, $G = \mathbb{G}(k)$, $B = \mathbb{B}(k)$ for the Borel subgroup defined over k, and $X = X_T$ as the set of k-rational points of the affine algebraic variety $\mathbb{X} = \mathfrak{X}_T/U(T)$, and we recall that relative invariants $f_{T,i}(x)$ and the spherical function $\omega_T(x; s)$ can be regarded as functions on X_T.

It is easy to see the present $(\mathbb{X}, \mathbb{B})$ satisfies the conditions ($A1$), ($A2$) and ($A4$) (cf. Lemma 1.1, (1.4) and (1.5)), in particular, the unique Zariski open $\mathbb{B}$-orbit is given by $\mathbb{X}^{op} = \{x \in \mathbb{X} \mid f_{T,i}(x) \neq 0,\ 1 \leq i \leq n\}$ (cf. (1.6)).

First we give an outline of a proof of Theorem 3.1, admitting the condition ($A3$). By Theorem 2.5, we obtain vector-wise functional equations for finer spherical functions $\omega_{T,u}(x; z)$, $u \in \mathcal{U} = (\mathbb{Z}/2\mathbb{Z})^{n-1}$ (cf. (1.12))

$$(\omega_{T,u}(x; z))_{u\in\mathcal{U}} = A^{-1} \cdot G(\sigma, z) \cdot \sigma A \cdot (\omega_{T,u}(x; \sigma(z)))_{u\in\mathcal{U}}, \qquad \sigma \in W, \tag{3.3}$$

where

$$A = (\chi(u))_{\chi,u}, \quad \sigma A = (\sigma(\chi)(u))_{\chi,u} \in GL_{2^{n-1}}(\mathbb{Z}),$$

χ runs over characters of $\mathcal{U}$, $u \in \mathcal{U}$, and $G(\sigma, z)$ is the diagonal matrix of size 2^{n-1} whose (χ, χ)-component is $\Gamma_\sigma(z_\chi)$. Here $\Gamma_\sigma(z)$ is given in Theorem 2.5 and z_χ is determined by the identity

$$\sum_{u \in \mathcal{U}} \chi(u) \omega_{T,u}(x; z) = \omega_T(x; z_\chi).$$

We denote by U the Iwahori subgroup of K compatible with B, take the normalized Haar measure du on U, and set

$$\begin{aligned} \delta_u(x_\lambda, z) &= \int_U |f_T(ux_\lambda)|_u^{s+\varepsilon}\, du \\ &= \begin{cases} (-1)^{\sum_i \lambda_i(n-i+1)} q^{-\sum_i \lambda_i(n-i+\frac{1}{2})} q^{-<\lambda, z>} & \text{if } x_\lambda \in \mathfrak{X}_{T,u} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Applying Theorem 2.6 in [H4] to our present case, we obtain

$$(\omega_{T,u}(x_\lambda; z))_{u \in \mathcal{U}} = \frac{1}{Q} \sum_{\sigma \in W} \gamma(\sigma(z)) \left(A^{-1} \cdot G(\sigma, z) \cdot \sigma A\right) (\delta_u(x_\lambda, \sigma(z)))_{u \in \mathcal{U}}, \qquad (3.4)$$

where

$$\begin{aligned} Q &= \sum_{\sigma \in W} [U\sigma U : U]^{-1} = \prod_{i=1}^{2n} \left(1 - (-1)^i q^{-i}\right) \Big/ (1 - q^{-2})^n, \\ \gamma(z) &= \prod_{\alpha \in \Sigma_s^+} \frac{1 - q^{2\langle \alpha, z \rangle - 2}}{1 - q^{2\langle \alpha, z \rangle}} \cdot \prod_{\alpha \in \Sigma_\ell^+} \frac{1 - q^{\langle \alpha, z \rangle - 1}}{1 - q^{\langle \alpha, z \rangle}}. \end{aligned}$$

Since $\omega_T(x_\lambda; z) = \sum_{u \in \mathcal{U}} \mathbf{1}(u) \omega_u(x_\lambda; z)$, we obtain the explicit formula for $\omega_\lambda(x_\lambda; z)$ from (3.4). ∎

Now we explain about the condition $(A3)$. We consider the action of $G \times U(T)$ on $\mathfrak{X}_T$ by $(g, h) \circ x = gxh^{-1}$. Then, the stabilizer B_y of B at $yU(T) \in X_T$ coincides with the image $B_{(y)}$ of the projection to B of the stabilizer $(B \times U(T))_y$ at $y \in \mathfrak{X}_T$ to B. Hence, in our case, the condition $(A3)$ is equivalent to the following:

(C) : For each $y \in \mathfrak{X}_T$ not contained in $\mathfrak{X}_T^{op}$, there exists $\psi \in \mathfrak{X}(\mathbb{B})$ whose restriction to the identity component of $B_{(y)}$ is not trivial.

It suffices to prove the condition $(A3)$ (or (C)) over the algebraic closure $\overline{k}$ of k, hence we may assume that $T = 1_n$; for simplicity of notation, we write $f_i(x)$ instead of $f_{T,i}(x)$. Until the end of this subsection, we consider algebraic sets over $\overline{k}$, extend the involution $*$ on k' to $\overline{k}$, indicate it by $^-$, and write $\overline{x} = (\overline{x_{ij}}) \in M_{\ell m}(\overline{k})$ for $x = (x_{ij}) \in M_{\ell m}(\overline{k})$.

Then, our situation is the following:

$$\begin{aligned} &\mathfrak{X} = \mathfrak{X}_{1_n} = \{x \in M_{2n,n} \mid H_n[x] = 1_n\}, \\ &(U(H_n) \times U(1_n)) \times \mathfrak{X} \longrightarrow \mathfrak{X}, \quad ((g,h), x) \longmapsto (g,h) \circ x = gxh^{-1}, \end{aligned}$$

and B is the Borel subgroup of $U(H_n)$ (as in (1.3)). We introduce a $(GL_{2n} \times GL_n)$-set $\widetilde{\mathfrak{X}}$ as follows:

$$\widetilde{\mathfrak{X}} = \{ (x,y) \in M_{2n,n} \oplus M_{2n,n} \mid {}^t y H_n x = 1_n \} \tag{3.5}$$
$$(g,h) \star (x,y) = (gxh^{-1}, \dot{g} y {}^t h), \qquad ((g,h) \in GL_{2n} \times G_n,\ \dot{g} = H_n {}^t g^{-1} H_n),$$

and we write an element of $\widetilde{\mathfrak{X}}$ as $(x,y) = (\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix})$ with $x_i, y_i \in M_n$. We take the Borel subgroup P of GL_{2n} by

$$P = \left\{ \begin{pmatrix} p & r \\ 0 & q \end{pmatrix} \in GL_{2n} \,\middle|\, p,\ {}^t q \in B_n,\ r \in M_n \right\},$$

where B_n is the Borel subgroup of GL_n consisting of the upper triangular matrices. The involution $g \longmapsto \dot{g} = H_n {}^t g^{-1} H_n$ on GL_{2n} induces an involution on P :

$$\begin{pmatrix} p & r \\ 0 & q \end{pmatrix} \longmapsto \begin{pmatrix} {}^t q^{-1} & 0 \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} 1_n & -{}^t r \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ 0 & {}^t p^{-1} \end{pmatrix}. \tag{3.6}$$

The embedding $\iota : \mathfrak{X} \longmapsto \widetilde{\mathfrak{X}},\ x \longmapsto (x, \overline{x})$ is compatible with action, i.e., we have the commutative diagram

$$\begin{array}{ccccc} (U(H_n) \times U(1_n)) & \times & \mathfrak{X} & \xrightarrow{\circ} & \mathfrak{X} \\ \big\downarrow{\scriptstyle incl.} & & \big\downarrow{\scriptstyle\iota} & \circlearrowleft & \big\downarrow{\scriptstyle\iota} \\ (GL_{2n} \times GL_n) & \times & \widetilde{\mathfrak{X}} & \xrightarrow{\star} & \widetilde{\mathfrak{X}}. \end{array}$$

For $(x,y) \in \widetilde{\mathfrak{X}}$ and $p \in P$, set

$$\widetilde{f}_i(x,y) = d_i(x_2 {}^t y_2), \quad \widetilde{\psi}_i(p) = \prod_{1 \le j \le i} p_j^{-1} p_{n+j}, \quad (1 \le i \le n), \tag{3.7}$$

where p_j is the j-th diagonal component of p. Then $\widetilde{f}_i(x,y)$'s are basic relative P-invariants on $\widetilde{\mathfrak{X}}$ associated with characters $\widetilde{\psi}_i$, $\widetilde{f}_i(x,\overline{x}) = f_i(x)$ for $x \in \mathfrak{X}$, and $\widetilde{\psi}_i|_B = \psi_i$. We set

$$\mathcal{S} = \left\{ (x,y) \in \widetilde{\mathfrak{X}} \cap (P \times GL_n) \star \mathfrak{X} \,\middle|\, \prod_{i=1}^{n} \widetilde{f}_i(x,y) = 0 \right\}.$$

For $\alpha = (x,y) \in \widetilde{\mathfrak{X}}$, we denote by H_α the stabilizer of $P \times GL_n$ at α, and by P_α the identity component of the image of H_α by the projection to P. In order to prove the condition (C), it is sufficient to show the following:

$(\widetilde{C})$: For each $\alpha \in \mathcal{S}$, there exists some $\psi \in \langle \widetilde{\psi}_i \mid 1 \le i \le n \rangle$ whose restriction to P_α is not trivial.

We have only to consider $(\widetilde{C})$ for representatives under the action of $P \times GL_n$. In the following we consider the case $n \ge 2$, since $\mathfrak{X}_T = \mathfrak{X}_T^{op}$ for $n = 1$ and there is nothing to prove. We denote by $\delta_i(a) \in GL_n$ the diagonal matrix whose j-th entry is 1 except the i-th which is $a \in GL_1$. We show $(\widetilde{C})$ according to the type of $\alpha \in \mathcal{S}$.

The case $\alpha = (x, y) \in \mathcal{S}$ with $\det(x_2) \neq 0$: Under $(P \times GL_n)$-action, we may assume that

$$\alpha = (\begin{pmatrix} 0 \\ 1_n \end{pmatrix}, \begin{pmatrix} 1_n \\ h \end{pmatrix}),$$

where $h = 1_r \perp h_1$, $0 \leq r < n$, and h_1 is a hermitian matrix such that

the first row and column are zero, or
for some i, $(1 < i \leq n - r)$, each entry in the first row and column or in the i-th row and column is 0 except at $(1, i)$ or $(i, 1)$ which are 1.

Then H_α contains the following elements, according to the above type of h_1,

$$(\left(\begin{array}{c|c} \delta_{r+1}(a) & \\ \hline & 1_n \end{array}\right), 1_n) \quad \text{or} \quad (\left(\begin{array}{c|c} \delta_{r+1}(a) & \\ \hline & \delta_{r+i}(a) \end{array}\right), \delta_{r+i}(a)) \quad (a \in GL_1),$$

and we see $\widetilde{\psi}_{r+1} \not\equiv 1$ on P_α.

The case $\alpha = (x, y) \in \mathcal{S}$ with $\det(y_2) \neq 0$ is reduced to the case $\det(x_2) \neq 0$.

The remaining case is $\alpha \in \mathcal{S}$ with $\det(x_2) = \det(y_2) = 0$. We set $J(i_1, i_2, \ldots, i_t)$ the matrix of size $n \times t$ such that $1 \leq i_1 < i_2 \cdots < i_t \leq n$ and whose (i_j, j)-entry is 1, $1 \leq j \leq t$, and all the other entries are 0.

Under $(P \times GL_n)$-action, we may assume that

$$\alpha = (\left(\begin{array}{c|c} 0 & J_1 \\ \hline J_2 & 0 \end{array}\right), \left(\begin{array}{c|c} z_1 & 0 \\ \hline z_2 & z_3 \end{array}\right)), \qquad (J_1, z_3 \in M_{n\ell},\ J_2, z_1, z_2 \in M_{nk}),$$

where

$$J_1 = J(r_1, r_2, \ldots, r_\ell), \quad J_2 = J(e_1, e_2, \ldots, e_k), \quad 1 \leq \ell, k < n, \quad \ell + k = n,$$

and

the e_j-th row of z_1 is the same as in J_2 and (i, j)-entry is 0 if $i < e_j$, $1 \leq j \leq k$,
the r_j-th row of z_2 is 0, $1 \leq j \leq \ell$, (3.8)
the r_j-th row of z_3 is the same as in J_1 and (i, j)-entry is 0 if $i > r_j$, $1 \leq j \leq \ell$.

We see, for any $a \in GL_1$,

$$(\left(\begin{array}{c|c} 1_n & 0 \\ \hline 0 & \delta_1(a) \end{array}\right), 1_n) \in H_\alpha \qquad \text{if } e_1 > 1,$$

$$(\left(\begin{array}{c|c} \delta_1(a) & 0 \\ \hline 0 & 1_n \end{array}\right), \delta_{k+1}(a)) \in H_\alpha \quad \text{if } r_1 = 1,$$

$$(\left(\begin{array}{c|c} a1_n & 0 \\ \hline 0 & 1_n \end{array}\right), a1_n) \in H_\alpha \qquad \text{if } z_2 = 0.$$

If $e_1 = 1, r_1 > 1$ and $z_2 \neq 0$, we modify z_i-part of α to satisfy not only (3.8) but also the following

if the i-th row of z_2 is nonzero, then the i-th row of z_3 is 0, and we still call it α. Then H_α contains the following (A_1, A_2) for any $a \in GL_1$

$$A_1 = Diag(a_1, \dots, a_n) \perp 1_n, \quad a_i = \begin{cases} a & \text{if the } i\text{-th row of } z_2 \text{ is } 0 \\ 1 & \text{if the } i\text{-th row of } z_2 \text{ is not } 0, \end{cases}$$
$$A_2 = 1_k \perp a1_\ell.$$

Hence $\widetilde{\psi}_n \not\equiv 1$ on P_α for $\alpha \in \mathcal{S}$ with $\det(x_2) = \det(y_2) = 0$. ∎

Thus we have shown the condition $(\widetilde{C})$ is satisfied for every $(x, y) \in \mathcal{S}$, which shows that our $(\mathbb{X}, \mathbb{B})$ satisfies the condition $(A3)$ and Theorem 3.1 is established.

§4 Spherical Fourier transform on $\mathcal{S}(K\backslash X_T)$

We consider the space $\mathcal{S}(K\backslash X_T)$ consisting of functions in $\mathcal{C}^\infty(K\backslash \mathfrak{X}_T/U(T))$ compactly supported modulo $U(T)$, which is an $\mathcal{H}(G,K)$-submodule (cf. (1.8)). We define the spherical Fourier transform F_T on $\mathcal{S}(K\backslash X_T)$ as follows

$$\begin{aligned} F_T : \mathcal{S}(K\backslash X_T) &\longrightarrow \mathbb{C}(q^{z_1}, \dots, q^{z_n}), \\ \xi &\longmapsto F_T(\xi)(z) = \hat{\xi}_T(z) = \int_X \xi(x)\Psi_T(x; z)dx, \end{aligned} \tag{4.1}$$

where $\Psi_T(x; z) = G(z) \cdot \omega_T(x; z)$ and dx is the G-invariant measure on X. By Theorem 2.8, we see the image of F_T is contained in

$$\mathcal{R} = \mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W.$$

We decompose $\mathcal{R}$ as follows

$$\mathcal{R} = \bigoplus_{\mathbf{e} \in \{0,1\}^n} s_1^{e_1} \cdots s_n^{e_n} \mathcal{R}_0,$$

where

$$\mathcal{R}_0 = \mathbb{C}[q^{\pm 2z_1}, \dots, q^{\pm 2z_n}]^W = \mathbb{C}[q^{2z_1} + q^{-2z_1}, \dots, q^{2z_n} + q^{-2z_n}]^{S_n},$$

and $s_i = s_i(z)$ is the i-th fundamental symmetric polynomial of $\{ q^{z_j} + q^{-z_j} \mid 1 \le j \le n\}$; $\mathcal{R}$ is a free $\mathcal{R}_0$-module of rank 2^n. We set

$$\mathcal{R}_{even} = \bigoplus_{\mathbf{e}: even} s_1^{e_1} \cdots s_n^{e_n} \mathcal{R}_0, \quad \mathcal{R}_{odd} = \bigoplus_{\mathbf{e}: odd} s_1^{e_1} \cdots s_n^{e_n} \mathcal{R}_0,$$

where $\mathbf{e} \in \{0,1\}^n$ is even (resp. odd) if $\sum_{i=1}^n ie_i$ is even (resp. odd), and for each $T \in \mathcal{H}_n^{nd}$, and define

$$\mathcal{R}_{\langle T \rangle}$$

to be $\mathcal{R}_{even}$ or $\mathcal{R}_{odd}$ according to the parity of $v_\pi(\det(T))$.

Theorem 4.1 *For each $T \in \mathcal{H}_n^{nd}$, one has a surjective $\mathcal{H}(G,K)$-module homomorphism*

$$F_T : \mathcal{S}(K\backslash X_T) \longrightarrow \mathcal{R}_{\langle T\rangle},$$

and a commutative diagram

$$\begin{array}{ccccc} \mathcal{H}(G,K) & \times & \mathcal{S}(K\backslash X_T) & \xrightarrow{*} & \mathcal{S}(K\backslash X_T) \\ \wr\downarrow & & \downarrow F_T & \circlearrowleft & \downarrow F_T \\ \mathcal{R}_0 & \times & \mathcal{R}_{\langle T\rangle} & \longrightarrow & \mathcal{R}_{\langle T\rangle}, \end{array} \tag{4.2}$$

where the upper horizontal arrow is given by the action of $\mathcal{H}(G,K)$ on $\mathcal{S}(K\backslash X_T)$, the left end vertical isomorphism is given by Satake isomorphism (1.11)

$$\mathcal{H}(G,K) \xrightarrow{\sim} \mathcal{R}_0, \ \phi \longmapsto \lambda_z(\check{\phi}), \qquad (\check{\phi}(g) = \phi(g^{-1})),$$

and the lower horizontal arrow is given by the ordinal multiplication in $\mathcal{R}$.

Outline of a proof. For $\phi \in \mathcal{H}(G,K)$ and $\xi \in \mathcal{S}(K\backslash X_T)$, it is easy to see

$$F_T(\phi * \xi)(z) \ = \ \lambda_z(\check{\phi})F_T(\xi)(z).$$

We may expand $\omega_T(x;z)$ in a region of absolute convergence of the integral (1.7)

$$\omega_T(x;z) = \sum_{\mu \in \mathbf{Z}^n} a_\mu q^{\langle \mu, z\rangle},$$

where $a_\mu = 0$ unless $|\mu| \, (= \sum_{i=1}^n \mu_i) \equiv v_\pi(\det(T)) \pmod 2$. Further we may expand $G(z)$ also in terms $q^{\langle \nu, z\rangle}$ with $|\nu|$ is even. Hence we see that $\mathrm{Im}(F_T) \subset \mathcal{R}_{\langle T\rangle}$. On the other hand, by Remark 3.2 and Theorem 3.3 we see

$$\mathrm{Im}(F_T) \supset \left\{ H_\lambda(z) \mid \lambda \in \Lambda_n^+, \ |\lambda| \equiv v_\pi(\det(T)) \pmod 2 \right\},$$

and the image of F_T coincides with $\mathcal{R}_{\langle T\rangle}$. ∎

Remark 4.2 We expect that the spherical Fourier transform F_T is injective, which is equivalent to the identity

$$\mathfrak{X}_T = \bigcup_{\substack{\lambda \in \Lambda_n^+ \\ |\lambda| \equiv v_\pi(\det(T)) \pmod 2}} Kx_\lambda h_\lambda U(T), \tag{4.3}$$

where disjointness in the right hand side is known by Theorem 3.3. If it is true, then $\mathcal{S}(K\backslash X_T)$ would be a free $\mathcal{H}(G,K)$-module of rank 2^{n-1} and the set $\{ \Psi_T(x; z + \widetilde{u}) \mid u \in \mathcal{U}\}$ would form a basis of spherical functions on X_T corresponding to $z \in \mathbb{C}^n$ through λ_z (cf. Proposition 1.3). This is true when $n = 1$ by Proposition 2.1, and we have the following.

Proposition 4.3 *Assume $n = 1$. Then the spherical transform F_T is injective and $\mathcal{S}(K\backslash X_T)$ is a free $\mathcal{H}(G,K)$-module of rank 1, in fact the image coincides with*

$$\mathbb{C}[q^{2z}+q^{-2z}] \text{ if } v_\pi(T) \text{ is even}, \quad (q^z+q^{-z})\mathbb{C}[q^{2z}+q^{-2z}] \text{ if } v_\pi(T) \text{ is odd.}$$

Any spherical function on X_T corresponding to $z \in \mathbb{C}$ through λ_z is a constant multiple of $\omega_T(x; z)$.

§5 Hermitian Siegel series

We recall p-adic hermitian Siegel series, and give those integral representation and a new proof of the functional equation as an application of spherical functions.

Let ψ be an additive character of k of conductor $\mathcal{O}_k$. For $T \in \mathcal{H}_n(k')$, the hermitian Siegel series $b_\pi(T; s)$ is defined by

$$b_\pi(T;t) = \int_{\mathcal{H}_n(k')} \nu_\pi(R)^{-t}\psi(\mathrm{tr}(TR))dR, \tag{5.1}$$

where tr( ) is the trace of matrix and $\nu_\pi(R)$ is defined as follows: if the elementary divisors of R with negative π-powers are $\pi^{-e_1}, \dots, \pi^{-e_r}$, then $\nu_\pi(R) = q^{e_1+\cdots+e_r}$, and $\nu_\pi(R) = 1$ otherwise (cf. [Sh]-§13). The right hand side of (5.1) is absolutely convergent if Re(t) is sufficiently large.

In the following we assume that T is nondegenerate, since the properties of $b_\pi(T;t)$ can be reduced to the nondegenerate case. We give an integral expression of $b_\pi(T;t)$ in a similar argument for Siegel series in [HS]-§2. We recall the set $\mathfrak{X}_T$ for $T \in \mathcal{H}_n^{nd}$ and take the measure $|\Theta_T|$ on it simultaneously as the fibre space of T by the polynomial map $M_{2n,n}(k') \longrightarrow \mathcal{H}_n(k')$, $x \longmapsto H_n[x]$.

Theorem 5.1 *If* Re(t) $> 2n$, *we have*

$$b_\pi(T;t) = \zeta_n(k';\frac{t}{2})^{-1} \times \int_{\mathfrak{X}_T(\mathcal{O}_{k'})} |N(\det(x_2))|^{\frac{t}{2}-n}\, |\Theta_T|\,(x),$$

where $\zeta_n(k';t)$ is the zeta function of the matrix algebra $M_n(k')$

$$\zeta(k';t) = \int_{M_n(\mathcal{O}_{k'})} |\det(x)|_{k'}^{t-n}\, dx = \prod_{i=1}^{n} \frac{1-q^{-2i}}{1-q^{-2(t-i+1)}},$$

and

$$\mathfrak{X}_T(\mathcal{O}_{k'}) = \{x \in M_{2n,n}(\mathcal{O}_{k'}) \mid H_n[x] = T\},$$

Since $\mathfrak{X}_T(\mathcal{O}_{k'})$ is compact, we obtain

Proposition 5.2 *Denote the K-orbit decomposition of $\mathfrak{X}_T(\mathcal{O}_{k'})$ as*

$$\mathfrak{X}_T(\mathcal{O}_{k'}) = \sqcup_{i=1}^r K x_i.$$

Then one has

$$b_\pi(T;t) = \zeta_n(k';\frac{t}{2})^{-1} |\det(T)|^{\frac{t}{1}-n} \times \sum_{i=1}^r c_i \cdot \omega_T(x_i; s_t),$$

where c_i is the volume of Kx_i and

$$s_t = (1,\ldots,1,\frac{t}{2}-n+\frac{1}{2}) + (\tfrac{\pi\sqrt{-1}}{\log q},\ldots,\tfrac{\pi\sqrt{-1}}{\log q}) \in \mathbb{C}^n.$$

Then, by Corollary 2.6, we obtain the functional equation of $b_\pi(T;t)$.

Theorem 5.3 *For any $T \in \mathcal{H}_n^{nd}$, one has*

$$b_\pi(T;t) = \chi_\pi(\det(T))^{n-1} |\det(T)|^{t-n} \times \prod_{i=0}^{n-1} \frac{1-(-1)^i q^{-t+i}}{1-(-1)^i q^{-(2n-t)+i}} \times b_\pi(T;2n-t),$$

where $\chi_\pi(a) = (-1)^{v_\pi(a)}$ for $a \in k^\times$.

Remark 5.4 The above functional equation is related to an element of the Weyl group of $U(H_n)$, which is not the case for Siegel series when n is odd. In [HS], even n is odd, we needed some harmonic analysis on $O(H_n)$ to establish the functional equation.

The existence of the functional equation of $b_\pi(T;t)$ was known in an abstract form as functional equations of Whittaker functions of p-adic groups by Karel [Kr]. Recently Ikeda [Ik] has given explicit functional equations on the basis of the results of Kudla-Sweet [KS] for all quadratic extensions over $\mathbb{Q}_p$ containing split cases. There is an error in the range of i in the definition of $t_p(K/\mathbb{Q};X)$ in [Ik] p.1112, and it is better to refer the original $f_\zeta(t)$ in [Sh] Theorem 13.6; if $K/\mathbb{Q}$ is unramified at p, $t_p(K/\mathbb{Q};X)$ is the product of $1-(-p)^iX$ from $i=0$ to $n-1$, and coincides with our case by taking $X = p^{-t}$.

References

[Bo] A. Borel: *Linear Algebraic Groups, Second enlarged edition*, Graduate Texts in Mathematics **126**, Springer, 1991.

[H1] Y. Hironaka: Spherical functions of hermitian and symmetric forms III, *Tôhoku Math. J.* **40**(1988), 651–671.

[H2] Y. Hironaka: Spherical functions and local densities on hermitian forms, *J. Math. Soc. Japan* **51**(1999), 553 - 581.

[H3] Y. Hironaka: Functional equations of spherical functions on p-adic homogeneous spaces, *Abh. Math. Sem. Univ. Hamburg* **75**(2005), 285 - 311.

[H4] Y. Hironaka: Spherical functions on p-adic homogeneous spaces, in *"Algebraic and Analytic Aspects of Zeta Functions and L-functions - Lectures at the French-Japanese Winter School (Miura, 2008)- ", MSJ Memoirs* **21**(2010), 50 - 72.

[HS] Y. Hironaka and F. Sato : The Siegel series and spherical functions on $O(2n)/(O(n) \times O(n))$, "Automorphic forms and zeta functions - Proceedings of the conference in memory of Tsuneo Arakawa -", World Scientific, 2006, p. 150 - 169.

[Ik] T. Ikeda: On the lifting of hermitian modular forms, *Comp. Math.***114** (2008), 1107-1154.

[Kr] M. L. Karel: Functional equations of Whittaker functions on p-adic groups, *Amer. J. Math.* **101**(1979), 1303 -1325.

[Mac] I. G. Macdonald: Orthogonal polynomials associated with root systems, *Séminaire Lotharingien de Combinatoire* **45**(2000), Article B45a.

[Om] O. T. O'Meara: *Introduction to quadratic forms*, Grund. math. Wiss. **117**, Springer-Verlag, 1973.

[KS] S. S .Kudla and W. J. Sweet: Degenerate principal series representations for $U(n,n)$, *Israel J. Math.* **98** (1997), 253 -306.

[Sch] W. Scharlau: *Quadratic and hermitian forms*, Grund. math. Wiss. **270**, Springer-Verlag, 1985.

[Sh] G. Shimura: *Euler products and Eisenstein series*, CBMS **93** (AMS), 1997.

[Ym] T. Yamazaki: Integrals defining singular series, *Memoirs Fac. Sci. Kyushu Univ.***37**(1983), 113 - 128.