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On the Kashaev invariant of twist knots

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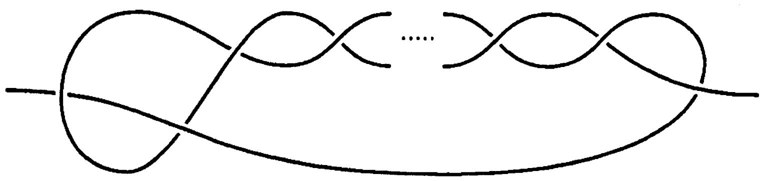
Let K denote the knot represented by the diagram with $n+3$ crossings below. In this talk, we compute its *Kashaev invariant*, which is a special value of the *colored Jones polynomial*, and study its asymptotic behavior.



1. Kashaev's invariant

Let N be a positive integer and

$$\mathcal{N}=\{0,1,\ldots,N-1\}.$$

Then, for $i,j,k,l\in\mathcal{N}$, we define θ_{kl}^{ij} by

$$\theta_{kl}^{ij}=\begin{cases}1 & \text{if } [i-j]+[j-l]+[l-k-1]+[k-i]=N-1,\\ 0 & \text{otherwise,}\end{cases}$$

where $[m]\in\mathcal{N}$ denotes the residue of m modulo N. Furthermore, for $x\in\mathbb{C}$, we define $(x)_m$ by

$$(x)_m=(1-x)(1-x^2)\cdots(1-x^{[m]}).$$

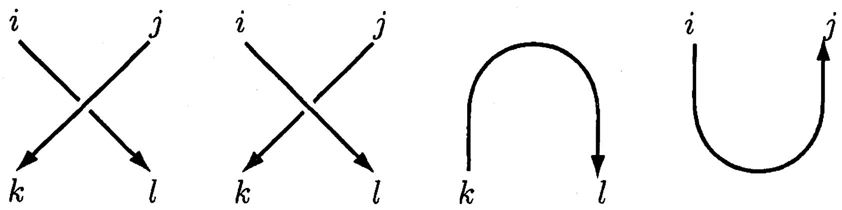
In what follows, we put

$$q=\exp\frac{2\pi\sqrt{-1}}{N}.$$

Then, the Kashaev invariant $\langle K\rangle_N$ of K is obtained by contracting the tensors associated to the following critical points, where

$$R_{kl}^{ij}=\frac{Nq^{-\frac{1}{2}+i-k}}{(q)_{[i-j]}(\bar{q})_{[j-l]}(q)_{[l-k-1]}(\bar{q})_{[k-i]}}\cdot\theta_{kl}^{ij},$$

$$\bar{R}_{kl}^{ij}=\frac{Nq^{\frac{1}{2}+j-l}}{(\bar{q})_{[i-j]}(q)_{[j-l]}(\bar{q})_{[l-k-1]}(q)_{[k-i]}}\cdot\theta_{kl}^{ij}.$$



R_{kl}^{ij} $\qquad$ $\bar{R}_{kl}^{ij}$ $\qquad$ $-q^{\frac{1}{2}}\delta_{k+1,l}$ $\qquad$ $-q^{-\frac{1}{2}}\delta_{i-1,j}$

Proposition. *The Kashaev invariant* $\langle K\rangle_N$ *of* K *is given by*

$$N^{n+1}\sum_{0\le i_1\le\cdots\le i_n<N}\frac{1}{(q)_{i_1}(\bar{q})_{i_n}}\prod_{\nu=1}^{n-1}\frac{1}{(\bar{q})_{i_\nu}(q)_{i_{\nu+1}-i_\nu}(\bar{q})_{N-1-i_{\nu+1}}}.$$

Example. Suppose $n=3$. Then,

$$\begin{aligned}\langle K\rangle_N=\sum_{c\le d\le b}&\frac{Nq^{-\frac12+b+1}}{(q)_{b-d}(\bar{q})_d(\bar{q})_{N-1-b}}\cdot\frac{Nq^{-\frac12-c}}{(\bar{q})_{N-1-d}(q)_{d-c}(\bar{q})_c}\\&\times\left(\sum_{a=c}^{N-1}\frac{Nq^{\frac12}}{(\bar{q})_{N-1-a}(q)_a}\cdot\frac{Nq^{\frac12+c}}{(\bar{q})_{a-c}(q)_c(q)_{N-1-a}}\right)\\&\times\left(\sum_{e=b}^{N-1}\frac{Nq^{-\frac12}}{(q)_e(\bar{q})_{N-1-e}}\cdot\frac{Nq^{-\frac12-b}}{(q)_{N-1-e}(q)_{e-b}(\bar{q})_b}\right).\end{aligned}$$

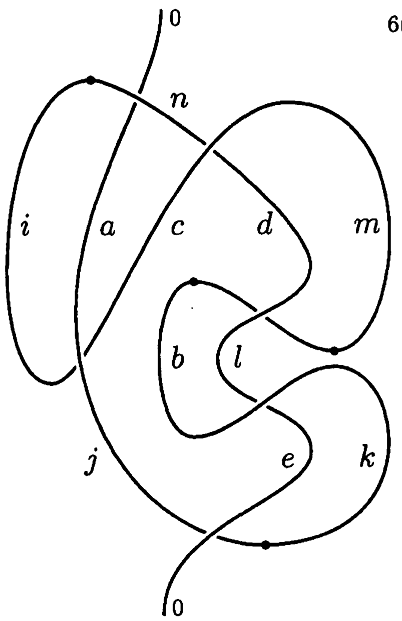
from the picture below. This is further equal to

$$\sum_{c\le d\le b}\frac{Nq^{-\frac12+b+1}}{(q)_{b-d}(\bar{q})_d(\bar{q})_{N-1-b}}\cdot\frac{Nq^{-\frac12-c}}{(\bar{q})_{N-1-d}(q)_{d-c}(\bar{q})_c}\cdot\frac{N^2q^{c-b}}{(q)_c(\bar{q})_b},$$

where we used the following lemma.

Lemma. $(q)_m(\bar{q})_{N-1-m}=N$ *and*

$$\sum_{i\le m\le j}\frac{1}{(q)_{j-m}(\bar{q})_{m-i}}=1.$$



$$\begin{aligned}6(N-1)=&([i+1]+[j-i-1]+[-j])\\&+([k]+[l-k]+[m-l]+[n-m]+[-n])\\&+([a-i-1]+[i-a])\\&+([n-a-1]+[c-n]+[a-c])\\&+([d-c-1]+[b+1-d]+[e-b-1]+[j-e]+[c-j])\\&+([m-d]+[d-m-1])\\&+([l-b-1]+[b-l])\\&+([k-e]+[e-k-1])\\\ge&([i+1]+[j-i-1]+[-j])\\&+([k]+[l-k]+[m-l]+[n-m]+[-n])\\&+6(N-1)\longrightarrow i=-1,\ j=k=l=m=n=0,\\&\qquad 0\le a<N,\\&\qquad 0\le c\le a<N,\\&\qquad 0\le c<d\le b+1\le e\le N,\\&\qquad 1\le d\le N,\\&\qquad 0\le b<N,\\&\qquad 1\le e\le N\end{aligned}$$

2. Quantum dilogarithms

Let

$$\psi_N(z) = \exp \int_{-\infty}^{\infty} \frac{e^{\sqrt{N}(2z+1)t} dt}{4t \sinh(t/\sqrt{N}) \sinh(\sqrt{N}t)},$$

and

$$p_k = \frac{2k+1}{2N}.$$

Then, the sets of poles and zeros of ψ_N are given by $\{p_k \mid k \geq N\}$ and $\{p_k \mid k < 0\}$ respectively, and

$$\frac{1}{(q)_k} = \frac{\psi_N(p_k)}{\psi_N(p_0)}, \quad \frac{1}{(\bar{q})_k} = \frac{\psi_N(1-p_0)}{\psi_N(1-p_k)},$$
$$\frac{\psi_N(p_0)}{\sqrt{N}} = \exp \frac{N}{2\pi\sqrt{-1}} \left(\frac{\pi^2}{6} - \frac{\pi^2}{2N} + \frac{\pi^2}{6N^2} \right) = \sqrt{N}\psi_N(1-p_0).$$

By using these quantum dilogarithms, we can write

$$\langle K \rangle_N = N^{n+1} \sum_{k_1 \leq \cdots \leq k_n} \frac{\psi_N(p_{k_1})}{\psi_N(p_0)} \frac{\psi_N(1-p_0)}{\psi_N(1-p_{k_n})}$$
$$\times \prod_{\nu=1}^{n-1} \frac{\psi_N(1-p_0)}{\psi_N(1-p_{k_\nu})} \frac{\psi_N(p_{k_{\nu+1}-k_\nu})}{\psi_N(p_0)} \frac{\psi_N(1-p_0)}{\psi_N(1-p_{k_{\nu+1}})}.$$

If we put

$$\Psi_N(z_1, \ldots, z_n) = e^{\frac{N(n-1)}{2\pi\sqrt{-1}}\left(\frac{\pi^2}{6} - \frac{\pi^2}{2N} + \frac{\pi^2}{6N^2}\right)} \frac{\psi_N(z_1)}{\psi_N(1-z_n)} \prod_{\nu=1}^{n-1} \frac{\psi_N(z_{\nu+1} - z_\nu + p_0)}{\psi_N(1-z_\nu)\psi_N(1-z_{\nu+1})},$$

we have

$$\langle K \rangle_N = N^{\frac{3-n}{2}} \sum_{k_1=0}^{N-1} \cdots \sum_{k_n=0}^{N-1} \Psi_N(p_{k_1}, \ldots, p_{k_n})$$

because $\psi_N(p_{k_{\nu+1}-k_\nu}) = 0$ if $k_{\nu+1} < k_\nu$.

3. Integrals

Let $Q_\nu = e^{2\pi\sqrt{-1}z_\nu}$ and

$$C = \left\{ x + y\sqrt{-1} \,\middle|\, (x - \tfrac{1}{2})^2 + y^2 = (\tfrac{1}{2})^2 \right\}.$$

Then, by the residue theorem,

$$\langle K \rangle_N = (-1)^n N^{\frac{n+3}{2}} \int_C \frac{dz_1}{1+Q_1^N} \cdots \int_C \frac{dz_n}{1+Q_n^N} \Psi_N(z_1, \ldots, z_n).$$

Let
$$A = \{z \in C \,|\, \mathrm{Im}\, z \geq 0\},\ B = \{z \in C \,|\, \mathrm{Im}\, z \leq 0\}.$$
Then, we have
$$\int_C \frac{dz_\nu}{1+Q_\nu^N} = \int_0^1 (Q_\nu^N - 1 + Q_\nu^{-N}) dz_\nu + \int_A \frac{Q_\nu^{2N}}{1+Q_\nu^N} dz_\nu - \int_B \frac{Q_\nu^{-2N}}{1+Q_\nu^{-N}} dz_\nu$$
because
$$\int_A \frac{dz_\nu}{1+Q_\nu^N} = \int_1^0 (1 - Q_\nu^N) dz_\nu + \int_A \frac{Q_\nu^{2N}}{1+Q_\nu^N} dz_\nu,$$
$$\int_B \frac{dz_\nu}{1+Q_\nu^N} = \int_B \frac{Q_\nu^{-N} dz_\nu}{1+Q_\nu^{-N}} = \int_0^1 Q_\nu^{-N} dz_\nu - \int_B \frac{Q_\nu^{-2N}}{1+Q_\nu^{-N}} dz_\nu.$$
In what follows, for $z \in \mathbb{C}$, we put
$$x_z = \mathrm{Re}\, 2\pi z,\ y_z = \mathrm{Im}\, 2\pi z,\ \omega_z = -\arg(1 - e^{2\pi\sqrt{-1}z}).$$

Lemma.
$$\lim_{y_{z_\nu} \to \pm\infty} \Psi_N(z_1, \ldots, z_n)\, Q_\nu^{\pm N} = 0.$$

Proposition.
$$\langle K \rangle_N \sim (-1)^n N^{\frac{n+3}{2}} \int_0^1 dz_1 \cdots \int_0^1 dz_n\, \Psi_N(z_1, \ldots, z_n).$$

4. Asymptotics

Define
$$\mathcal{L}(z) = \mathrm{Li}_2(e^{2\pi\sqrt{-1}z}) + \beta_z(2\pi z - \tfrac{1}{2}\beta_z - \pi),$$
where Li_2 denotes Euler's dilogarithm function and
$$\beta_z = \begin{cases} 2\pi\lfloor \mathrm{Re}\, z \rfloor & \text{if } \mathrm{Im}\, z < 0, \\ 0 & \text{otherwise.} \end{cases}$$
Then, we have
$$\psi_N(z) \sim e^{\frac{N}{2\pi\sqrt{-1}}\{\mathcal{L}(z) + O(N^{-2})\}}.$$
Furthermore, we put
$$V(z) = \Lambda(x_z) + \Lambda(\omega_z) - \Lambda(x_z + \omega_z),$$
where Λ denotes Lobachevsky's function. Then, we have
$$\mathrm{Im}\, \mathcal{L}(z) = y_z(\beta_z - \omega_z) + V(z), \quad \frac{\partial}{\partial y_z} \mathrm{Im}\, \mathcal{L}(z) = \beta_z - \omega_z.$$

Define $H(z_1, \ldots, z_n)$ by

$$\mathcal{L}(z_1) - \mathcal{L}(z_n) + \frac{(n-1)\pi^2}{6} + \sum_{\nu=1}^{n-1} \{\mathcal{L}(z_{\nu+1} - z_\nu) - \mathcal{L}(1 - z_\nu) - \mathcal{L}(z_{\nu+1})\}.$$

For simplicity, we put $f(z_1, \ldots, z_n) = \operatorname{Im} H(z_1, \ldots, z_n)$.

Proposition.

$$\Psi_N(z_1, \ldots, z_n) \sim e^{\frac{N}{2\pi\sqrt{-1}}\{H(z_1,\ldots,z_n)+O(N^{-1})\}}.$$

Example. Suppose $n = 3$. Then, $f(z_1, z_2, z_3)$ is equal to

$$\begin{aligned} &V(z_1, z_2, z_3) + y_{z_1}(-\omega_{z_1} - \omega_{1-z_1} - \beta_{z_2-z_1} + \omega_{z_2-z_1}) \\ &\quad + y_{z_2}(\beta_{z_2-z_1} - \omega_{z_2-z_1} + \pi - x_{z_2} - \beta_{z_3-z_2} + \omega_{z_3-z_2}) \\ &\quad + y_{z_3}(\beta_{z_3-z_2} - \omega_{z_3-z_2} + \pi - x_{z_3}), \end{aligned}$$

where $V(z_1, \ldots, z_n)$ is defined by

$$V(z_1) - V(z_n) + \sum_{\nu=1}^{n-1} \{V(z_{\nu+1} - z_\nu) - V(1 - z_\nu) - V(z_{\nu+1})\}.$$

How does $f(z_1, z_2, z_3)$ behave when $y_{z_1}^2 + y_{z_2}^2 + y_{z_3}^2 \to \infty$? Since

$$\lim_{y_z \to \infty} (\beta_z - \omega_z) = 0, \qquad \lim_{y_z \to -\infty} (\beta_z - \omega_z) = x_z - \pi,$$

we check its behavior along the following 3 lines;

$$y_{z_2-z_1} = y_{z_3-z_2} = 0, \quad y_{z_1} = y_{z_3-z_2} = 0, \quad y_{z_1} = y_{z_2-z_1} = 0$$

with $x_{z_1}, x_{z_2}, x_{z_3}$ fixed. For simplicity, we put

$$\lambda(y) = \tfrac{1}{2}(y - |y|).$$

Then, $f(z_1, z_2, z_3)$ is approximated by

$$\lambda(y_{z_1})(x_{z_1} - x_{z_2} - x_{z_3} + \pi) + \lambda(-y_{z_1})(x_{z_1} + x_{z_2} + x_{z_3} - 3\pi)$$

when $y_{z_2-z_1} = y_{z_3-z_2} = 0$, by

$$\lambda(y_{z_2})(-x_{z_1} - x_{z_3} + \pi) + \lambda(-y_{z_2})(x_{z_2} + x_{z_3} - 2\pi)$$

when $y_{z_1} = y_{z_3-z_2} = 0$, and by

$$\lambda(y_{z_3})(-x_{z_2}) + \lambda(-y_{z_3})(x_{z_3} - \pi)$$

when $y_{z_1} = y_{z_2-z_1} = 0$. Therefore, we can observe

$$\lim_{y_{z_1}^2 + y_{z_2}^2 + y_{z_3}^2 \to \infty} f(z_1, z_2, z_3) = \infty$$

if $x_{z_1}, x_{z_2}, x_{z_3}$ satisfy the following conditions.

$$\begin{gathered} \pi < -x_{z_1} + x_{z_2} + x_{z_3} < x_{z_1} + x_{z_2} + x_{z_3} < 3\pi, \\ \pi < x_{z_1} + x_{z_3} < x_{z_2} + x_{z_3} < 2\pi, \ x_{z_3} < \pi. \end{gathered}$$

This region will play an important role in the following argument.

Let Δ be the set of $(z_1,\dots,z_n)\in[0,1]^n$ satisfying

$$\tfrac{1}{2}(n-\nu)<z_{\nu-1}+\textstyle\sum_{k=\nu+1}^n z_k<\sum_{k=\nu}^n z_k<\tfrac{1}{2}(n-\nu+1)$$

for $1<\nu\leq n$ and

$$\tfrac{1}{2}(n-2)<-z_1+\textstyle\sum_{k=2}^n z_k<\sum_{k=1}^n z_k<\tfrac{1}{2}n.$$

The main purpose of this note is to show

Proposition. *Let* $(\zeta_1,\dots,\zeta_n)$ *be the solution to*

$$\frac{\partial H}{\partial z_\nu}\equiv 0 \mod 2\pi\sqrt{-1}.$$

satisfying $(\mathrm{Re}\,\zeta_1,\dots,\mathrm{Re}\,\zeta_n)\in\Delta$. *Then,*

$$\int_\Delta \Psi_N(z_1,\dots,z_n)\,dz_1\wedge\cdots\wedge dz_n\sim N^{-\frac{n}{2}}e^{\frac{N}{2\pi\sqrt{-1}}H(\zeta_1,\dots,\zeta_n)}.$$

Note that $f(\zeta_1,\dots,\zeta_n)$ *is equal to the complex volume of* K.

Proof. Define $p:\mathbb{C}^n\to\mathbb{R}^n$ by

$$p(z_1,\dots,z_n)=(\mathrm{Re}\,z_1,\dots,\mathrm{Re}\,z_n).$$

Let Σ be the set of $(z_1,\dots,z_n)\in p^{-1}(\Delta)$ satisfying

$$\begin{aligned}
y_{z_1}&=\log\frac{\sin\frac{1}{2}(n\pi-x_{z_1}+x_{z_2}+\cdots+x_{z_n})}{\sin\frac{1}{2}(n\pi+x_{z_1}+x_{z_2}+\cdots+x_{z_n})},\\
y_{z_2}&=\log\frac{\sin(x_{z_1}+x_{z_3}+\cdots+x_{z_n})}{\sin(x_{z_2}+x_{z_3}+\cdots+x_{z_n})}+y_{z_1},\\
&\cdots\\
y_{z_n}&=\log\frac{\sin x_{z_{n-1}}}{\sin x_{z_n}}+y_{z_{n-1}},
\end{aligned}$$

which is the unique solution to

$$\frac{\partial f}{\partial y_{z_\nu}}=0.$$

Then, $f|_\Sigma$ takes its unique maximum at $(\zeta_1,\dots,\zeta_n)\in\Sigma$. Let

$$E_\pm=f^{-1}((-\infty,f(\zeta_1,\dots,\zeta_n)\pm\varepsilon])\cap p^{-1}(\Delta)$$

and I the set of $(z_1(t),\dots,z_n(t))\in E_+-E_-$ satisfying

$$\frac{dz_\nu}{dt}=\frac{\partial f}{\partial\bar z_\nu},\quad \lim_{t\to-\infty}(z_1(t),\dots,z_n(t))=(\zeta_1,\dots,\zeta_n).$$

Then, $\operatorname{Re} H(z_1, \ldots, z_n)$ is constant on I because

$$\begin{aligned}\frac{d(\operatorname{Re} H)}{dt} &= \sum_{\nu=1}^{n} \operatorname{Re}\left\{\frac{\partial H}{\partial z_\nu}\cdot\frac{dz_\nu}{dt} + \frac{\partial H}{\partial \bar{z}_\nu}\cdot\frac{d\bar{z}_\nu}{dt}\right\}\\ &= \sum_{\nu=1}^{n} \operatorname{Re}\left\{\frac{\partial(\operatorname{Re} H + \sqrt{-1}\, f)}{\partial z_\nu}\cdot\frac{dz_\nu}{dt}\right\}\\ &= \sum_{\nu=1}^{n} \operatorname{Re}\left\{2\sqrt{-1}\cdot\frac{\partial f}{\partial z_\nu}\cdot\frac{\partial f}{\partial \bar{z}_\nu}\right\} = 0.\end{aligned}$$

Since $E_+ \simeq E_- \cup I \simeq \Sigma$, by the saddle point method, we have

$$\int_\Delta \Omega_N = \int_\Sigma \Omega_N \sim \int_I \Omega_N \sim N^{-\frac{n}{2}} e^{\frac{N}{2\pi\sqrt{-1}} H(\zeta_1,\ldots,\zeta_n)},$$

where $\Omega_N = \Psi_N(z_1, \ldots, z_n)\, dz_1 \wedge \cdots \wedge dz_n$. □