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Asymptotics of spin networks

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Abstract

In this survey paper we introduce spin networks and show how they provide a convenient framework for studying the colored Jones polynomial. The main idea is that by generalizing from knots to graphs (i.e. spin networks) we gain flexibility and make contact with the recoupling theory of quantum angular momentum. We survey recent results on the asymptotics of spin networks that are related to the volume conjecture.

1 Introduction

An important problem in low dimensional topology is to gain a better understanding of quantum invariants of knots. The simplest of these invariants is the Jones polynomial [O]. Many aspects of the Jones polynomial become more apparent when considering the colored Jones polynomial instead. A well known example of this is the volume conjecture that relates the Jones polynomial to hyperbolic geometry.

Conjecture 1 (Volume conjecture [K, MM]). *For any knot K we have*

$$\lim_{N\to\infty} \frac{2\pi}{N} \log |J_N(K)(e^{\frac{2\pi i}{N}})| = \mathrm{Vol}(\mathbb{S}^3 - \Gamma)$$

To better study problems such as the volume conjecture we propose to generalize from knots to the context of spin networks. Spin networks are defined as follows.

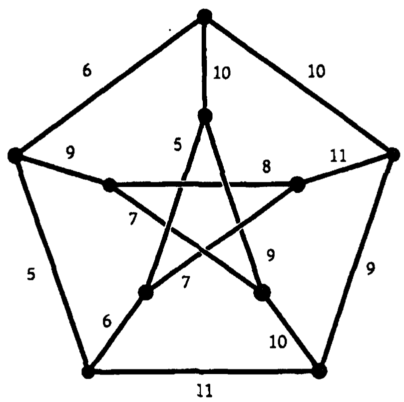

Figure 1: An example of a spin network.

Definition 1. *A spin network is pair (Γ, γ) where Γ is a trivalent ribbon graph embedded into $\mathbb{S}^3$ together with a coloring $\gamma : E(\Gamma) \to \mathbb{N}$.*

In the above definition we allow spin networks without vertices so that knots and links are special cases of spin networks. See figure 1 for an example of a more complicated spin network.

In the next section we will define an evaluation of spin networks that assigns to each spin network a rational function in the variable A. In case the spin network is a knot, the evaluation will be the colored Jones polynomial. If on the other hand we set $A = \pm 1$, the evaluation will equal the value of the network in quantum angular momentum theory [VMK]. The reason behind this is that the Jones polynomial is based on the quantum group $U_q(sl_2)$ while quantum angular momentum theory works with sl_2 instead. Thus the study of spin networks connects the theory of the Jones polynomial to the older theory of recoupling of angular momentum.

In this paper we will survey recent results on spin networks all of which are motivated by the following main question:

Question 1. *What is the behavior of* $\langle \Gamma, N\gamma \rangle$ *as* $N \to \infty$*?*

The volume conjecture is a special case of this question where the variable A is specialized at a root of unity depending on N. The plan of the paper is as follows. After defining the evaluation of a general spin network we show that (once properly normalized) the evaluations are actually polynomials in A so that their value at roots of unity is well defined. Next we state a theorem on the existence of an asymptotic expansion of the evaluations where one fixes the root of unity. Specializing even further we discuss the case of the classical spin networks where $A = \pm 1$. Finally we generalize the original volume conjecture to spin networks and show it holds in many cases. This also settles the conjecture for a large class of links. Thus showing the utility of the spin network point of view to knot theory.

2 Evaluation of spin networks

In this section we define the evaluation of spin networks. The evaluation is a direct generalization of the Jones and colored Jones polynomials to (trivalent) graphs. For simplicity we choose to build our definition on the Kauffman bracket. We start by defining quantum integers and the Jones-Wenzl idempotents. They serve to distillate the irreducible representations from the tensor products of the standard representation of quantum sl_2.

By an n-tangle we mean a tangle with n ends on one side and n ends on the other. Define the quantum integer $[k]$ to be the following symmetric k-term geometric series:

$$[k] = \frac{A^{2k} - A^{-2k}}{A^2 - A^{-2}}$$

Definition 2. *For $a \in \mathbb{N}$ we define the Jones-Wenzl idempotent of size a to be the following linear combination of n-tangles. The idempotent of size* 1 *is simply a single arc and the idempotent of size $a \geq 2$ is defined by the recursion shown in figure 2 below. Here the idempotents are depicted as boxes.*

For more properties of the Jones-Wenzl idempotents see for example [MV].

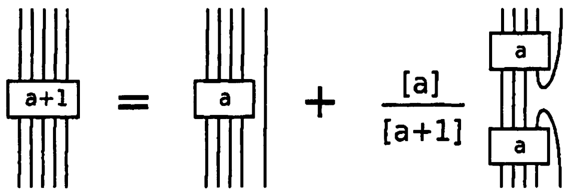

Figure 2: The defining relation of the Jones-Wenzl idempotent of size $a+1$ (the box on the left) in terms of the one of size a (boxes on the right).

Definition 3. *The evaluation $\langle\Gamma,\gamma\rangle^K$ of a quantum spin network (Γ,γ) is defined to be the Kauffman bracket of the linear combination of link diagrams obtained as follows:*

1. *Replace every edge e by a Jones-Wenzl idempotent of size $\gamma(e)$.*
2. *Connect the diagrams of the idempotents at the vertices in the only planar way without U-turns.*

If this is not possible to connect the idempotents then we define $\langle\Gamma,\gamma\rangle^K = 0$

Example 1. *Denote by $(\Theta,(a,b,c))$ the spin network for which Γ is the theta graph and the edges are colored a,b,c. Its evaluation is [MV]:*

$$\langle\Theta,(a,b,c)\rangle^K = (-1)^{\frac{a+b+c}{2}} \frac{[\frac{a+b+c+2}{2}]![\frac{-a+b+c}{2}]![\frac{a-b+c}{2}]![\frac{a+b-c}{2}]!}{[a]![b]![c]!}$$

Remark 1. *In the special case where Γ has no vertices Γ is a framed knot. If Γ is 0-framed then the evaluation $\langle\Gamma,\gamma\rangle^K$ is by definition the γ-colored Jones polynomial of the knot Γ. With these definitions the 1-colored Jones polynomial of a knot coincides with the original unnormalized Jones polynomial. However, it is quite common to redefine the colored Jones polynomial to be $J_N(K) = \langle K, N-1\rangle^K$. It is also usual to write $q = A^4$ and possibly to normalize the colored Jones polynomial so that it is 1 on the unknot. One has to be careful to always check which conventions are being used.*

There exist many formulas for reducing the evaluation of a spin network to the evaluation of several smaller ones. These go under the name of recoupling or fusion formulas [MV].

It is possible to generalize the evaluation to spin networks whose underlying graph is not necessarily trivalent, see [GV2].

2.1 Normalizations

We define two normalizations that are important in discussing spin network evaluations.

Definition 4. *Let the colors of the edges around a vertex $v \in V(\Gamma)$ be a_v, b_v, c_v.*

1. *The* integral normalization *of a spin network is defined by*

$$\langle\!\langle \Gamma, \gamma \rangle\!\rangle = \frac{\prod_{e \in E(\Gamma)} [\gamma(e)]!}{\prod_{v \in V(\Gamma)} [-a_v + b_v + c_v]! [a_v - b_v + c_v]! [a_v + b_v - c_v]!} \langle \Gamma, \gamma \rangle^K$$

2. *The* unitary normalization *of a spin network is defined by*

$$\langle \Gamma, \gamma \rangle = \frac{\prod_{e \in E(\Gamma)} [\gamma(e)]!}{\prod_{v \in V(\Gamma)} \langle \Theta, (a_v, b_v, c_v) \rangle^K} \langle \Gamma, \gamma \rangle^K$$

Proposition 1. *For every spin network* (Γ, γ) *there exist a* $\mathbb{Z}/4\mathbb{Z}$*-valued quadratic form* $Q(\gamma)$ *such that*

$$\langle\!\langle \Gamma, \gamma \rangle\!\rangle(q) \in A^{Q(\gamma)} \mathbb{Z}[A^{\pm 4}]. \tag{1}$$

This proposition is proven in [GV2] and extends earlier work of Costantino [C1]. From the integral normalization $\langle\!\langle \Gamma, \gamma \rangle\!\rangle$ we can get well-defined numerical invariants, by specializing the variable A to some value. This is important since in the next section we will set A to be a root of unity.

3 Spin networks at a fixed root of unity

In this section we consider spin network evaluations where A is a fixed root of unity. We will show that an asymptotic expansion exists for all such evaluations. This is in stark contrast with the case of the original volume conjecture where it is unknown whether the limit exists. The additional complication in the volume conjecture is that the roots of unity do not remain fixed. Finally we will state a new conjecture concerning the dependence of the leading order term in the expansion on the chosen root of unity.

3.1 Asymptotics

In order to discuss asymptotics we need to describe Nilsson type asymptotic expansions first. Sequences of Nilsson type are discussed in detail in [Ga1].

Definition 5. *We say that a sequence* (a_n) *is of* Nilsson type *if it has an* asymptotic expansion *of the form as* $n \to \infty$

$$a_n \sim \sum_{\lambda, \alpha, \beta} \lambda^n n^\alpha (\log n)^\beta S_{\lambda,\alpha,\beta} h_{\lambda,\alpha,\beta}\left(\frac{1}{n}\right) \tag{2}$$

where

(a) the summation is over a finite set of triples (λ, α, β)

(b) the growth rates λ *are algebraic numbers of equal absolute value,*

(c) the exponents α *are rational and the* nilpotency exponents β *are natural numbers,*

(d) *the* Stokes constants $S_{\lambda,\alpha,\beta}$ *are complex numbers,*

(e) *the formal power series* $h_{\lambda,\alpha,\beta}(x) \in K[[x]]$ *are Gevrey-1 (i.e., the coefficient of* x^n *is bounded by* $C^n n!$ *for some* $C > 0$*),*

(f) K *is a* number field *generated by the coefficients of* $h_{\lambda,\alpha,\beta}(x)$ *for all* λ, α, β.

It can be shown that a sequence of Nilsson type uniquely determines the constants in points (a)-(f) of the above definition. The meaning of the asymptotic expansion is roughly as follows. Expanding all the power series h, the asymptotic expansion becomes a sum of monomials of the form $C\lambda^n n^x (\log n)^y$. We order these the monomials according to how fast they grow as $n \to \infty$. It follows from the above assumptions that for any monomial M of this form, the set $F(M)$ of monomials occurring in the expansion that grow faster or equally fast, is finite. The asymptotic expansion now means that for any monomial M, if we subtract from a_n the sum of all monomials in $F(M)$ we obtain a sequence that grows slower than M.

Theorem 1 (Garoufalidis, vdV).
For every quantum spin network (Γ, γ)*, every complex root of unity* ζ *the sequence* $\langle\langle \Gamma, n\gamma \rangle\rangle(\zeta)$ *is of Nilsson type.*

The proof uses the theory of G-functions and results on q-holonomicity, see [GV2].

Note that the theorem still holds if one replaces the integral normalization by one of the two other normalizations $\langle \Gamma, \gamma \rangle$ and $\langle \Gamma, \gamma \rangle^K$, because they only differ by a factor for which the entire asymptotic expansion can be written down in closed form.

3.2 Conjecture on growth rates

Fix any spin network (Γ, γ). Let Λ_r be the absolute value of the growth rates of $\langle \Gamma, n\gamma \rangle$ at the $4r$-th root of unity. We conjecture that

Conjecture 2.

$$\Lambda_r^r = \Lambda_1$$

The above conjecture is surprising in that it tells us that an important part of the evaluation does not depend on the embedding, but only by the abstract graph itself. Indeed, Λ_1 is the absolute value of the growth rates of the classical spin network, that we will meet in the next section. It was shown in [A] that $\Lambda_1 \leq 1$, so if true the conjecture would prove the same for any root of unity. For more on the case of $A = \pm 1$ see the next section.

For knots the conjecture is trivially true since the colored Jones at a fixed root of unity is known to be periodic in the colors. The conjecture was shown to be true for some special $6j$-symbols and experimentally for few other simple networks using the following formula for the quantum multinomial:

Lemma 1. *Let* A *be a* $4r$*-th root of unity, and set* $b_j = B_j N + \beta_j$*. We have*

$$\begin{bmatrix} b_1 + \ldots + b_n \\ b_1, b_2, \ldots, b_n \end{bmatrix} = \pm \begin{pmatrix} B_1 + \ldots + B_n \\ B_1, B_2, \ldots, B_n \end{pmatrix} \begin{bmatrix} \beta_1 + \ldots + \beta_n \\ \beta_1, \beta_2, \ldots, \beta_n \end{bmatrix} \tag{3}$$

This was proven for Gaussian (asymmetric) binomial coefficients in [De]. The extension to the multinomial case is straightforward and the sign appears when one converts to symmetric multinomial coefficients.

It might be possible to extend the conjecture to the actual growth rates instead of only their absolute value. For subleading terms such a conjecture seems unlikely.

4 Classical spin networks

By a classical spin network we mean the specialization of a spin network to $A = \pm 1$. Note that at this value the embedding of the graph into $\mathbb{S}^3$ does not matter anymore. This is because from the Kauffman bracket relation one sees that there is no distinction anymore between the under and the over crossing. So one can say that a classical spin network is an abstract trivalent ribbon graph with a coloring γ as before.

Historically the classical spin network evaluations are much older than the Jones polynomial, as they arose from computations in quantum angular momenta of particle physics from the first half of the 20th century. Some general references on classical spin networks are [BL1], [BL2], [VMK].

Many of the conjectures about the colored Jones polynomial have counterparts for classical spin networks. For example, an analogue of the volume conjecture was already stated in 1968 [PR]:

If Γ is the tetrahedron graph ($6j$-symbol) and γ is such that there exists a Euclidean tetrahedron T with side lengths and 1-skeleton equal (Γ, γ) then:

$$\langle \Gamma, n\gamma \rangle = \frac{\sqrt{2}}{\sqrt{3\pi n^3 \mathrm{Vol}}} \cos\left(n \sum_e \gamma(e) \frac{\theta_e}{2} + \sum_e \frac{\theta_e}{2} + \frac{\pi}{4} \right) \left(1 + O\left(\frac{1}{n}\right)\right)$$

where Vol is the Euclidean volume of T and θ_e is the dihedral angle corresponding to edge e of T. This formula was rigorously proven in [Rb]. There exist analogous formulas for general γ, and some experimental results on the cube spin network, see [GV1]. For other recent approaches, see [CM] and [AH].

4.1 Generating function

One of the special features of classical spin networks is that their evaluations can be collected conveniently into a generating function as follows. We introduce the generating series S_Γ of a fixed graph Γ with cyclic ordering. S_Γ is a formal power series in commuting variables z_e where $e \in E(\Gamma)$, the set of edges of Γ.

Definition 6. *The* spin generating function S_Γ *of a trivalent ribbon graph* Γ *is defined to be*

$$S_\Gamma = \sum_\gamma \langle\langle \Gamma, \gamma \rangle\rangle \prod_{e \in E(\Gamma)} z_e^{\gamma_e} \in \mathbb{Q}[[z_e, e \in E(\Gamma)]] \tag{4}$$

where the sum ranges over all colorings γ *of the graph* Γ.

We will express the spin generating function of an arbitrary graph in terms of the cycle polynomial that we will define now. A *cycle* in a graph Γ is a 2-regular subgraph. By definition the empty set is also a cycle. The set of all cycles in Γ is called C_Γ.

Definition 7. *Given a trivalent ribbon graph Γ and $X \subset C_\Gamma$ we define*

(a) a polynomial

$$P_{\Gamma,X} = \sum_{c \in C_\Gamma} \epsilon_X(c) \prod_{e \in c} z_e \tag{5}$$

where $\epsilon_X(c) = -1$ (resp. 1) when $c \in X$ (resp. $c \notin X$).

(b) We will call $P_{\Gamma,\emptyset}$ the cycle polynomial *of Γ.*

Define the function Q_Γ on the subsets of C_Γ as follows. Let $Q_\Gamma(X)$ be the number of unordered pairs $\{c,d\} \subset X$ with the property that c and d intersect in an odd number of places when drawn on the thickening of Γ. Note that the cyclic orientation of Γ defines a unique thickening.

Theorem 2 (Garoufalidis, vdV).
For every trivalent ribbon graph Γ we have

$$S_\Gamma = \sum_{X \subset C_\Gamma} a_X P_{\Gamma,X}^{-2} \in \mathbb{Z}[[z_e, e \in E(\Gamma)]] \cap \mathbb{Q}(z_e, e \in E(\Gamma)). \tag{6}$$

where the coefficients are given by

$$a_X = \frac{1}{2^{|C_\Gamma|}} \sum_{Y \subset C_\Gamma} (-1)^{Q_\Gamma(Y) + |X \cap Y|}$$

The proof of the theorem is combinatorial, see [GV1]. An alternative approach is given by [CM].

Corollary 1. *For every spin network (Γ, γ), the evaluation $\langle\langle \Gamma, \gamma \rangle\rangle$ is an integer number.*

Corollary 2 ([We]). *When Γ is planar with the counterclockwise orientation, then all cycles intersect an even number of times so $(-1)^{Q(X)} = 1$ and hence only $a_\emptyset$ is non-zero. It follows that*

$$S_\Gamma = \frac{1}{P_{\Gamma,\emptyset}^2}$$

recovering an earlier theorem by Westbury [We].

5 Volume conjecture for spin networks

Finally we consider a direct generalization of the volume conjecture to spin networks. For simplicity we choose to restrict to the case where all colors are equal. So in this section by a spin network we always mean a spin network of the form: (Γ, N) for some color $N \in \mathbb{N}$. When the color is unimportant we will simply denote the spin network by Γ.

The volume conjecture for spin networks we consider is the following:

Conjecture 3.

$$\lim_{N\to\infty} \frac{2\pi}{N} \log |\frac{\langle \Gamma, N\rangle}{\langle O, N\rangle}(e^{\frac{\pi i}{2N}})| = \mathrm{Vol}(\mathbb{S}^3 - \Gamma)$$

Note how this precisely coincides with the original volume conjecture in case Γ is a knot or a link. (Note we are using the variable A here instead of the more common choice $q = A^4$). Some care should be taken in defining correctly the volume of the complement of a graph in the three sphere. The basic idea is that every edge becomes an annular cusp and every vertex a totally geodesic thrice punctured sphere, [V].

To state our result on the volume conjecture we need to define several moves on spin networks that are called KTG moves. Using these moves we will define a class of spin networks called augmented spin networks for which the volume conjecture can be proven.

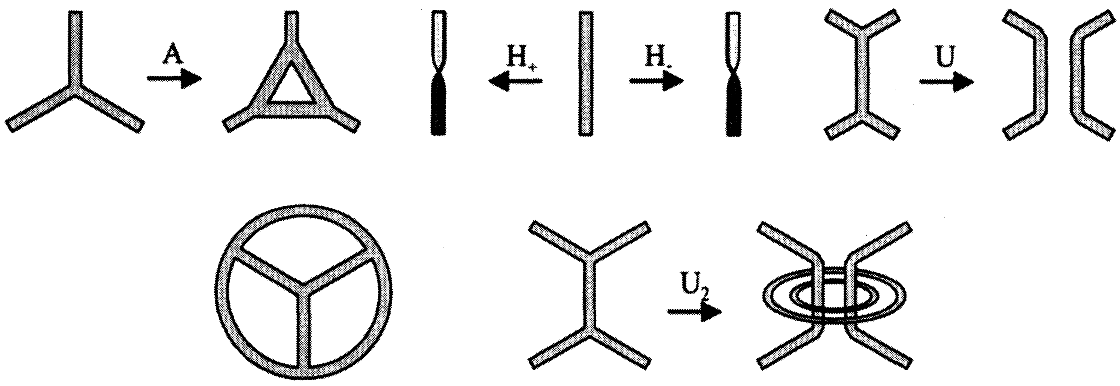

Figure 3: First row: the four KTG moves: triangle A, positive and negative half twists $H_\pm$ and Unzip U. Second row: the standard tetrahedron and the b–unzip U_b (we have drawn the case $b = 2$).

Definition 8. *The following four operations on spin networks will be called the KTG moves, see figure 3. The triangle move A replaces a vertex by a triangle, the positive half twist move H_+ inserts a positive half twist into an edge, the negative half twist H_- inserts a negative half twist and finally the unzip move U takes an edge and splices it into two parallel edges.*

We also define the following variations on the unzip move called the b–unzip U_b. This is the unzip together with the addition of b parallel rings encircling the two unzipped strands.

The four KTG moves defined above are sufficient to generate all spin networks (Γ, n) starting from the standard n-colored tetrahedron graph shown in figure 3.

Lemma 2 ([V]). *Any spin network (Γ, n) can be obtained from the standard n-colored tetrahedron using the KTG moves only.*

According to this lemma we can work with spin networks by studying sequences of KTG moves. Of course there are many inequivalent ways to produce the same spin network using the KTG moves.

Now we can define the notion of an augmented spin network.

Definition 9. *Let S be a sequence of KTG moves. Define the singly augmented spin network corresponding to S to be the spin network obtained from the standard tetrahedron by the moves of S except that all unzip moves are to be replaced by $1-$unzip moves. We will denote the singly augmented spin network corresponding to S by Γ'_S.*

Likewise the b-augmented spin networks corresponding to S are defined to be all the spin networks that can be produced from the standard tetrahedron by the moves of S except that every unzip move is to be replaced by an m-unzip move, where $m \geq n$. Note that one may choose a different m for all unzip moves in S.

Let Γ_S be the spin network obtained from a sequence of KTG moves S and let Ψ be a b–augmented spin network corresponding to S. Then Γ_S is contained in Ψ and $\Psi - \Gamma_S$ is an r–fold unlink. Here is r the number of rings that were added to Γ_S to obtain the augmented spin network Ψ. The number r is called the number of augmentation rings of Ψ.

With all definitions in place we can now formulate the result.

Theorem 3 (vdV).
Let S be a sequence of KTG moves. There exists an $n \in \mathbb{N}$ such that all n-augmented spin networks Γ corresponding to S satisfy the following.

1) *Let t be the number of triangle moves in S and let r be the number of augmentation rings of Γ. Let θ be the number of half twists counted with sign and define the following numbers.*

$$\phi_N = (-1)^{\frac{N-1}{2}} e^{\frac{N^2-1}{4N}\pi i} \quad \text{and} \quad \mathrm{sixj}_N = \sum_{k=0}^{\frac{N-1}{2}} \begin{bmatrix} \frac{N-1}{2} \\ k \end{bmatrix}^4 (e^{\frac{\pi i}{2N}})$$

The normalized N-colored Jones invariant of Γ satisfies:

$$\frac{\langle \Gamma, N\rangle}{\langle O, N\rangle}(e^{\frac{\pi i}{2N}}) = \begin{cases} \phi_N^\theta N^r \mathrm{sixj}_N^{t+1} & \text{if } N \text{ is odd} \\ 0 & \text{if } N \text{ is even} \end{cases}$$

2) *The JSJ–decomposition of the complement of Γ consists of the complement of Γ'_S and a Seifert fibered piece for every n–unzip used in the construction of Γ such that $n \geq 2$. It follows that* $\mathrm{Vol}(\Gamma) = \mathrm{Vol}(\Gamma'_S)$

   Moreover the exterior of Γ'_S is hyperbolic with geodesic boundary and can be obtained explicitly by gluing $2t + 2$ regular ideal hyperbolic octahedra.

3) *Γ satisfies the volume conjecture (restricting to even colors).*

The proof consists of interpreting the KTG moves in two ways. First in terms of the spin network evaluations (fusion or recoupling theory) and second in terms of a specific geometric decomposition of the complement of the network [V]. This reduces the case of augmented spin networks to the case of the tetrahedron or $6j$-symbol for which a the conjecture was already known before [C2].

As can be seen from the above theorem, it is essential to restrict to even colors N. Otherwise limit will not exist or not equal the volume. The even N correspond with the representations that come from $SO(3)$.

Finally let us note an immediate corollary.

Corollary 3. *For every spin network Γ there is a spin network Ψ containing Γ such that $\Psi - \Gamma$ is an unlink and Ψ satisfies the volume conjecture. If Γ happens to be a link then so is Ψ.*

The volume conjecture for spin networks can also be stated for spin networks for which γ is not constant. This will be the subject of future research.

Acknowledgement

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