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Pallets and coloring invariants for spatial graphs

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Abstract

We introduce the notion of pallets of quandles and define coloring invariants for spatial graphs which give a generalization of Fox colorings studied in [4]. And we show a result for pallets of dihedral quandles, which implies that all possible coloring conditions around vertices for Fox colorings are classified.

1 Spatial graphs

For an embedding of a graph to the 3-dimensional Euclidean space $\mathbb{R}^3$, the image is called a spatial graph. Two spatial graphs are *equivalent* if we can deform by ambient isotopy one onto the other. A diagram of a spatial graph G is an image of G by a regular projection onto a plane with a crossing information at each double point. It is known that two spatial graph diagrams represent an equivalent spatial graph if and only if they are related by a finite sequence of the R1-5 moves depicted in Figure 1. Each edge of a spatial graph is separated into some pieces in a diagram. We call each piece an arc of the diagram.

2 Pallets of quandles and coloring invariants

A *quandle* [5, 6] is a set X equipped with a binary operation $(a,b)\mapsto a^b$ on X satisfying the following conditions: (i) For any $a\in X$, the formula $a^a=a$ holds, (ii) for any $a\in X$, the map $S_a:X\to X$ defined by $S_a(x)=x^a$ is a bijection, and (iii) for any $a,b,c\in X$, the formula $(a^b)^c=(a^c)^{(b^c)}$ holds.

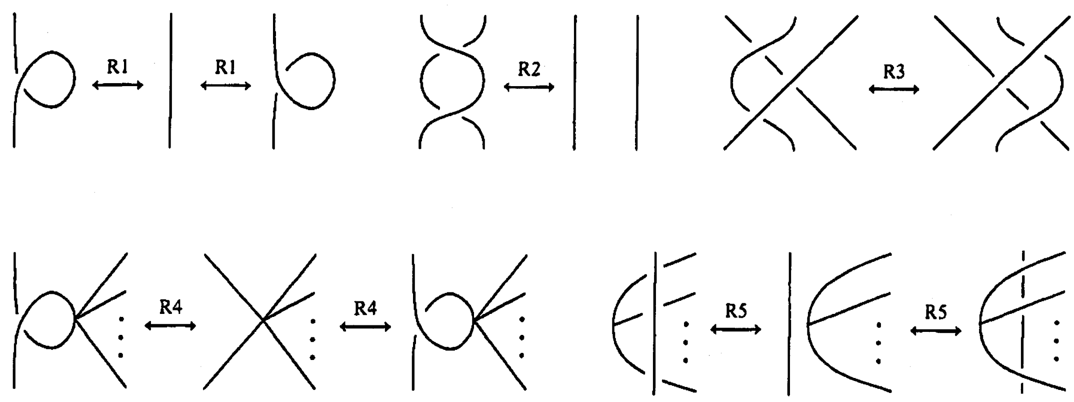

Figure 1: Elementary moves

We omit round brackets throughout this paper and we call the bijection S_a ($a \in X$) defined in (ii) the *symmetry by* a. A *dihedral quandle* of order p ($p \geq 3$) is the set $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ equipped with the quandle operation $a^b = 2b - a$. We denote it by R_p. We see that all symmetries of R_p are involutions of R_p. We say a quandle such that all symmetries are involutions an *involutory quandle.*

We mean by $\mathbb{Z}_+$ the set of the positive integers throughout this paper.

Definition 2.1 Let X be a quandle. For any element a in X, we denote simply by a^{+1} the pair (a, S_a) of a and the symmetry S_a by a, and by a^{-1} the pair (a, S_a^{-1}) of a and the inverse map of S_a. Let

$$\mathcal{X} = \{a^{+1} \mid a \in X\} \cup \{a^{-1} \mid a \in X\}.$$

A *pallet* of X is a subset P of $\cup_{n \in \mathbb{Z}_+} \mathcal{X}^n$ satisfying the following conditions:

(i) for any $(a_1^{\epsilon_1}, \cdots, a_n^{\epsilon_n}) \in P$, it holds that

$$(a_2^{\epsilon_2}, \cdots, a_n^{\epsilon_n}, a_1^{\epsilon_1}) \in P$$

(ii) for any $(a_1^{\epsilon_1}, \cdots, a_n^{\epsilon_n}) \in P$, it holds that

$$S_{a_n}^{\epsilon_n} \circ \cdots \circ S_{a_1}^{\epsilon_1} = \mathrm{id},$$

(iii) for any $(a_1^{\epsilon_1}, \cdots, a_n^{\epsilon_n}) \in P$ and any $x \in X$, it holds that

$$(S_x(a_1)^{\epsilon_1}, \cdots, S_x(a_n)^{\epsilon_n}) \in P \text{ and } (S_x^{-1}(a_1)^{\epsilon_1}, \cdots, S_x^{-1}(a_n)^{\epsilon_n}) \in P,$$

(iv) for any $(a_1^{\epsilon_1}, \cdots, a_n^{\epsilon_n}) \in P$, it holds that

$$(a_2^{\epsilon_2}, S_{a_2}^{\epsilon_2}(a_1)^{\epsilon_1}, a_3^{\epsilon_3}, \cdots, a_n^{\epsilon_n}) \in P \text{ and } (S_{a_1}^{-\epsilon_1}(a_2)^{\epsilon_2}, a_1^{\epsilon_1}, a_3^{\epsilon_3}, \cdots, a_n^{\epsilon_n}) \in P.$$

For any $n \in \mathbb{Z}_+$, we call a pallet which is a non-empty subset of $\mathcal{X}^n$ an *n-pallet.*

Example 2.2 For any $n \in \mathbb{Z}_+$, let

$$U_n = \{(a_1^{\epsilon_1}, \cdots, a_n^{\epsilon_n}) \in \mathcal{X}^n \mid S_{a_n}^{\epsilon_n} \circ \cdots \circ S_{a_1}^{\epsilon_1} = \mathrm{id}\}.$$

This set is a pallet of X and we call it the *universal n-pallet of X*. Let

$$U = \cup_{n \in \mathbb{Z}_+} U_n.$$

This is also a pallet of X. Since it includes any pallet as a subset, we call it the *universal pallet of X*.

Assume that X is an involutory quandle. Since it holds that $a^{+1} = a^{-1}$ for any $a \in X$, we may omit the superscripts $+1$ or -1 of the elements of $\mathcal{X}$. For any $n \in \mathbb{Z}_+$, let

$$C_n = \{(a_1, \ldots, a_n) \in \mathcal{X}^n \mid S_{a_n}^{\epsilon_n} \circ \cdots \circ S_{a_1}^{\epsilon_1} = \mathrm{id} \text{ and } a_1 = \cdots = a_n\}.$$

This is a pallet of X and we call it the *classical n-pallet of X*. Let

$$C = \cup_{n \in \mathbb{Z}_+} C_n.$$

This is also a pallet of X and we call it the *classical pallet of X*.

Each pallet gives a coloring invariant for spatial graphs:

Let G be an oriented spatial graph embedded in $\mathbb{R}^3$ and D be a diagram of G. Let P be a pallet of a quandle X.

An *X-coloring* of D associated with P is an assignment of an element of X to each arc of D satisfying the following conditions:

- Around a crossing c, let e_o be the over arc, e_r the under arc which is on the right side of e_o along the orientation of e_o, and e_l be the other under arc. Suppose that the arcs e_o, e_r and e_l are colored by a_1, a_2 and a_3, respectively. Then it holds that $a_2^{a_1} = a_3$ (Figure 2).

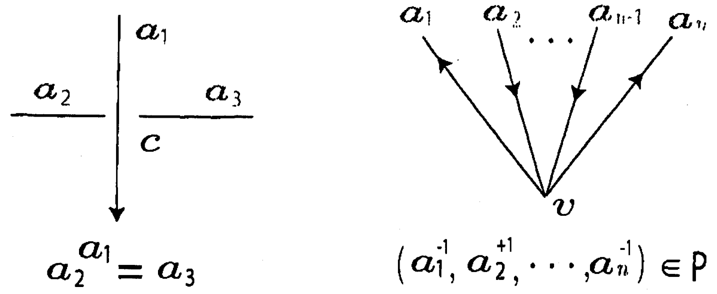

Figure 2: Coloring conditions

- For an n-valent vertex v, let $e_1, \ldots, e_n$ be the arcs which are situated clockwise around v. Let $a_1, \ldots, a_n$ be the elements of X assigned to the arcs $e_1, \ldots, e_n$, respectively. Then it holds that $(a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n}) \in P$, where for each $i \in \{1, \ldots, n\}$, ϵ_i is $+1$ if the arc e_i is directed in toward v, and it is -1 if the arc is directed out (Figure 2).

Let $\mathrm{Col}_{X,P}(D)$ be the set of X-colorings of D associated with P. We have the following proposition:

Proposition 2.3 *Let D and D' be diagrams which represent the same spatial graph. Then there is a bijection between $\mathrm{Col}_{X,P}(D)$ and $\mathrm{Col}_{X,P}(D')$.*

Proof. Suppose that D and D' are diagrams related by a single move among R1-5 moves shown in Figure 1. Let E be a 2-disk in $\mathbb{R}^2$ in which the single move is applied. For each X-coloring of D associated with P, its restriction to $D \backslash E (= D' \backslash E)$ can be uniquely extended to an X-coloring of D' associated with P. Thus there is a bijection between the sets $\mathrm{Col}_{X,P}(D)$ and $\mathrm{Col}_{X,P}(D')$. ∎

By Proposition 2.3, we see that the number of the elements of $\mathrm{Col}_{X,P}(D)$ is an invariant for spatial graphs. Therefore we also denote the invariant by $\#\mathrm{Col}_{X,P}(G)$.

Remark 2.4 When X is an involutory quandle, for any pallet P of X, the coloring invariant by X and P does not depend on orientations of spatial graphs. Therefore we can define a coloring invariant for un-oriented spatial graphs.

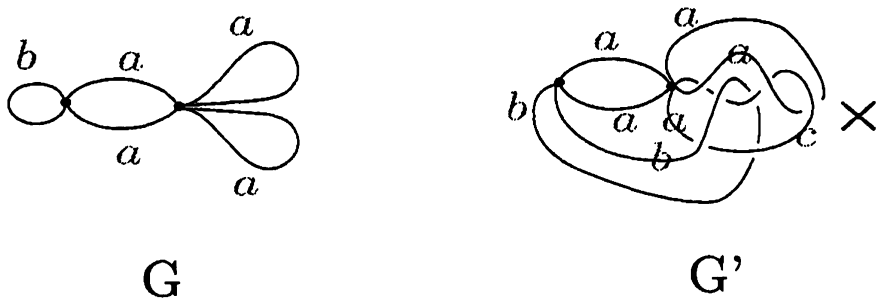

Figure 3: Coloring invariants

Example 2.5 Let G and G' be the spatial graphs shown in Figure 3. Let D and D' be diagrams of G and G', respectively. We consider R_3-colorings of D and D' with some pallets. Since the dihedral quandle R_3 is the involutory quandle, it holds that $a^{+1} = a^{-1}$ for any $a \in R_3$. Hence we omit the superscripts $+1$ or -1 of the elements of $\mathcal{X} = \{0^{+1}(= 0^{-1}), 1^{+1}(= 1^{-1}), 2^{+1}(= 2^{-1})\}$.

Let P be the classical pallet of R_3. Then we can not distinguish the spatial graphs G and G' with the above coloring invariant because it holds that $\#\mathrm{Col}_{R_3,P}(G) = \#\mathrm{Col}_{R_3,P}(G')$.

Replace the pallet P as follows: Let

$$\begin{aligned} P_4 = \ & \{(0,0,1,1),(0,0,2,2),(0,1,0,2),(0,1,1,0),(0,1,2,1),(0,2,0,1), \\ & (0,2,1,2),(0,2,2,0),(1,0,0,1),(1,0,1,2),(1,0,2,0),(1,1,0,0), \\ & (1,1,2,2),(1,2,0,2),(1,2,1,0),(1,2,2,1),(2,0,0,2),(2,0,1,0), \\ & (2,0,2,1),(2,1,0,1),(2,1,1,2),(2,1,2,0),(2,2,0,0),(2,2,1,1)\} \end{aligned}$$

and

$$P_6 = \{(a_1, \dots, a_6) \in R_3^6 \mid a_1 = \cdots = a_6\},$$

and let $P = P_4 \cup P_6$. Then $\#\mathrm{Col}_{R_3,P}(G) = 6$ and $\#\mathrm{Col}_{R_3,P}(G') = 0$, see Figure 3. Hence the spatial graphs G and G' are not equivalent.

We can also distinguish the spatial graphs G and G' with the universal pallet of R_3. But the calculation is complicated compared with that using the above pallet. Thus, choosing a pallet makes the calculation simple.

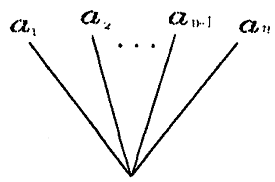

Figure 4: Coloring conditions

3 Fox colorings

Fox colorings [1, 2, 3] are defined for diagrams of classical links. For an integer $p \geq 3$, we consider an assignment of an element of $\mathbb{Z}_p$ to each arc of a classical link diagram. It is called a *Fox p-coloring* if a coloring condition for crossings is satisfied. Then the coloring condition is given as follows: It holds that $a + c = 2b$ in $\mathbb{Z}_p$ near each crossing, where the lower arcs are colored by a and c and the upper arc is colored by b.

As a generalization, Ishii and Yasuhara [4] introduced Fox colorings for spatial graphs such that the valency of each vertex is even. The additional coloring condition is to satisfy $a_1 = \cdots = a_n$ for an n-valent vertex whose arcs are colored as shown in Figure 4. The Fox colorings are the same as the dihedral quandle colorings with the classical pallets. And they also studied Fox colorings for spatial graphs such that the coloring condition for vertices is given as $\sum_{i=1}^{n}(-1)^i a_i = 0$ for an n-valent vertex whose arcs are colored as shown in Figure 4. The Fox colorings are also given by using pallets, that is, we use the following pallet of R_p for R_p-colorings:

$$P = \{(a_1, \ldots, a_n) \in U_p \mid n \in 2\mathbb{Z}_+, \sum_{i=1}^{n}(-1)^i a_i = 0\}.$$

Thus, Fox colorings for spatial graphs are translated as dihedral quandle colorings with pallets, and each pallet gives a coloring condition for vertices. Now, we have the following question: *For Fox colorings of spatial graphs, is it possible to give any other coloring conditions for vertices?* The question is translated as the following question: *For dihedral quandles, is there any other pallets except for the above two pallets?* Our main theorem in the section 4 says "Yes".

4 Main theorem

In this section, we classify all n-pallets of dihedral quandles.

For any integers $n > 0$ and $p \geq 3$, define $\varphi_{n,p} : R_p^n \to \{1, \ldots, p\}$ by

$$\varphi_{n,p}(a_1, \cdots, a_n) = \max\{k \in \{1, \ldots, p\} \mid k|p,\ a_1 \equiv \cdots \equiv a_n \pmod{k}\}.$$

When p is an even number, define $\kappa_{n,p} : R_p^n \to \mathbb{Z}_p$ by

$$\kappa_{n,p}(a_1, \ldots, a_n) = \sum_{i=1}^{n} (-1)^i a_i,$$

and define $\mu_{n,p} : R_p^n \to \mathbb{Z}$ by

$$\mu_{n,p}(a_1, \ldots, a_n) = E[(a_1, \ldots, a_n)] - O[(a_1, \ldots, a_n)],$$

where

$$E[(a_1, \ldots, a_n)] = \#\{i \in \{1, \cdots, n\} \mid a_i \equiv 0 \pmod{2}\}$$

and

$$O[(a_1, \ldots, a_n)] = \#\{i \in \{1, \cdots, n\} \mid a_i \equiv 1 \pmod{2}\}.$$

Let $k \in \{1, \ldots, p\}$ be an even divisor of p and let

$$S_k = \{\mathbf{a} \in R_p^n \mid \varphi_{n,p}(\mathbf{a}) = k\}.$$

We define $\epsilon_{n,p,k} : S_k \to \{0, 1\}$ by

$$\epsilon_{n,p,k}(a_1, \ldots, a_n) = \begin{cases} 0 & \text{if } a_1 \equiv \cdots \equiv a_n \equiv 0 \pmod{2}, \\ 1 & \text{if } a_1 \equiv \cdots \equiv a_n \equiv 1 \pmod{2}. \end{cases}$$

Let $k \in \{1, \ldots, p\}$ be an even divisor of p such that p/k is an even number. We define $\mu_{n,p,k} : S_k \to \mathbb{Z}$ by

$$\mu_{n,p,k}((a_1, \ldots, a_n)) = \left| \mu_{n,\frac{p}{k}}\left(0, \frac{a_2 - a_1}{k}, \cdots, \frac{a_n - a_1}{k}\right) \right|$$

We have the following theorem:

Theorem 4.1 *Let n and p be integers such that $n > 0$ and $p \geq 3$.*

(i) When n is an odd number, there is no n-pallet of R_p.

(ii) When n is an even number, the set of the n-pallets of R_p is equal to the set which consists of the non-empty subsets of a set V:

$$\{n\text{-}pallets\ of\ R_p\} = \{ \bigcup_{w \in W} w \mid W \subset V, W \neq \emptyset\},$$

where V is the following set.

(1) When $n = 2$ and p is an odd number,

$$V = \{\{(a,a) \mid a \in R_p\}\}.$$

When $n = 2$ and p is an even number such that $p/2$ is an odd number,

$$\begin{aligned} V = \{ & \{(a,a) \mid a \in R_p, a \equiv 0 \pmod 2\}, \\ & \{(a,a) \mid a \in R_p, a \equiv 1 \pmod 2\}, \\ & \{(a, a + \tfrac{p}{2}) \mid a \in R_p\}\}. \end{aligned}$$

When $n = 2$ and p is an even number such that $p/2$ is an even number, we have

$$\begin{aligned} V = \{ & \{(a,a) \mid a \in R_p, a \equiv 0 \pmod 2\}, \\ & \{(a,a) \mid a \in R_p, a \equiv 1 \pmod 2\}, \\ & \{(a, a + \tfrac{p}{2}) \mid a \in R_p, a \equiv 0 \pmod 2\}, \\ & \{(a, a + \tfrac{p}{2}) \mid a \in R_p, a \equiv 1 \pmod 2\}\}. \end{aligned}$$

(2) When n is an even number other than 2 and p is an odd number, we have

$$V = \{\eta_k \mid k \in \{1, \cdots, p\}, k|p\},$$

where $\eta_k = \{\mathbf{a} \in R_p^n \mid \varphi_{n,p}(\mathbf{a}) = k, \kappa_{n,p}(\mathbf{a}) = 0\}$.

(3) When n is an even number other than 2 and p is an even number, we have

$$V = \left\{ \alpha_{k,\kappa,\mu} \;\middle|\; \begin{array}{l} k \in \{1, \cdots, p\},\ k|p,\ k \text{ is odd; } \kappa \in \{0, \frac{p}{2}\}; \\ -n < \mu < n,\ \mu \text{ is even, } \frac{n-|\mu|}{2} \equiv \kappa \pmod 2 \end{array} \right\}$$

$$\cup \left\{ \beta_{k,\epsilon} \;\middle|\; k \in \{1, \cdots, p\},\ k|p,\ k \text{ is even, } \tfrac{p}{k} \text{ is odd; } \epsilon \in \{0,1\} \right\}$$

$$\cup \left\{ \gamma_{k,\kappa,\mu,\epsilon} \;\middle|\; \begin{array}{l} k \in \{1, \cdots, p\},\ k|p,\ k \text{ and } \frac{p}{k} \text{ are even; } \kappa \in \{0, \frac{p}{2}\}; \\ 0 \leq \mu < n,\ \mu \text{ is even, } \frac{n-\mu}{2} \equiv \frac{\kappa}{k} \pmod 2; \epsilon \in \{0,1\} \end{array} \right\},$$

where

$$\alpha_{k,\kappa,\mu} = \{\mathbf{a} \in R_p^n \mid \varphi_{n,p}(\mathbf{a}) = k, \kappa_{n,p}(\mathbf{a}) = \kappa, \mu_{n,p}(\mathbf{a}) = \mu\},$$

$$\beta_{k,\epsilon} = \{\mathbf{a} \in R_p^n \mid \varphi_{n,p}(\mathbf{a}) = k, \kappa_{n,p}(\mathbf{a}) = 0, \epsilon_{n,p,k}(\mathbf{a}) = \epsilon\}, \ \textit{and}$$

$$\gamma_{k,\kappa,\mu,\epsilon} = \left\{ \ \mathbf{a} \in R_p^n \ \middle| \ \varphi_{n,p}(\mathbf{a}) = k, \kappa_{n,p}(\mathbf{a}) = \kappa, \mu_{n,p,k}(\mathbf{a}) = \mu, \epsilon_{n,p,k}(\mathbf{a}) = \epsilon \ \right\}.$$

By the above theorem, we have the following properties:

Corollary 4.2 *When n is an even number such that $n \geq 4$ and p is an odd number, the number of the n-pallets of R_p is equal to $2^t - 1$, where t is the number of the divisors of p. Especially, when p is a prime, we have exactly three n-pallets of R_p: One is the universal n-pallet U_n, another pallet is the classical n-pallet C_n, and the other is the difference set $U_n \setminus C_n$.*

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