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EXCEPTIONAL SURGERIES ON COMPONENTS OF TWO-BRIDGE LINKS

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1. Introduction

We report on the recent result by the author concerning the classification of exceptional Dehn surgeries on a component of a hyperbolic two-bridge link in the 3-sphere S^3.

1.1. **Backgrounds.** Let us start with viewing backgrounds on the study of Dehn surgery.

A 3-dimensional manifold, simply called 3-manifold, is one of the central objects to study in low-dimensional topology. Originally, in 1904, Poincaré raised the famous Poincaré Conjecture for the characterization of the 3-dimensional sphere. It had been a guiding principle in the early study of 3-manifolds. Extending the Poincaré Conjecture, the Geometrization Conjecture was conjectured by Thurston in [17, Conjecture 1.1]. This gave a relationship between the 3-manifold theory and complex analysis, hyperbolic geometry, foliation theory, differential geometry, and so on. Eventually the Geometrization Conjecture was established by Perelman in his celebrated preprints in [11, 12, 13].

Now, for example, we have a following classification of 3-manifolds as a consequence of the geometrization. That is, every closed orientable 3-manifold is a reducible manifold

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(containing essential 2-sphere), a toroidal manifold (containing essential torus), Seifert fibered space (foliated by circles), or a hyperbolic manifold (admitting a Riemmanian metric of constant sectional curvature -1).

Beyond the classification of 3-manifolds, there would be several directions in the future study of 3-manifolds. For example, we can consider:

- Attack the remaining Open Problems. (e.g., Virtually Haken Conjecture [6, Problem 3.2], "Heegaard genus VS rank of π_1" problem [6, Problem 3.92], etc.)
- Relate geometric and topological invariants (quantities) (e.g., Volume conjecture [10, Conjecture 5.1])
- Study the Relationships between 3-manifolds.

One of such relationship between 3-manifolds is given by Dehn surgery defined as follows.

1.2. **Dehn surgery.** A Dehn surgery on a link L in a 3-manifold M is an operation to construct a 3-manifold from M and L as follows. Take the exterior $E(L)$ of L, i.e., remove the interior of the tubular neighborhood $N(L)$ of L from M, and then, glue solid tori to $E(L)$.

This gives an interesting subject to study; for example, it was shown by [7, 18] that any pair of closed orientable 3-manifolds are related by a Dehn surgery on a link.

1.3. **Exceptional surgery.** Another motivation to study Dehn surgery was given by Thurston. He proved the following theorem [16, Theorem 5.8.2], now called the Hyperbolic Dehn Surgery Theorem: On each component of a hyperbolic link, there are only finitely many Dehn surgeries yielding non-hyperbolic manifolds. Remark that this is just a consequence of the original form. In view of this, we say that a Dehn surgery on a hyperbolic link giving a non-hyperbolic manifold an *exceptional surgery.* We here remark that a link is called hyperbolic if the complement $M - L$ admits a complete hyperbolic structure.

In the study of exceptional surgery, one of the most important problems, related to Knot theory, is the following:

Problem 1. *Completely classify the exceptional surgeries on hyperbolic links in the 3-sphere S^3.*

This seems to be considerably challenging, and the following much easier to tackle.

Problem 2. *Completely classify the exceptional surgeries on hyperbolic 2-bridge links in the 3-sphere S^3.*

Actually the class of 2-bridge links gives one of the most well-known and most well-studied family of links in S^3.

1.4. **2-bridge link.** A link in S^3 is called a *2-bridge link* if it admits a diagram with exactly two maxima and minima. See [5] for more details. We will follow the definition and notations about 2-bridge link from [4, 19]. In the following, we denote by $L_{p/q}$ the 2-bridge link associated to a rational number p/q.

We here recall the results about exceptional surgeries on hyperbolic 2-bridge links. Remark that a 2-bridge link is hyperbolic unless it is equivalent to $L_{1/n}$, that is, $(2, n)$-torus link by [8].

On hyperbolic 2-bridge knots, Brittenham and Wu gave in [1] a complete classification of exceptional surgeries. For example, they showed that only 2-bridge knots $K_{[b_1,b_2]}$ admits exceptional surgeries. Here, by $[a_1, a_2, \cdots, a_n]$, we mean a continued fraction expansion following [4].

For 2-bridge links, it follows from the result obtained by Wu in [19]: If a 3-manifold obtained by a Dehn surgery on a component of a 2-bridge link L contains an essential disk, annulus, or 2-sphere, then L is equivalent to $L_{[b_1,b_2]}$. Recall that an embedded disk, annulus, 2-sphere in a 3-manifold is called essential if it is incompressible and not boundary-parallel. We remark that Dehn surgery on a hyperbolic link yielding 3-manifolds with essential disk, annulus, or 2-sphere, is a typical example of exceptional surgery. See the next subsection for details.

Further, in [4], Goda, Hayashi and Song obtained a complete classification (resp. a necessary condition) of 2-bridge links on a component of which a Dehn surgery yields a nontrivial, non-core torus knot exterior or a cable knot exterior (resp. a prime satellite knot exterior) in a lens space.

Based on these results, we set out target the following:

Problem 3. *Completely classify the exceptional surgeries on a component of hyperbolic 2-bridge links in* S^3.

2. Result

To state our result, we fix our notation as follows.

For a knot K in the 3-sphere S^3, by using a standard meridian-longitude system, we have a one-to-one correspondence between the set of slopes on the peripheral torus of K and the set of rational numbers, or equivalently irreducible fractions, with $1/0$. See [14] for example.

Let L be a 2-bridge link. We denote $L(r)$ the manifold obtained by Dehn surgery on a component of L along the slope r, i.e., the r corresponds to the slope determined by the meridian of the attached solid torus.

Next we recall the classification of exceptional surgery on a component of a hyperbolic link. A Dehn surgery on one component of a 2-component hyperbolic link is exceptional, i.e., it yields a non-hyperbolic 3-manifold with torus boundary, if and only if the obtained manifold contains an essential disk, annulus, 2-sphere, or torus. See [16] as the original reference.

Now we give our classification theorem of exceptional surgeries on components of 2-bridge links.

Theorem. *Let L be a hyperbolic 2-bridge link in S^3 and $L(r)$ denote the 3-manifold obtained by Dehn surgery on a component of L along the slope r. Then the following hold.*

(1) *$L(r)$ contains neither essential disk nor 2-sphere.*

(2) *$L(r)$ contains an essential torus if and only if L is equivalent to $L_{[2w,v,2u]}$ and $r = -w - u$ with*
(a) $w = 1, u = -1, |v| \geq 2$,
(b) $w \geq 2, |u| \geq 2, |v| = 1$.
(c) $w \geq 2, |u| \geq 2, |v| \geq 2$.
In all the cases, $L(r)$ is never Seifert fibered, and $L(r)$ gives a graph manifold if and only if the parameters u, v, w satisfies the first and the second conditions.

(3) *$L(r)$ contains an essential annulus, but contains no essential tori, equivalently $L(r)$ is a small Seifert fibered space if and only if L is equivalent to*
(a) *$L_{[3,2u+1]}$ and $r = u$,*
(b) *$L_{[2w+1,3]}$ and $r = -w - 1$,*
(c) *$L_{[3,-3]}$ and $r = -1$, or,*
(d) *$L_{[2w+1,2u+1]}$ and $r = -w + u$*
with $w \geq 1, u \neq 0, -1$.

3. Surfaces in 2-bridge link exterior

Our proof is heavily based on the results on [4] and [2]. In [2], Floyd and Hatcher studied meridionally incompressible essential surfaces in 2-bridge link exteriors, and gave a complete description of such surfaces. See [2] and [4] for details. In the following, we assume that the readers are familiar to a certain extent.

Here a surface F in $E(L)$ is called *meridionally incompressible* if, for any disk $D \subset S^3$ with $D \cap F = \partial D$ and D meeting L transversely in one point in the interior of D, there is a disk $D' \subset F \cup L$ with $\partial D' = \partial D$, D' also meeting L transversely in one interior point.

To prove our theorem, a key investigation is to study essential surfaces embedded in 2-bridge link exteriors of genus at most one. Most parts of such studies have been achieved in [4]. Our advantage is the following lemma obtained by using the machinery of [2].

Lemma 1. *If a hyperbolic 2-bridge link exterior contains a meridionally incompressible essential planer surface F with at most two meridional boundaries on a component of the link and non-empty boundary on the other component if and only if the link is equivalent to $L_{[2,n,-2]}$ with $|n| \geq 2$ and F is an essential two punctured disk with two meridional punctures on a component on the link and a single longitudinal boundary on the other component.*

4. Outline of Proof

Let $L = K_1 \cup K_2$ be a hyperbolic 2-bridge link in S^3 and $L(r)$ denote the 3-manifold obtained by Dehn surgery on $K_1 \subset L$ along the slope r. Note that, since the component K_2 remains unfilled, $L(r)$ has a torus boundary component. Also note that it is known by [8] that L is hyperbolic if and only if L is not equivalent to $L_{1/n}$ for some integer n.

Now suppose that $L(r)$ is non-hyperbolic. Then, as remarked before, $L(r)$ contains an essential disk, sphere, annulus or torus.

In the following, we give our proof of the theorem divided into four claims.

Claim 1. *There are no essential sphere in $L(r)$.*

Proof. Suppose for a contrary that there exists an essential sphere in $L(r)$. Then, by the standard argument, the link exterior $E(L)$ contains a connected, orientable, essential (i.e., incompressible and ∂-incompressible), properly embedded planer surface F. The surface F has non-empty boundary components on $\partial N(K_1)$ with boundary slope r and no boundary components on $\partial N(K_2)$.

First suppose that F is meridionally incompressible. Then, by [2, Theorem 3.1 (a)], the surface F is carried by a branched surface Σ_γ for some minimal edge-path γ in the diagram D_t in [2]. See also [4]. In this case, we can apply the argument given in [4, Lemma 12.1]. Then we see that the minimal edge-path γ is in D_∞ and is composed of only two edges with label B with endpoints $1/0$ and p/q, where $L_{p/q}$ is equivalent to L. However, as seen in [2, Figure 1.1] or [4, Figure 2], it implies that $L_{p/q}$ is equivalent to $L_{\pm 1/m}$ for some m, contradicting L is hyperbolic.

Next suppose that F is meridionally compressible. Perform meridional compressions as possible. It can be checked by the standard argument that meridional compressions preserve essentiality of surfaces. Then, since any boundary curve of a meridionally compressing disk is separating on F, there must exist some component which is meridionally incompressible essential planer surface with single meridional boundary on $\partial N(K_2)$ and with non-empty boundaries on $\partial N(K_1)$. However, by Lemma 1, such a surface must have exactly two meridional boundaries on $\partial N(K_2)$. A contradiction occurs. □

Claim 2. *There are no essential disk in $L(r)$.*

Proof. Suppose for a contrary that there exists an essential disk in $L(r)$. It follows that there is a compressible disk for $\partial L(r)$ in $L(r)$. By compression, $L(r)$ must be a solid torus. Otherwise we would have an essential sphere in $L(r)$ contradicting Claim 1.

Then, considering the exterior of K_2, we can regard K_1 as a knot in a handlebody. Since the surgery on K_1 yields a solid torus again, by the result given in [3], K_1 is either a 0 or 1-bridge braid in the solid torus $E(K_2)$. Then, together with the result of [9, Proposition 3.2], K_1 must be knotted in S^3. This contradicts that L is a 2-bridge link. □

Claim 3. *There exists an essential torus in $L(r)$ if and only if L is equivalent to $L_{[2w,v,2u]}$ and $r = -w - u$ with*

(1) $w = 1, u = -1, |v| \geq 2$,
(2) $w \geq 2, |u| \geq 2, |v| = 1$.
(3) $w \geq 2, |u| \geq 2, |v| \geq 2$.

In all the cases, $L(r)$ is never Seifert fibered, and $L(r)$ gives a graph manifold if and only if the parameters u, v, w satisfies the first and the second conditions.

Proof. Suppose that there exists an essential torus in $L(r)$.

As seen in the proof of Claim 1, the link exterior $E(L)$ contains a connected, orientable, essential properly embedded surface F of genus one with non-empty boundaries on $\partial N(K_1)$ with boundary slope r and no boundary components on $\partial N(K_2)$.

First suppose that F is meridionally incompressible. Then, by [2, Theorem 3.1 (a)], the surface F is carried by a branched surface Σ_γ for some minimal edge-path γ in the diagram D_t in [2]. See also [4]. Again we can apply the argument given in [4, Lemma 12.1]. Then, in this case, γ has length 4 in D_∞ with endpoints $1/0$ and p/q, where $L_{p/q}$ is equivalent to L. As claimed in the proof of [4, Theorem 1.5], $L_{p/q}$ must be equivalent to $L_{[2w,v,2u]}$ with $w \geq 2, |v| \geq 1, |u| \geq 2$.

It remains to show that $L_{[2w,v,2u]}$ actually contains essential torus for $w \geq 2, |v| \geq 1, |u| \geq 2$. By imitating the arguments used in the proofs of [19, Theorem 5.1] and [4, Theorem 11.1], it can be checked directly from the illustration that the manifold obtained by the surgery is homeomorphic to the exterior of a satellite knot in a lens space. We here omit the details. Moreover, in the case where

$|v| \neq 1$ (resp. $|v| = 1$), we can see that the companion knot is a torus knot and the pattern knot is a hyperbolic knot (resp. a cable knot). See also [4, Theorem 11.1] in the case where $|v| = 1$. Note that we have $L_{[2w,\pm 1,2u]}(-w-u) \equiv L_{[2w'+1,2u'+1]}(-w'+u' \pm 1)$ for some w' and u'.

Next suppose that F is meridionally compressible. As in the proof of Claim 1, perform meridional compressions as possible. It can be checked by the standard argument that meridional compressions preserve essentiality of surfaces. If some boundary curve of a meridionally compressing disk on F is separating, then the same contradiction could occur as in Claim 1, and so, there must be single meridional compression for F along the non-separating curve on F. Then, by Lemma 1, the link is equivalent to $L_{[2,n,-2]}$ with $|n| \geq 2$ and F is an essential two punctured disk with two meridional punctures on $\partial N(K_2)$ and a single longitudinal boundary on $\partial N(K_1)$. Actually, by tubing operation, we can find a once-punctured torus or klein bottle embedded in $E(L)$ coming from a spanning surface for K_1.

Conversely, we can see that 0-surgery on $K_1 \subset L_{[2,n,-2]}$ with $|n| \geq 2$ gives the exterior of a knot in $S^2 \times S^1$. This knot intersects the level horizontal sphere in $S^2 \times S^1$ transversely twice. This implies that the knot exterior contains a meridional incompressible annulus. By tubing operation, we have a non-separating incompressible torus or klein bottle in the knot exterior.

It can be checked by the Montesinos trick technique for the surgery on $K_1 \subset L_{[2,n,-2]}$ that the manifold so obtained is a graph manifold. The verification of the details are remained to the reader. □

Claim 4. *There exists an essential annulus, but no essential torus in $L(r)$ if and only if $L(r)$ is a small Seifert fibered space and L is equivalent to*

(1) $L_{[3,2u+1]}$ *and* $r = u$,
(2) $L_{[2w+1,3]}$ *and* $r = -w - 1$,
(3) $L_{[3,-3]}$ *and* $r = -1$, *or,*
(4) $L_{[2w+1,2u+1]}$ *and* $r = -w + u$

with $w \geq 1, u \neq 0, -1$.

Proof. Suppose that there exists an essential annulus but no essential torus in $L(r)$. Then it is known that $L(r)$ must be a small Seifert fibered space.

Let r_2 be the slope on $\partial N(K_2)$ determined by the boundary of the essential annulus. Then it is shown that $r_2 \neq 1/0$ as follows. Suppose for a contrary that $r_2 = 1/0$, i.e., r_2 is meridional. Now we are assuming that $L(r)$ is a Seifert fibered space, and the essential annulus coming from the surface F must be vertical. This implies that the meridian of K_2 is a regular fiber of the Seifert fibration of $L(r)$. Then, as shown in [15, Proof of Corollary 2.6], K_2 must be a core knot in the lens space. However it contradicts that $L(r)$ is not a solid torus as claimed before.

Thus we see that $r_2 \neq 1/0$. Then, as also shown in [15, Proof of Corollary 2.6], K_2 gives a non-trivial non-core torus knot in a lens space. In this case, if we perform suitable surgery on K_2, we have a reducible manifold, equivalently, a suitable surgery on the 2-bridge link L yields a reducible manifold. Then, as a consequence of [19, Theorem 5.1], L must be equivalent to a 2-bridge link corresponding to a continued irreducible fraction of length two.

Now we can apply [4, Theorem 11.1], which establishes a complete classification of such 2-bridge links and surgery slopes on which surgeries yield non-trivial non-core torus knots in lens spaces. This gives us the desired conclusions. □

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