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Chern numbers associated with semi-quantum systems with symmetry

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Abstract

Most of Hamiltonians of semi-quantum systems treated in this article take the form of 2×2 traceless Hermitian matrices defined on S^2 and are invariant under subgroups of $SO(3)$, where S^2 is viewed as a space of classical variables and the Hermitian matrices as quantum variables. A subgroup G of $SO(3)$ which is assigned to a semi-quantum system acts simultaneously both on S^2 and on the Hermitian matrix by adjoint action. Subgroups of $SO(3)$ to be considered here are $SO(2)$, D_3, and O, where D_3 and O denote a symmetry group for an equilateral triangle and that for a regular octahedron (or for a cube, equivalently), respectively, and are isomorphic with the symmetric groups S_3 and S_4, respectively.

While an invariant Hamiltonian is not unique to each group G, suitable one will be chosen with control parameters included. For each Hamiltonian thus chosen, associated are two real eigenvalues and eigenvectors. The eigenvalues, which are functions on the sphere, are non-degenerate on the sphere in general, and the associated eigenvectors determine respective line bundles over the sphere S^2, for which Chern numbers are attached. Since the Hamiltonian depends on parameters, it may occur that the eigenvalues are degenerate on some points of the sphere for some parameter values and thereby the line bundles associated with the degenerate eigenvalues cannot be assigned over the whole sphere. Such a set of parameters is called a degeneracy point of the parameter space, and there may exist a finite set of or a continuum of degeneracy points in the parameter space. The points other than degeneracy ones in the parameter space are called regular points, for each of which a line bundles is associated with an eigenvalue together with a Chern number. Interest will center on the parameter space which is divided into several regular regions and on the Chern numbers assigned to respective regions.

1 Introduction

The present article results from joint works with Prof. Boris Zhilinskii [1], [2]. However, the content of this article is rather a supplement to those.

The organization of this article is as follows: Section 2 is concerned with quantum systems describing the coupling between the angular momentum operators and the spin operators. The Clebsh-Gordan formula for the tensor product representation of $SU(2)$ is a key theme in this section. The interest of this section centers on a one-parameter family of Hamiltonians describing the coupling and on the redistribution of eigenvalues which occurs depending on the variation of the parameter.

Section 3 deals with semi-quantum systems which are obtained by averaging the angular momentum operators with coherent states for $SU(2)$. With a choice of the representation of the spin operators, we obtain a Hermitian matrix defined on the two-sphere with the radius depending on the representation parameter for the averaged operators. We are interested in line bundles associated with eigenvalues of the Hermitian matrix mentioned above. It is shown that the Chern number of the line bundle is related to the redistribution of the eigenvalues of the Hamiltonian treated in Sec. 2.

In Secs. 4 to 6, Hamiltonians take the form of 2×2 traceless Hermitian matrices. Section 4 is concerned with Chern numbrs associated with a Hamiltonian with $U(1)$ symmetry. In Section 5, Chern numbrs are studied in the presence of D_3 symmetry. In Section 6, Chern numbrs are treated in the presence of O symmetry. Results stated in Secs. 4 and 5 are taken from [1]. The result stated in Sec. 6 is the same as that given in [2], but the proof of it is different from that in [2].

2 One-parameter quantum systems

We start with a review of the Clebsh-Gordan formula. Let (V_ℓ, D^ℓ) be a unitary irreducible representation of $SU(2)$, where $\dim V_\ell = 2\ell + 1$ with $\ell \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\}$. Let J_k and S_k denote $su(2)$ operators acting on V_j and V_s, respectively, $k = 1, 2, 3$. We view $\boldsymbol{J} = (J_k)$ and $\boldsymbol{S} = (S_k)$ as the angular momentum operators and the spin operators, respectively.

$SU(2)$ acts on $V_j \otimes V_s$ unitarily, whose infinitesimal generators are expressed as

$$\boldsymbol{N} = \boldsymbol{J} \otimes \mathbf{1} + \mathbf{1} \otimes \boldsymbol{S}. \tag{1}$$

As is well known, according to the Clebsh-Gordan formula, the representation space $V_j \otimes V_s$ is decomposed into

$$V_j \otimes V_s \cong V_{j+s} \oplus \cdots \oplus V_{|j-s|}. \tag{2}$$

The V_n, $|j - s| \leq n \leq j + s$, are known to be the eigenspaces of the squared operator $\boldsymbol{N}^2$ associated with the eigenvalue $n(n+1)$. The $\boldsymbol{N}^2$ is expanded and expressed as

$$\boldsymbol{N}^2 = \boldsymbol{J}^2 \otimes \mathbf{1} + 2\boldsymbol{J} \otimes \boldsymbol{S} + \mathbf{1} \otimes \boldsymbol{S}^2. \tag{3}$$

The V_n are also eigenspaces of the coupled operator $\boldsymbol{J}\otimes\boldsymbol{S}$ associated with the eigenvalue [3]

$$\frac{1}{2}\big(n(n+1)-j(j+1)-s(s+1)\big).$$

We proceed to study the one-parameter quantum operator given by

$$H_\tau = (1-\tau)\mathbf{1}\otimes S_z + \tau \boldsymbol{J}\otimes\boldsymbol{S}, \quad 0\le\tau\le 1. \tag{4}$$

We here give an example. For $j=1$ and $s=\frac{1}{2}$, the Hamiltonian H_τ takes the form

$$H_\tau = \begin{pmatrix} \frac{1}{2} & & & & & \\ & -\frac{1}{2} & \frac{\tau}{\sqrt{2}} & & & \\ & \frac{\tau}{\sqrt{2}} & \frac{1}{2}-\frac{\tau}{2} & & & \\ & & & -\frac{1}{2}+\frac{\tau}{2} & \frac{\tau}{\sqrt{2}} & \\ & & & \frac{\tau}{\sqrt{2}} & \frac{1}{2}-\tau & \\ & & & & & -\frac{1}{2}+\tau \end{pmatrix}. \tag{5}$$

The six eigenvalues as functions of τ with the range $0\le\tau\le 1$ are given by

$$\begin{cases} \dfrac{1}{2}, \\ -\dfrac{1}{2}+\tau, \\ -\dfrac{\tau}{4}\pm\sqrt{\dfrac{1}{4}-\dfrac{t}{4}+\dfrac{9\tau^2}{16}}, \\ -\dfrac{\tau}{4}\pm\sqrt{\dfrac{1}{4}-\dfrac{3t}{4}+\dfrac{17\tau^2}{16}}, \end{cases} \tag{6}$$

and shown in Fig.1.

This figure illustrates how the eigenspaces of the Hamiltonian changes according to $V_1\otimes|\frac{1}{2}\rangle\oplus V_1\otimes|-\frac{1}{2}\rangle \to V_{\frac{1}{2}}\oplus V_{\frac{3}{2}}$, as τ varies from 0 to 1, where $|\pm\frac{1}{2}\rangle$ denote a basis of $V_{\frac{1}{2}}$. The multiplicity of the upper level and that of the lower level at $\tau=0$ are both as may as $\dim V_1=3$. When $\tau=1$, they are $\dim V_{\frac{3}{2}}=4$ and $\dim V_{\frac{1}{2}}=2$, respectively. From Fig. 1, we observe that one eigenvalue leaves the lower level at $\tau=0$ and get into the upper level at $\tau=1$. This is also seen from the dimensionality relation

$$\dim V_{\frac{1}{2}} = \dim V_1 - 1, \quad \dim V_{\frac{3}{2}} = \dim V_1 + 1. \tag{7}$$

In general, as the parameter τ varies from 0 to 1, the eigenspace decomposition of the tensor product space $V_j\otimes V_s$ with respect to H_τ changes according to

$$\begin{array}{c} V_j\otimes|s\rangle\oplus\cdots\oplus V_j\otimes|-s\rangle \\ \text{w.r.t. } H_0=\mathbf{1}\otimes S_z \end{array} \longrightarrow \begin{array}{c} V_{j+s}\oplus\cdots\oplus V_{|j-s|} \\ \text{w.r.t. } H_1=\boldsymbol{J}\otimes\boldsymbol{S} \end{array}, \tag{8}$$

where $|r\rangle := |s\ r\rangle$, $|r|\le s$ are the eigenvectors of S_z, and $j>s$. The number of the component spaces of each decomposition is the same, but the dimensions of respective

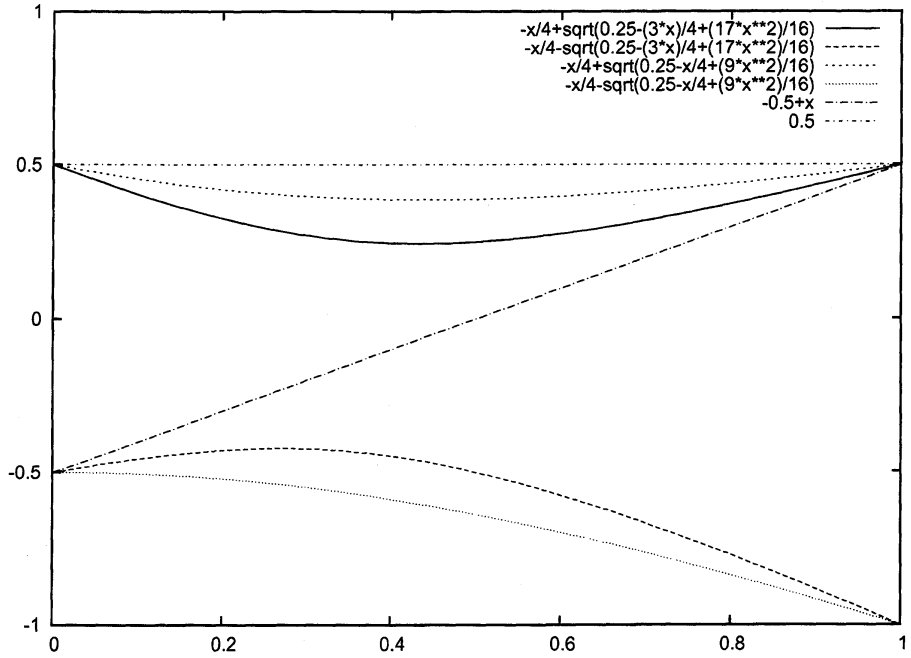

Figure 1: The redistribution of eigenvalues for $j = 1, s = \frac{1}{2}$

spaces are different from one another. According to the Clebsch-Gordan decomposition, we have the relation among their dimensions,

$$\dim(V_j \otimes V_s) = \sum_{n=|j-s|}^{j+s} \dim V_n, \quad \dim V_n = \dim V_j + 2r, \quad |r| \leq s. \tag{9}$$

In the paper [3], they claims that $2r$ can be interpreted as Chern numbers, if the quantum system is transformed into a semi-quantum system by averaging $\boldsymbol{J}$. However, they did not prove it mathematically, but gave a physical reason for that statement. They gave an interpretation to Eq. (9) from the viewpoint of rotation-vibration coupling in molecular spectra. In place of viewing S_j as the spin operators, one may view the basis of the representation space V_s as spanning a vibrational energy band. From this point of view, the tensor product space $V_j \otimes V_s$ may be thought of as a space in which a rotation-vibration coupling is described. Hence, the change in the eigenspace decomposition

$$V_j \otimes |s\rangle \oplus \cdots \oplus V_j \otimes |-s\rangle \longrightarrow V_{j+s} \oplus \cdots \oplus V_{|j-s|} \tag{10}$$

is interpreted as a a reorganization of energy bands from $\{V_j \otimes |r\rangle\}_{|r|\leq s}$ into $\{V_n\}_{n=j+s}^{|j-s|}$ through the rotation-vibration interaction, and the equation $\dim V_n = \dim V_j + 2r$ allows of the interpretation that the number $2r$ is characteristic of rovibration in each molecular band V_n, $n = j + s, \cdots, |j - s|$.

3 One-parameter semi-quantum systems

We now consider that $j >> s$. This means that we may use the method of averaging for the operators $\boldsymbol{J}$ with coherent states. The $SU(2)$ coherent states are defined to be the $SU(2)$ orbit of the lowest weight vector of the representation D^j,

$$\mathbf{J} = D^j(g)|j\rangle, \quad g \in SU(2). \tag{11}$$

Averaging J_k with coherent states results in

$$\langle \mathbf{J}|J_x|\mathbf{J}\rangle = \langle j|D^j(g)^* J_x D^j(g)|j\rangle = j\cos\phi\sin\theta = x, \tag{12a}$$
$$\langle \mathbf{J}|J_y|\mathbf{J}\rangle = \langle j|D^j(g)^* J_y D^j(g)|j\rangle = j\sin\phi\sin\theta = y, \tag{12b}$$
$$\langle \mathbf{J}|J_z|\mathbf{J}\rangle = \langle j|D^j(g)^* J_z D^j(g)|j\rangle = j\cos\theta = z, \tag{12c}$$

where

$$D^j(g) = e^{-i\phi J_z} e^{-i\theta J_y} e^{-i\psi J_z}, \quad \sum_{k=1}^{3} x_k^2 = \rho^2, \ \rho = j, \tag{13}$$

and where J_k are expressed as Hermitian matrices acting on V_j.

Then, the operator H_τ is made into a semi-quantum operator acting on V_s,

$$\overline{H}_\tau(\boldsymbol{x}) := \langle \mathbf{J}|H_\tau|\mathbf{J}\rangle = (1-\tau)S_z + \tau \sum_k x_k S_k, \quad \boldsymbol{x} \in S^2(\rho) \subset \mathbb{R}^3. \tag{14}$$

For each $\boldsymbol{x} \in S^2(\rho)$, we can assign the eigenspace of $\overline{H}_\tau(\boldsymbol{x})$ associated with one of eigenvalues. If this procedure were possible on the whole $S^2(\rho)$ for all τ, one might determine a one-parameter family of complex line bundles over $S^2(\rho)$. However, this procedure fails to be valid at some parameter value. To see this, we consider the above Hamiltonian with $s = \frac{1}{2}$;

$$\overline{H}_\tau(\boldsymbol{x}) = \frac{1}{2}\begin{pmatrix} 1-\tau+\tau z & \tau(x-iy) \\ \tau(x+iy) & -1+\tau-\tau z \end{pmatrix}, \quad \boldsymbol{x} \in S^2(1) \subset \mathbb{R}^3. \tag{15}$$

The eigenvalues are easily obtained as

$$\lambda(\tau) = \pm\sqrt{\frac{1}{4} - \frac{1-z}{2}(\tau-\tau^2)}, \tag{16}$$

which are not degenerate if $z \neq -1$, but degenerate at $z = -1$ for $\tau = \frac{1}{2}$,

$$\lambda(\tau) = \pm|\tau - \frac{1}{2}|. \tag{17}$$

If $z \neq -1$, the graph of eigenvalues $\lambda(\tau)$ at z are shown in Fig.2. It turns out that we cannot assign a line bundle associated with an eigenvalue of the present Hamiltonian at $\tau = \frac{1}{2}$ because of the presence of the degeneracy point $z = -1$ when $\tau = \frac{1}{2}$.

In general, we can associate a complex line bundle with each of non-degenerate eigenvalues except for a singular value of the parameter τ. We denote by $S^2(\rho) \times \mathbb{C}|r\rangle$

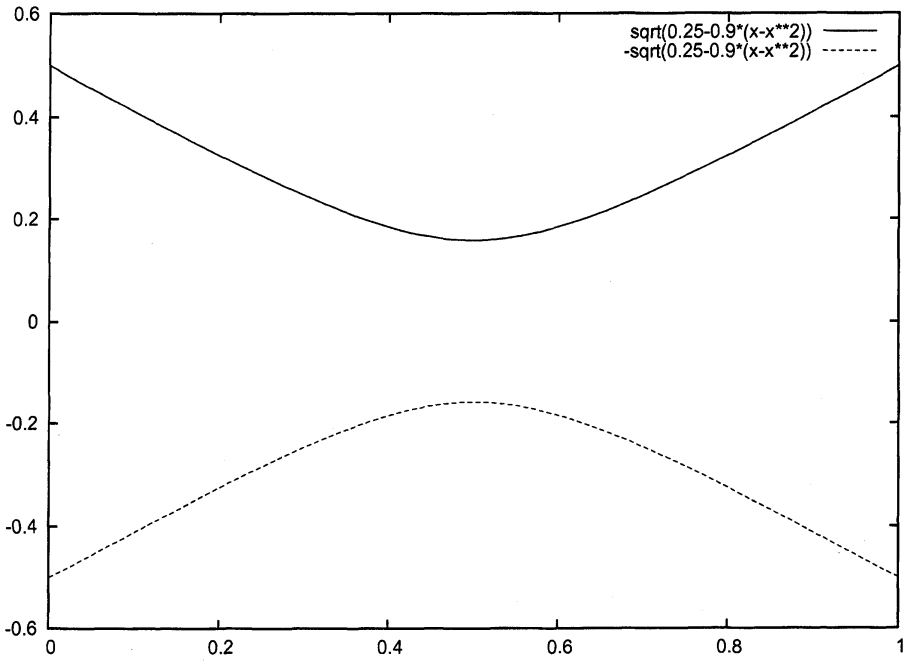

Figure 2: The eigenvalues of $\overline{H}_\tau(\boldsymbol{x})$ against τ with $\boldsymbol{x}$ fixed at a a point of the small circle determined by $z \neq -1$

and $L^{(r)}$ the complex line bundles associated with the eigenvalues r of S_z and ρr of $\boldsymbol{x} \cdot \boldsymbol{S}$, respectively. When τ passes a critical value, the bundle structure is expected to change, like

$$\begin{array}{c} \sum_{|r| \leq s}^{\oplus} S^2(\rho) \times \mathbb{C}|r\rangle \\ \text{w.r.t. } \overline{H}_0(\boldsymbol{x}) = S_z \end{array} \longrightarrow \begin{array}{c} \sum_{|r| \leq s}^{\oplus} L^{(r)} \\ \text{w.r.t. } \overline{H}_1(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{S} \end{array} \tag{18}$$

Our question is now described as follows: Does the line bundle $L^{(r)}$ have the first Chern number $2r$? We refer to $L^{(r)}$ as an eigen-line bundle in what follows.

Before treating a generic bundle $L^{(r)}$, we here give examples of eigen-line bundle $L^{(\pm 1)} \to S^2(1)$ for $s = \pm\frac{1}{2}$. The semi-quantum Hamiltonian in this case is expressed as

$$\overline{H}_1(\boldsymbol{x}) = \frac{1}{2}\begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}, \quad \boldsymbol{x} \in S^2(1) \subset \mathbb{R}^3, \tag{19}$$

which has normalized eigenvectors associated with the eigenvalues $1/2$, $-1/2$,

$$\boldsymbol{u}_+^{(-\frac{1}{2})} := \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}, \quad \boldsymbol{u}_+^{(\frac{1}{2})} := \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}, \quad \boldsymbol{x} \in U_+, \tag{20}$$

$$\boldsymbol{u}_-^{(-\frac{1}{2})} := \begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}, \quad \boldsymbol{u}_-^{(\frac{1}{2})} := \begin{pmatrix} e^{-i\phi}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}, \quad \boldsymbol{x} \in U_-, \tag{21}$$

where

$$U_+ = \{\boldsymbol{x} \in S^2(1) | \, \theta \neq \pi\}, \quad U_- = \{\boldsymbol{x} \in S^2(1) | \, \theta \neq 0\}. \tag{22}$$

The transformation rules for local eigenvectors on $U_+ \cap U_-$ are

$$\boldsymbol{u}_-^{(-\frac{1}{2})} = e^{i\phi}\boldsymbol{u}_+^{(-\frac{1}{2})}, \quad \boldsymbol{u}_-^{(\frac{1}{2})} = e^{-i\phi}\boldsymbol{u}_+^{(\frac{1}{2})} \quad \text{on } U_+ \cap U_-. \tag{23}$$

We now proceed to the eigen-line bundle $L^{(r)} \to S^2(\rho)$ with the aim of evaluating the first Chern number of $L^{(r)}$. The normalized eigenvector associated with the eigenvalue $r\rho$ of the matrix $\boldsymbol{x} \cdot \boldsymbol{S} = \sum_k x_k S_k$ with $\boldsymbol{x} \in S^2(\rho)$ is expressed locally as

$$\begin{aligned} \boldsymbol{u}_+^{(r)} &:= D^s(g)|r\rangle|_{\psi=-\phi} = e^{-i\phi S_z} e^{-i\theta S_y} e^{i\phi S_z}|r\rangle \quad \text{on} \quad U_+, \\ \boldsymbol{u}_-^{(r)} &:= D^s(g)|r\rangle|_{\psi=\phi} = e^{-i\phi S_z} e^{-i\theta S_y} e^{-i\phi S_z}|r\rangle \quad \text{on} \quad U_-, \end{aligned} \tag{24}$$

where S_k's with $k = x, y, z$, are represented as Hermitian matrices acting on V_s, where $|r| \leq s$, and where

$$U_+ = \{\boldsymbol{x} \in S^2(\rho)|, z \neq -\rho\}, \quad U_- = \{\boldsymbol{x} \in S^2(\rho)|, z \neq \rho\}. \tag{25}$$

The transformation rule between $\boldsymbol{u}_+^{(r)}$ and $\boldsymbol{u}_-^{(r)}$ on $U_+ \cap U_-$ are given by

$$e^{-2ir\phi}\boldsymbol{u}_+^{(r)} = \boldsymbol{u}_-^{(r)}. \tag{26}$$

Thus, the eigen-line bundle $L^{(r)} \to S^2(\rho)$ is defined.

The local connection forms of the eigen-line bundle $L^{(r)} \to S^2(\rho)$ are defined by and expressed as

$$\omega_+^{(r)} := \langle r|D^s(g)^* dD^s(g)|r\rangle|_{\psi=-\phi} = -ir(-d\phi + \cos\theta d\phi) \quad \text{on } U_+, \tag{27a}$$

$$\omega_-^{(r)} := \langle r|D^s(g)^* dD^s(g)|r\rangle|_{\psi=\phi} = -ir(d\phi + \cos\theta d\phi) \quad \text{on } U_-. \tag{27b}$$

On the intersection $U_+ \cap U_-$, the local connection forms $\omega_+^{(r)}$ and $\omega_-^{(r)}$ are related by

$$\omega_+^{(r)} - 2ird\phi = \omega_-^{(r)}. \tag{28}$$

Since $d\omega_+^{(r)} = d\omega_-^{(r)}$ on $U_+ \cap U_-$, the curvature form is defined globally on $S^2(\rho)$ by

$$\Omega = \begin{cases} d\omega_+^{(r)} & \text{on } U_+, \\ d\omega_-^{(r)} & \text{on } U_-. \end{cases} \tag{29}$$

We are now in a position to evaluate the first Chern number of $L^{(r)}$. Let C denote the equator of $S^2(\rho)$, and $S_+^2(\rho)$ and $S_-^2(\rho)$ the northern and southern hemispheres, respectively. Then, by applying Stokes' theorem, one obtains

$$\begin{aligned} \int_{S^2(\rho)} \Omega &= \int_{S_+^2(\rho)} d\omega_+^{(r)} + \int_{S_-^2(\rho)} d\omega_-^{(r)} \\ &= \int_C \omega_+^{(r)} - \int_C \omega_-^{(r)} = 2ir \int_C d\phi = 4\pi ir \end{aligned} \tag{30}$$

Proposition 1. The Chern number of the eigen-line bundle $L^{(r)}$ is given by

$$\frac{i}{2\pi}\int_{S^2(\rho)}\Omega = -2r. \tag{31}$$

Remark. The first Chern numbers associated with the Hamiltonian $\boldsymbol{x}\cdot\boldsymbol{S}$ with $\boldsymbol{x}\in\mathbb{R}^3\backslash\{0\}$ are calculated in another method in connection with the "adiabatic" connection and curvature [4].

4 Chern numbers in the presence of $U(1)$ symmetry

4.1 $U(1)$ invariance of H_τ and $\overline{H_\tau(\boldsymbol{x})}$

We look into Eq. (4) from the viewpoint of symmetry. Let $U(1)$ act on $V_j\otimes V_s$ in the manner,

$$e^{-itJ_z}\otimes e^{-itS_z}. \tag{32}$$

Since J_k and S_k transform according to

$$e^{-itJ_z}J_k(\text{rep. }S_k)e^{itJ_z} = \sum_\ell a_{k\ell}J_\ell(\text{resp. }S_\ell),\quad (a_{k\ell}) = e^{-t\hat{\boldsymbol{e}}_3}, \tag{33}$$

both $\mathbf{1}\otimes S_z$ and $\sum_k J_k\otimes S_k$ are invariant under this $U(1)$ action, so that the Hamiltonian $H_\tau = (1-\tau)\mathbf{1}\otimes S_z + \tau\sum_k J_k\otimes S_k$ is invariant under $U(1)$ as well.

We turn to the averaged Hamiltonian $\overline{H}_\tau(\boldsymbol{x})$. The transformation (33) is averaged to yield

$$\langle\mathbf{J}|e^{-itJ_z}J_ke^{itJ_z}|\mathbf{J}\rangle = \sum_\ell a_{k\ell}\langle\mathbf{J}|J_\ell|\mathbf{J}\rangle = \sum_\ell a_{k\ell}x_\ell,\quad (a_{k\ell}) = e^{-t\hat{\boldsymbol{e}}_3}, \tag{34}$$

which defines an $SO(2)$ action on $S^2(\rho)$. Thus, the induced $U(1)$ action on the semi-quantum system is described as

$$x_k\mapsto\sum a_{k\ell}x_\ell,\quad S_k\mapsto\sum a_{k\ell}S_\ell. \tag{35}$$

Since S_z and $\sum_k x_kS_k$ are both invariant under the $U(1)$ action, the semi-quantum Hamiltonian $\overline{H}_\tau(\boldsymbol{x}) = (1-\tau)S_z + \tau\sum_k x_kS_k$ is invariant as well.

4.2 $U(1)$-invariant Hamiltonians

In place of $\sum_k x_kS_k$, we consider a Hamiltonian of the from $H(\boldsymbol{x}) = \sum_k f_k(\boldsymbol{x})S_k$. Since $H(\boldsymbol{x})$ transforms according to

$$\sum_k f_k(\boldsymbol{x})S_k\mapsto\sum_{k,\ell} f_k(A_t\boldsymbol{x})a_{k\ell}S_\ell,\quad A_t = (a_{k\ell}) = e^{-t\hat{\boldsymbol{e}}_3}, \tag{36}$$

the $U(1)$ invariance condition for $H(\boldsymbol{x})$ yields

$$f_k(A_t\boldsymbol{x}) = \sum_\ell a_{k\ell} f_\ell(\boldsymbol{x}), \tag{37}$$

which implies that the $\mathbb{R}^3$-valued function $\boldsymbol{F}(\boldsymbol{x}) = \sum_k f_k(\boldsymbol{x})\boldsymbol{e}_k$ is $SO(2)$-equivariant; $\boldsymbol{F}(A_t\boldsymbol{x}) = A_t\boldsymbol{F}(\boldsymbol{x})$. Equivalently, one has the invariance condition for $H(\boldsymbol{x})$,

$$H(e^{-t\hat{\boldsymbol{e}}_3}\boldsymbol{x}) = D(e^{-it/2})H(\boldsymbol{x})D(e^{it/2}), \tag{38}$$

where

$$D(e^{-it/2}) = \mathrm{diag}(e^{-it/2}, e^{it/2}). \tag{39}$$

We now look for Hamiltonians satisfying the above invariance condition. Let

$$w = x + iy, \quad h(\boldsymbol{x}) = f_1(\boldsymbol{x}) + if_2(\boldsymbol{x}). \tag{40}$$

Then, the $SO(2)$-equivariance condition (37) takes the form

$$h(A_t\boldsymbol{x}) = e^{-it}h(\boldsymbol{x}), \quad f_3(A_t\boldsymbol{x}) = f_3(\boldsymbol{x}), \tag{41}$$

or

$$h(e^{-it}w, e^{it}\overline{w}, z) = e^{-it}h(w, \overline{w}, z), \quad f_3(e^{-it}w, e^{it}\overline{w}, z) = f_3(w, \overline{w}, z). \tag{42}$$

By differentiating the above equations with respect to t at $t = 0$, we obtain

$$w\frac{\partial h}{\partial w} = h, \quad \frac{\partial h}{\partial \overline{w}} = 0, \quad \frac{\partial f_3}{\partial w} = \frac{\partial f_3}{\partial \overline{w}} = 0. \tag{43}$$

Examples of h, f_3 are given, respectively, by

$$h(w, \overline{w}, z) = h(z)(x + iy), \quad f_3(w, \overline{w}, z) = f(z). \tag{44}$$

4.3 Weighted $U(1)$-invariant Hamiltonians

If we start with the weighted $U(1)$ action on $V_j \otimes V_s$ expressed as

$$e^{-itJ_z} \otimes e^{-itKS_z}, \quad K \in \{0, 1, 2, \cdots\}, \tag{45}$$

the invariance condition (38) should be replaced by

$$H(e^{-t\hat{\boldsymbol{e}}_3}\boldsymbol{x}) = D(e^{-iKt/2})H(\boldsymbol{x})D(e^{iKt/2}), \tag{46}$$

and (42) by

$$h(e^{-it}w, e^{it}\overline{w}, z) = e^{-iKt}h(w, \overline{w}, z), \quad f_3(e^{-it}w, e^{it}\overline{w}, z) = f_3(w, \overline{w}, z). \tag{47}$$

Consequently, we obtain, in place of (43),

$$w\frac{\partial h}{\partial w} = Kh, \quad \frac{\partial h}{\partial \overline{w}} = 0, \quad \frac{\partial f_3}{\partial w} = \frac{\partial f_3}{\partial \overline{w}} = 0. \tag{48}$$

We then obtain, for example,

$$h(w,\overline{w},z)=h(z)(x+iy)^K,\quad f_3(w,\overline{w},z)=f(z). \tag{49}$$

Examples of weighted $U(1)$-invariant Hamiltonians $\sum_k f_k(\boldsymbol{x})S_k$ with $\boldsymbol{x}\in S^2(\rho)$, $\rho=j$ are given by

$$\begin{pmatrix} f(z) & h(z)(x-iy)^K \\ h(z)(x+iy)^K & -f(z) \end{pmatrix} \quad \text{for} \quad s=\frac{1}{2}, \tag{50}$$

and

$$\begin{pmatrix} f(z) & \frac{1}{\sqrt{2}}\Big(h(z)(x-iy)^K\Big) & 0 \\ \frac{1}{\sqrt{2}}\Big(h(z)(x+iy)^K\Big) & 0 & \frac{1}{\sqrt{2}}\Big(h(z)(x-iy)^K\Big) \\ 0 & \frac{1}{\sqrt{2}}\Big(h(z)(x+iy)^K\Big) & -f(z) \end{pmatrix} \quad \text{for } s=1. \tag{51}$$

We can evaluate the first Chern number for the model Hamiltonian (50). Since the calculation is too long, we here do not reproduce it and give the result only from [1].

Proposition 2. Let K be an integer, and f and h any real polynomials in $z=\cos\theta$. Suppose that for the weighted $U(1)$ invariant Hamiltonian

$$H=\begin{pmatrix} f(\cos\theta) & h(\cos\theta)\ \sin^K\theta\ \exp(iK\phi) \\ h(\cos\theta)\ \sin^K\theta\ \exp(-iK\phi) & -f(\cos\theta) \end{pmatrix}, \tag{52}$$

the matrix elements f and $h\sin^K\theta$ do not share zeros. Then H has two eigenvalues, positive and negative, without degeneracy. For $K\neq 0$, the complex line bundle associated with each eigenvalue is defined over the two-sphere S^2. The first Chern number, which characterizes each eigen-line bundle, is equal to 0 or $\pm K$, depending on whether the number of zeros of the diagonal element, counted with their multiplicities, is even or odd.

We note that we may set $\rho=1$ without changing Chern numbers and that the complex conjugation of the Hamiltonian, $H\mapsto\overline{H}$, alter the sign of the Chern number since the eigenvalue is real-valued and the connection form is pure-imaginary valued.

5 Chern numbers in the presence of D_3 symmetry

This section starts with a review of the dihedral group D_3. The D_3 is a symmetry group of an equilateral triangle, which is known to be isomorphic with the symmetric group S_3. Let

$$\begin{aligned} &\pi_1=(1),\quad \pi_2=(1\ 2\ 3),\quad \pi_3=(1\ 3\ 2),\\ &\pi_4=(1\ 2),\quad \pi_5=(2\ 3),\quad \pi_6=(1\ 3). \end{aligned} \tag{53}$$

Then, as is well known, the E representation of D_3 is given by

$$\begin{aligned} &D^E(\pi_1)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ D^E(\pi_2)=\begin{pmatrix}-\frac{1}{2}&-\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix},\ D^E(\pi_3)=\begin{pmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix},\\ &D^E(\pi_4)=\begin{pmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix},\ D^E(\pi_5)=\begin{pmatrix}-1&0\\0&1\end{pmatrix},\ D^E(\pi_6)=\begin{pmatrix}\frac{1}{2}&\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix}, \end{aligned} \tag{54}$$

and A_2 representation by

$$D^{A_2}(\pi_j) = 1, \quad j = 1,2,3, \quad D^{A_2}(\pi_k) = -1, \quad k = 4,5,6. \tag{55}$$

The expression of the D_3 action on $\mathbb{R}^3$ depends on the choice of a basis of $\mathbb{R}^3$. To fix the expression, we first consider the above E representation of D_3 and make it act on the set $\mathcal{H}_0(2)$ of 2×2 traceless Hermitian matrices by the adjoint action. Then we get a representation equivalent to $E \oplus A_1$ on account of the isomorphism $\mathcal{H}_0(2) \cong \mathbb{R}^3$. Taking the Pauli basis $\sigma_1, \sigma_2, \sigma_3$ of $\mathcal{H}_0(2)$ as $\sigma'_y, \sigma'_z, \sigma'_x$, respectively, we identify the subspace spanned by σ'_x, σ'_y with the x-y plane as the E representation space and the subspace spanned by σ'_z with the z axis as the A_2 representation space. For example, one has the representation matrices such as

$$D^{E\oplus A_2}(\pi_2) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D^{E\oplus A_2}(\pi_4) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{56}$$

We now look for Hamiltonians invariant under the D_3 group. The condition for invariance is expressed as

$$D^E(g)H(\boldsymbol{x})D^E(g)^{-1} = H(D^{E\oplus A_2}(g)\boldsymbol{x}) \quad \text{for} \quad g \in D_3. \tag{57}$$

To find such a Hamiltonian, we refer to the representation theory for D_3. As is well known, the sets of functions

$$\begin{pmatrix} y^2 - x^2 \\ 2xy \end{pmatrix}, \quad \begin{pmatrix} zy \\ -zx \end{pmatrix} \tag{58}$$

are E-equivariant and the functions

$$z, \quad y(y^2 - 3x^2) \tag{59}$$

are A_2-equivariant. The functions

$$z^2, \quad x(x^2 - 3y^2) \tag{60}$$

are known to be A_1-equivariant or simply invariant. Since the equivariance of the above-mentioned functions and the invariance condition (57) are equivalent, we obtain the invariant Hamiltonian of the form

$$H(\boldsymbol{x}) = \begin{pmatrix} X & Y + iZ \\ Y - iZ & -X \end{pmatrix}, \quad \boldsymbol{x} \in S^2(1) \subset \mathbb{R}^3, \tag{61}$$

where

$$X(\boldsymbol{x}) = b_1(y^2 - x^2) + b_2 zy, \tag{62a}$$
$$Y(\boldsymbol{x}) = 2b_1 yx - b_2 zx, \tag{62b}$$
$$Z(\boldsymbol{x}) = -(a_1 z + a_2 y(y^2 - 3x^2)), \tag{62c}$$

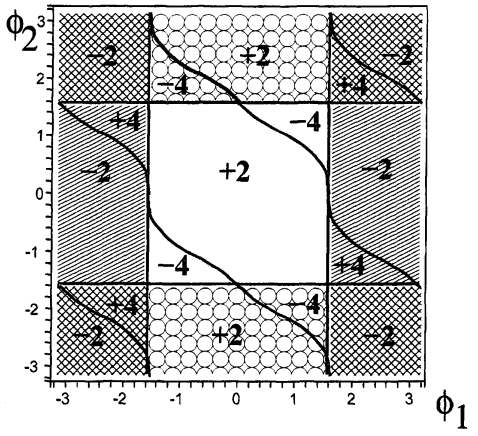

Figure 3: The iso-Chern diagram for the D_3 invariant Hamiltonian

and where (a_1, a_2, b_1, b_2) are real constants. We assume that $(a_1, a_2) \neq (0, 0)$ and $(b_1, b_2) \neq (0, 0)$.

We can evaluate the first Chern numbers for the eigen-line bundle for the above Hamiltonian. Since the calculation is too long, we here cite the results only from [1].

Proposition 3. For the D_3 invariant Hamiltonian $H(\boldsymbol{x})$ given in (61), owing to the invariance of the Chern numbers with respect the scaling of the parameters (a_1, a_2, b_1, b_2), the parameter space $\dot{\mathbb{R}}^2 \times \dot{\mathbb{R}}^2$ reduces to the two-torus T^2, where $\dot{\mathbb{R}}^2 = \mathbb{R}^2 - \{0\}$. The torus T^2 is endowed with the coordinates (ϕ_1, ϕ_2) determined by $a_1 = \cos\phi_1, a_2 = \sin\phi_1$ and $b_1 = \cos\phi_2, b_2 = \sin\phi_2$. The iso-Chern diagram for the eigen-line bundle associated with the positive eigenvalue is described on T^2, as is shown in Fig. 3. The read and blue lines ($\phi_1 = \pm\frac{\pi}{2}, \phi_2 = \pm\frac{\pi}{2}$) and black curves ($\cos\phi_1 \cos\phi_2 = \sin\phi_1 \sin^3\phi_2$) are the sets of degeneracy points, on which no eigen-line bundle is defined. In each iso-Chern domain separated by the degeneracy lines and curves, the Chern number for the eigen-line bundle with positive eigenvalue is indicated.

The iso-Chern diagram for the eigen-line bundle associated with the negative eigenvalue is obtained by opposing the sign of the Chern number assigned to each iso-Chern domain.

6 Chern numbers in the presence of O symmetry

In this section, we start with a brief review of the octahedral group O. The octahedral group O is the orientation-preserving symmetry group for the regular octahedron, and is known to be isomorphic to the symmetric group S_4 on a finite set of four symbols. Further, the group O is known to be generated by the following three elements,

$$C_4^Z \mapsto \begin{pmatrix} & -1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \quad C_3^{[-1-1-1]} \mapsto \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}, \quad C_2^X \mapsto \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \tag{63}$$

where C_4^Z denotes a rotation about the z-axis with the rotation angle $2\pi/4$, and the other symbols are defied in a similar manner. The matrix representation given in Eq. (63) is known as the T_1 (or F_1) representation on $\mathbb{R}^3$. The two-dimensional representation E of the group O is generated by the matrices

$$C_4^Z \mapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad C_3^{[-1-1-1]} \mapsto \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad C_2^X \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}. \tag{64}$$

The E representation of the group O acts on the set $\mathcal{H}_0(2)$ of traceless 2×2 Hermitian matrices by the adjoint transformation. This action gives rise to the reducible representation $E \oplus A_2$ on $\mathcal{H}_0(2)$, where the representation space for E is spanned by σ_3, σ_1 and that for A_2 by σ_2 and where σ_j, $j = 1, 2, 3$, denote the Pauli matrices. According to the representation theory for the group O, the functions

$$\begin{pmatrix} 2z^2 - x^2 - y^2 \\ \sqrt{3}(x^2 - y^2) \end{pmatrix}, \quad xyz \tag{65}$$

are E-equivariant and A_2-equivariant, respectively, where the group O acts on $\mathbb{R}^3$ by T_1 (or F_1) representation and on $\mathbb{R}^2$ by the E representation.

In terms of these equivariant functions, we can form an O-invariant Hamiltonian satisfying

$$D^E(g)H(\boldsymbol{x})D^E(g)^{-1} = H(D^{T_1}(g)\boldsymbol{x}) \quad \text{for} \quad g \in O. \tag{66}$$

Let

$$\phi_1 = 2z^2 - x^2 - y^2, \quad \phi_2 = \sqrt{3}(x^2 - y^2), \quad \phi_3 = xyz. \tag{67}$$

Then, the Hamiltonian

$$H(\boldsymbol{x}) = \begin{pmatrix} a\phi_1 & a\phi_2 - ib\phi_3 \\ a\phi_2 + ib\phi_3 & -a\phi_1 \end{pmatrix} \tag{68}$$

proves to be invariant under the O group action, where a, b are real parameters with $(a, b) \neq (0, 0)$. For this Hamiltonian with O-symmetry, we have the following result on the Chern number [2].

Proposition 4. The parameter space $\mathbb{R}^2 - \{0\}$ for the O-invariant Hamiltonian $H(\boldsymbol{x})$ given in (68) reduces to a circle, and the degeneracy points on this circle are $(a, b) = (\pm 1, 0), (0, \pm 1)$. The Chern numbers of the eigen-line bundle associated with the positive eigenvalue are shown in Fig. 4.

In what follows, we give a sketch of the proof, which is other than that given in [2] The condition of the degeneracy in eigenvalues is described as

$$\det H(\boldsymbol{x}) = 0 \quad \Leftrightarrow \quad a^2(\phi_1^2 + \phi_2^2) = 0, \; b^2\phi_3^2 = 0. \tag{69}$$

Since the condition is scale invariant, we may restrict the parameters (a, b) to the unit circle $a^2 + b^2 = 1$. There are four degeneracy points $(a, b) = (0, \pm 1), (\pm 1, 0)$ on this circle, for which the eigenvalues of $H(\boldsymbol{x})$ are degenerate on some points of S^2. For

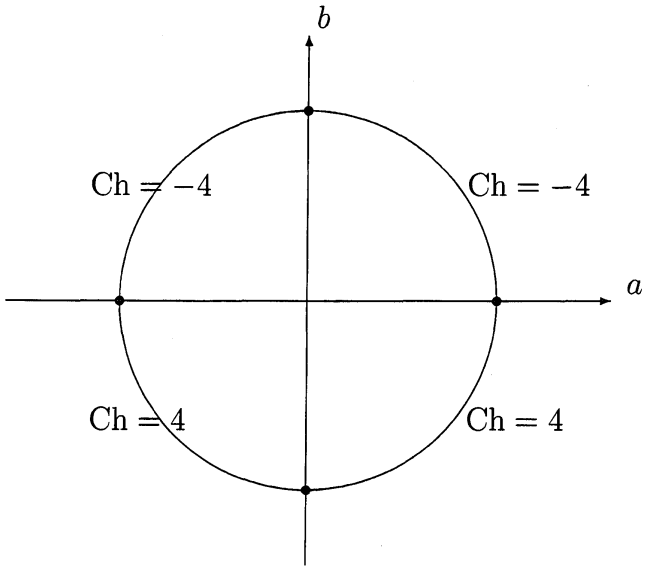

Figure 4: Chen numbers on the unit circle

$(a, b) = (0, \pm 1)$, the corresponding degeneracy points on S^2 form three orthogonal circles;

$$x = 0 \text{ or } y^2 + z^2 = 1, \quad y = 0 \text{ or } z^2 + x^2 = 1, \quad z = 0 \text{ or } x^2 + y^2 = 1. \tag{70}$$

For $(a, b) = (\pm 1, 0)$, the corresponding degeneracy points on S^2 are

$$\begin{pmatrix} \pm\frac{1}{\sqrt{3}} \\ \pm\frac{1}{\sqrt{3}} \\ \pm\frac{1}{\sqrt{3}} \end{pmatrix}, \tag{71}$$

which form an orbit of the group O. For regular values of the parameter, line bundles are associated with each eigenvalue. The exceptional points at which the normalized eigenvector for the positive eigenvalue is not defined are

$$\boldsymbol{n}_\pm = \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}, \quad \boldsymbol{a}_\pm = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad \boldsymbol{b}_\pm = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}. \tag{72}$$

In the case of $a > 0$, the domains of normalized eigenvectors $\boldsymbol{v}_\pm$ associated with the positive eigenvalue are

$$U_+ = S^2 - \{\boldsymbol{n}_\pm\}, \quad U_- = S^2 - \{\boldsymbol{a}_\pm, \boldsymbol{b}_\pm\}, \tag{73}$$

respectively. The $\boldsymbol{v}_\pm$ are related by

$$\boldsymbol{v}_+ = \Phi \boldsymbol{v}_-, \quad \Phi = \frac{a\phi_2 - ib\phi_3}{\sqrt{a^2\phi^2 + b^2\phi_3^2}} \quad \text{on} \quad U_+ \cap U_-. \tag{74}$$

Thus, the line bundle associated with the positive eigenvalue is defined over S^2. The local connection forms of this bundle are defined on U_+ and U_- to be

$$\omega_+ = \boldsymbol{v}_+^\dagger d\boldsymbol{v}_+, \quad \omega_- = \boldsymbol{v}_-^\dagger d\boldsymbol{v}_-, \tag{75}$$

respectively, and related by

$$\omega_+ = \Phi^{-1} d\Phi + \omega_- \quad \text{on} \quad U_+ \cap U_-. \tag{76}$$

Since Φ is $U(1)$-valued, one has $d\omega_+ = d\omega_-$ on $U_+ \cap U_-$, so that the curvature form Ω is defined on S^2 to be

$$\Omega = \begin{cases} d\omega_+ & \text{on} \quad U_+, \\ d\omega_- & \text{on} \quad U_-. \end{cases} \tag{77}$$

Let C_1 and C_2 be two circles at the levels $z = \pm h$ with $0 < h < 1$. Let S_+^2 and S_-^2 be regions separated by C_1 and C_2. The S_+^2 is the region containing the equator and S_-^2 is the union of two regions containing either of the north or the south pole. The orientation of C_k is in keeping with that of S_+^2.

The Chern number is then defined and evaluated as

$$c_1 = \frac{i}{2\pi} \int_{S^2} \Omega = -\frac{1}{2\pi i} \int_{C_1 + C_2} \Phi^{-1} d\Phi, \tag{78}$$

where the Stokes' theorem has been applied in the second equality.

The right-hand side of (78) is minus the sum of the winding numbers of the maps $C_k \to U(1)$ by Φ with $k = 1, 2$. The winding numbers are computable directly in the following manner. The curve C_1 is expressed as

$$x = \sqrt{1 - h^2} \cos t, \quad y = \sqrt{1 - h^2} \sin t, \quad z = h. \tag{79}$$

Then, ϕ_2 and ϕ_3 restricted on C_1 are put in the form

$$\begin{aligned} \phi_2(t) &= \sqrt{3}(1 - h^2)(\cos^2 t - \sin^2 t) = \sqrt{3}(1 - h^2) \cos 2t, \\ \phi_3(t) &= h(1 - h^2) \sin t \cos t = \tfrac{1}{2} h(1 - h^2) \sin 2t. \end{aligned} \tag{80}$$

In the case of $a > 0, b < 0$, we may take Φ as $\Phi = \frac{\phi_2 + i\phi_3}{|\phi_2 + i\phi_3|}$, after a scaling of a, b. The orientation of C_1 is in the direction of t decreasing, *i.e.*, clockwise. Hence the winding number is -2. We can do the same reasoning for C_2 to obtain the winding number -2. The sum is -4. Hence the Chern number is 4.

There is another method for evaluating the Chern number. A linearization method is applicable if the circles are deformed so suitably that the value of the contour integrals along C_k, $k = 1, 2$, may be equal to the sum of the contour integrals along respective small circles centered at exceptional points concerned. For details of the linearization method, see [1]. We now apply the linearization method to evaluate the right-hand side of (78). In the case of $a > 0$, $b < 0$, after a suitable scaling of parameters a, b, we may put the Hamiltonian in the form

$$H(\boldsymbol{x}) = \begin{pmatrix} F & X + iY \\ X - iY & -F \end{pmatrix}, \tag{81}$$

where

$$X = \sqrt{3}(x^2 - y^2), \quad Y = xyz, \quad F = 2z^2 - x^2 - y^2. \tag{82}$$

We note here that the parameter scaling does not alter the value of the right-hand side of (78). Further, we have to point out that the linearization method cannot be applied for the exceptional points $\boldsymbol{n}_\pm$. In fact, each of the linear approximations of X and Y vanishes at $\boldsymbol{n}_\pm$. We choose to take up $\boldsymbol{a}_\pm$ and $\boldsymbol{b}_\pm$ and deform the contour integrals along C_1 and C_2 into the sum of the contour integrals along small circles around $\boldsymbol{a}_\pm$ and $\boldsymbol{b}_\pm$ to apply the linearization method. The local coordinate systems we use are (z, x) for the exceptional points $\boldsymbol{a}_+$ and $\boldsymbol{b}_+$, and (x, z) for the exceptional points $\boldsymbol{a}_-$ and $\boldsymbol{b}_-$ in view of the orientation of the respective coordinate systems.

The first-order derivatives of X and Y with respect to (z, x) are

$$\begin{aligned} \frac{\partial X}{\partial z} &= -2\sqrt{3}y\frac{\partial y}{\partial z}, & \frac{\partial X}{\partial x} &= 2\sqrt{3}x, \\ \frac{\partial Y}{\partial z} &= x\frac{\partial y}{\partial z}z + xy, & \frac{\partial Y}{\partial x} &= yz + \frac{\partial y}{\partial x}z. \end{aligned}$$

Using the equality $\partial y/\partial z = -z/y$, we obtain

$$\det\begin{pmatrix} \frac{\partial X}{\partial z} & \frac{\partial X}{\partial x} \\ \frac{\partial Y}{\partial z} & \frac{\partial Y}{\partial x} \end{pmatrix}_{\boldsymbol{a}_+} = \begin{vmatrix} 0 & \sqrt{6} \\ \frac{1}{2} & 0 \end{vmatrix} = -\sqrt{\frac{3}{2}} < 0,$$

$$\det\begin{pmatrix} \frac{\partial X}{\partial z} & \frac{\partial X}{\partial x} \\ \frac{\partial Y}{\partial z} & \frac{\partial Y}{\partial x} \end{pmatrix}_{\boldsymbol{b}_+} = \begin{vmatrix} 0 & -\sqrt{6} \\ -\frac{1}{2} & 0 \end{vmatrix} = -\sqrt{\frac{3}{2}} < 0,$$

$$\det\begin{pmatrix} \frac{\partial X}{\partial z} & \frac{\partial X}{\partial x} \\ \frac{\partial Y}{\partial z} & \frac{\partial Y}{\partial x} \end{pmatrix}_{\boldsymbol{a}_-} = \begin{vmatrix} 0 & \sqrt{6} \\ -\frac{1}{2} & 0 \end{vmatrix} = \sqrt{\frac{3}{2}} > 0,$$

$$\det\begin{pmatrix} \frac{\partial X}{\partial z} & \frac{\partial X}{\partial x} \\ \frac{\partial Y}{\partial z} & \frac{\partial Y}{\partial x} \end{pmatrix}_{\boldsymbol{b}_-} = \begin{vmatrix} 0 & -\sqrt{6} \\ \frac{1}{2} & 0 \end{vmatrix} = \sqrt{\frac{3}{2}} > 0.$$

On account of the orientation of the coordinate system adopted and of the orientation of small circles enclosing respective exceptional points, we observe that the winding number at each exceptional points is -1, so that the sum is -4. This implies that the Chern number is 4.

In the case of $a > 0, b > 0$, Eq. (82) undergoes the transformation $(X, Y) \mapsto (X, -Y)$ and F is kept unchanged, and thereby the definition of U_+ and U_- is also left unchanged. Hence, the determinants evaluated at the exceptional points changes in sign, so that the linearization method provides the Chern number -4.

In the case of $a < 0, b < 0$, Eq. (82) undergoes the transformation $(X, Y) \mapsto (-X, Y)$ and $F \mapsto -F$. We then have $F(\boldsymbol{n}_\pm) < 0$ and $F(\boldsymbol{a}_\pm) = F(\boldsymbol{b}_\pm) > 0$, so that the definition of U_+ and U_- are changed into

$$U_- = S^2 - \{\boldsymbol{n}_\pm\}, \quad U_+ = S^2 - \{\boldsymbol{a}_\pm, \boldsymbol{b}_\pm\}. \tag{83}$$

We take two circles C_1 and C_2 in a similar manner as above to divide the sphere S^2 into three regions. In the present case, S^2_+ is defined to be the union of the regions in which

either of $\boldsymbol{n}_{\pm}$ is contained, and S^2_- is the region containing the equator. The orientation of each of C_1 and C_2 is determined so as to be consistent with the orientation of S^2_+. As a result, the orientations of C_1 and of C_2 are opposite to those in the case of $a > 0, b < 0$. However, the map Φ undergoes the change $\Phi = \frac{\phi_2(t)+i\phi_3(t)}{|\phi_2(t)+i\phi_3(t)|} \mapsto \Phi = \frac{-\phi_2(t)+i\phi_3(t)}{|-\phi_2(t)+i\phi_3(t)|}$, according to the change in the sign of a. Hence, the winding number of each of C_1 and C_2 is shown to be -2, the same as that in the case of $a > 0, b < 0$. The sum of the winding numbers is -4, so that the Chern number is 4.

In the case of $a < 0, b > 0$, we have the Chern number -4. This is because the transformation $(a, b) \mapsto (a, -b)$ results in the change of Chern numbers in the sign, as is already seen above, and because we have the Chern number 4 in the case of $a < 0, b < 0$.

We have already observed that in the case of $b = 0$ there are three mutually orthogonal degeneracy circles given in (70) and that in the case of $a = 0$ there are eight degeneracy points given in (71). Figure 4 shows that the difference of Chern numbers between that on the upper right (resp. left) arc and that on the lower right (resp. left) arc is eight. This difference is equal to the number of degeneracy points corresponding to $(a, b) = (\pm 1, 0)$. In contrast with this, there is no difference in Chern numbers between that on the upper (resp. lower) left arc and that on the upper (resp. lower) right arc. This may be because the degeneracy points corresponding to $(a, b) = (0, \pm 1)$ are not a finite set but three circles, continua.

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