数理解析研究所講究録
第 1775 巻 2012 年 85-93

GEOMETRIC CAUCHY PROBLEMS FOR SPACELIKE AND TIMELIKE CMC SURFACES IN $\mathbb{R}^{2,1}$

DAVID BRANDER

ABSTRACT. We discuss recent work of the author and collaborators on generalizations of Björling's classical problem to the case of constant non-zero mean curvature surfaces in $2+1$-dimensional spacetime. The aim is to give an overview, and to point out the similarities and differences between the two cases of timelike and spacelike surfaces. Applications to the construction of CMC surfaces with prescribed singularities are also described.

1. INTRODUCTION

This article discusses results of work of the author and collaborators, [2], [5] and [6], on the generalization of Björling's problem for minimal surfaces to the case of constant non-zero mean curvature (CMC) surfaces in the 3-dimensional Minkowski space $\mathbb{R}^{2,1}$. The classical Björling problem is to find the minimal surface in $\mathbb{R}^3$ which contains a prescribed curve with the surface tangent distribution prescribed along the curve. It has a unique solution via analytic extension because the data is enough to determine the (holomorphic) Weierstrass data along the curve. The generalization of this problem to other surface classes besides minimal is sometimes called the *geometric Cauchy problem.*

The problem has also been considered for zero mean curvature surfaces in $\mathbb{R}^{2,1}$, both spacelike [1] and timelike [7]. There is a substantial difference between the spacelike and timelike cases, as the former is an elliptic problem and has a Weierstrass representation in terms of holomorphic data, as in the case of minimal surfaces in $\mathbb{R}^3$, whilst the timelike case is hyperbolic and has a d'Alembert type solution in terms of two functions of one variable each. The timelike case is complicated by the fact that there is no unique solution to the Björling problem if the initial curve is characteristic.

All of the last comments also apply to the corresponding problems for constant *non-zero* mean curvature surfaces. Moreover, the non-zero situation has the additional property that the Weierstrass/d'Alembert type representations are now in terms of maps into a loop group [9, 8], and obtaining the solutions from the data entails a loop group decomposition (Iwasawa or Birkhoff). The loop group decomposition is really an LU decomposition of infinite matrices, which can easily be approximated numerically, but cannot be written down explicitly, in general. Therefore some work is necessary in going back and forth between the surface and the corresponding Weierstrass data.

The Björling problem for non-minimal CMC surfaces in $\mathbb{R}^3$, was solved by the author and J Dorfmeister in [3]. Perhaps the essential point of the solution is that, although the Iwasawa decomposition cannot be computed explicitly for general Weierstrass data, one has considerable flexibility in one's choice of Weierstrass data. It turns out to be possible to choose the Weierstrass data such that the Iwasawa decomposition is trivial along the initial curve in question. This allows one to write down explicitly the Weierstrass data corresponding to given Björling data along a curve. The exact same argument can be used to solve the problem for spacelike CMC surfaces in $\mathbb{R}^{2,1}$, so that problem was essentially solved by the $\mathbb{R}^3$ case. We discuss this below in Section 2.

The geometric Cauchy problem for timelike CMC surfaces was solved by the author and M Svensson in [5]. Although the d'Alembert construction is rather different from the Weierstrass construction of the spacelike case, the solution of the problem again depends essentially on the fact that one can choose d'Alembert data such that the loop group decomposition (Birkhoff in this case) is trivial along the curve in question. This is discussed in Section 3.

Finally there is the *singular* geometric Cauchy problem, which arises naturally for CMC surfaces in $\mathbb{R}^{2,1}$. Both timelike and spacelike CMC surfaces almost always have generic singularities. Most of these arise at points where the loop group decomposition has a middle term, and cannot be avoided in general. An analogous situation exists for minimal/maximal surfaces in $\mathbb{R}^{2,1}$, and the singularities have been studied recently by various authors. In particular, the singular Björling problem, which is to find the minimal/maximal surface with a prescribed singular curve, and a further parallel vector field prescribed along the curve (in analogue with the Gauss map), has been studied by S-D Yang, YW Kim, S-E Koh and H Shin in [11] and [10].

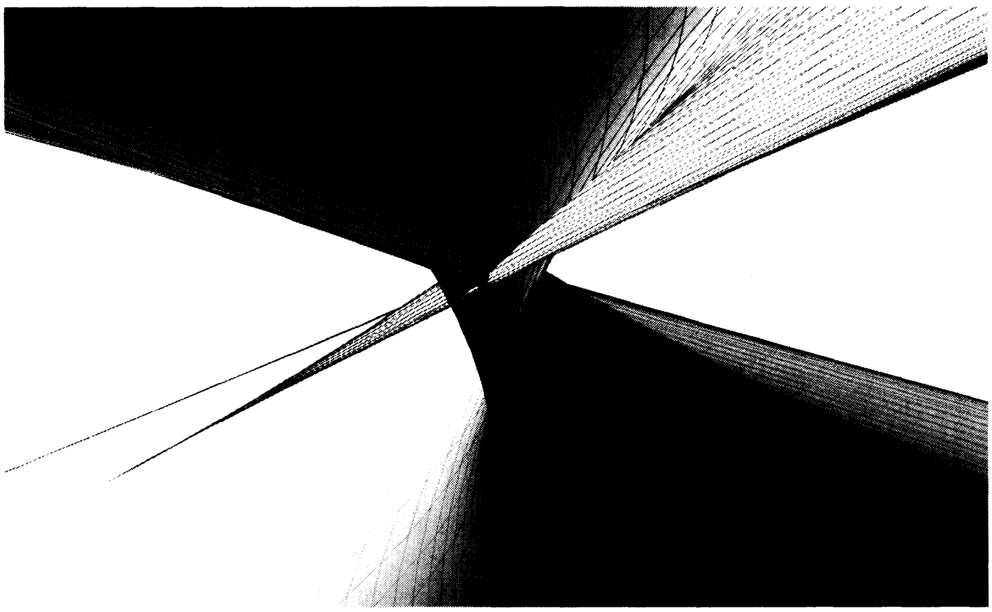

FIGURE 1. The singular geometric Cauchy construction can be used to produce CMC surfaces with arbitrary prescribed singular curves. The timelike CMC surface shown has infinitely many swallowtails along a cuspidal edge. The first few can be seen in this plot of a finite domain. The geometric Cauchy data are $s(v) = \sin(v)$, $t(v) = 2$, $\theta'(v) = 0.0001$. The swallowtails occur when $s = 0$.

The singular geometric Cauchy problem for spacelike CMC surfaces is solved by the author in [2]. The essential problem here is that the $SU(1,1)$ frame for the surface, which is critical for solving the problem in the regular situation of [3], *blows up* as the singular curve is approached. The problem is solved by multiplying all the maps by a certain element of the loop group, which moves all the data into the *big cell*, that is the set on which the Iwasawa decomposition does not have a middle term. It is then possible to solve the problem with a new "singular frame" which arises from this idea. An analogous device is used to solve the problem for timelike CMC surfaces in joint work with M Svensson in [6]. This will be discussed in Section 4.

2. Spacelike CMC surfaces

For explicit details of the following discussion, the reader is referred to [4] and [3].

2.1. **The generalized Weierstrass representation.** The so called *DPW* (Dorfmeister/Pedit/Wu [9]), or generalized Weierstrass representation, for spacelike CMC surfaces is sketched below. We will discuss it locally for simplicity. Let $U \subset \mathbb{C}^2$ be an open set.

- A conformal CMC immersion $f : U \to \mathbb{R}^{2,1}$ is represented by an *extended frame* $\hat{F} : U \to \Lambda G$, where $G = SU(1,1)$, and ΛG is the group of twisted loops in G.

- The surface f is recovered from $\hat{F}$ by a simple formula, called the Sym-Bobenko formula, which is essentially of the form $\mathscr{S}(\hat{F}) = \lambda(\partial_\lambda \hat{F})\hat{F}^{-1}|_{\lambda=1}$.

- The extended frame is *characterized*[1] among smooth maps into ΛG by the fact that its Maurer-Cartan form $\hat{F}^{-1}\mathrm{d}\hat{F}$ can be written:

$$\hat{F}^{-1}\mathrm{d}\hat{F} = A_{-1}\lambda^{-1}\mathrm{d}z + \alpha_0 + \tau(A_{-1})\lambda \mathrm{d}\bar{z},$$

  where τ is the involution determining the real form $\mathfrak{su}(1,1)$ in $\mathfrak{sl}(2,\mathbb{C})$, and α_0 is a 1-form which does not depend on λ.

- Finally (and this is the DPW part of the construction), the extended frame can be obtained from a *holomorphic frame* $\hat{\Phi} : U \to \Lambda G^{\mathbb{C}}$ by a pointwise Iwasawa decomposition

$$\hat{\Phi} = \hat{F}\hat{B}_+, \qquad \hat{F} \in \Lambda G, \quad \hat{B}_+ \in \Lambda^+ G^{\mathbb{C}}, \tag{2.1}$$

  the last group denoting those loops in $SL(2,\mathbb{C})$ which can be holomorphically extended to the unit disc.

- By comparing the Maurer-Cartan forms of $\hat{F}$ and $\hat{\Phi}$, and using that $\hat{B}_+$ has a Taylor expansion in λ, it is easy to deduce that $\hat{\Phi}$ must be *holomorphic*, and is *characterized* by its Maurer-Cartan form being of the form

$$\hat{\Phi}^{-1}\mathrm{d}\hat{\Phi} = (\psi_{-1}\lambda^{-1} + \psi_0 + O(\lambda))\mathrm{d}z.$$

[1] This is standard for harmonic maps into symmetric spaces, and the Gauss map is harmonic into $\mathbb{H}^2$

The holomorphic functions in the matrices ψ_i are arbitrary, apart from a regularity condition on one of them. This is the *generalized Weierstrass representation* for (spacelike) CMC surfaces.

2.2. **Solution of the Björling problem.** Now the idea that is used in [3] to solve the Björling problem is actually very simple: The factor $\hat{F} \in \Lambda\mathcal{G}$ in the decomposition 2.1 is essentially *unique*. So suppose that $\hat{\Phi}(z_0)$ is already ΛG-valued, i.e. already takes values in the real form. At such a point, the Iwasawa decomposition (2.1) is trivial:

$$\hat{\Phi}(z_0) = \hat{F}(z_0).$$

We can use this to construct a special holomorphic frame from the knowledge of $\hat{F}$ along a curve as follows: suppose we know the value of $\hat{F}$ along some curve in U, say $\hat{F}(x+0.i) = \hat{F}_0(x)$ along an interval $J = U \cap \mathbb{R}$. Let us set

$$\hat{\Phi}_0(x) = \hat{F}_0(x), \quad x \in J,$$

and then *extend this holomorphically* to a map $\hat{\Phi} : V \to \Lambda G^{\mathbb{C}}$, from some open subset V containing J. It turns out that this can be done, and moreover this map $\hat{\Phi}$ has all the required properties of a holomorphic extended frame for a CMC surface. Further, the corresponding extended frame $\hat{F}$ agrees with $\hat{F}_0(x)$ along J, because the Iwasawa decomposition is trivial along this curve. Thus we have solved the Björling problem, provided that we can construct $\hat{F}_0(x)$ along J from the Björling data. It turns out that this can be done in a fairly straightforward manner (see [3]), and one finally obtains a simple formula for the *holomorphic potential* $\hat{\Phi}^{-1}\mathrm{d}\hat{\Phi}$ in terms of the Björling data.

Using a unique local correspondence between Weierstrass data and extended frames, obtained via a normalized Birkhoff splitting, it is also not difficult to show that the Björling problem has a unique solution.

3. Timelike CMC surfaces

We turn now to the timelike case. The d'Alembert representation given in [8] differs from the Weierstrass representation described in Section 2 as follows:

- The extended frame this time takes values in a subgroup of $\Lambda SL(2,\mathbb{C})$ consisting of loops which are in $G = SL(2,\mathbb{R})$ for *real* values of λ.

- The Maurer-Cartan form of the extended frame $\hat{F}(x,y)$, where x and y are lightlike coordinates for $U \subset \mathbb{R}^{1,1}$, has the form

$$\hat{F}^{-1}\mathrm{d}\hat{F} = A_1\lambda\mathrm{d}x + \alpha_0 + A_{-1}\lambda^{-1}\mathrm{d}y.$$

- The extended frame $\hat{F}(x,y)$ is obtained from a pair of maps $\hat{X}(x)$ and $\hat{Y}(y)$, where

$$\hat{X}^{-1}\mathrm{d}\hat{X} = \sum_{j=-\infty}^{1} \psi_i^X \mathrm{d}x, \qquad \hat{Y}^{-1}\mathrm{d}\hat{Y} = \sum_{j=-1}^{\infty} \psi_i^Y \mathrm{d}y,$$

and the coefficients ψ_i^X and ψ_i^Y are essentially arbitrary functions.

- $\hat{F}$ is obtained from $\hat{X}$ and $\hat{Y}$ by performing a pointwise *Birkhoff* splitting

$$\hat{X}^{-1}(x)\hat{Y}(y) = \hat{H}_-(x,y)\hat{H}_+(x,y), \tag{3.1}$$

where $\hat{H}_-(x,y) \in \Lambda^- G$, and $\hat{H}_+(x,y) \in \Lambda^+ G$. [2]

Then

$$\hat{F}(x,y) = \hat{Y}(y)\hat{H}_+^{-1}(x,y). \tag{3.2}$$

3.1. **Solution of the geometric Cauchy problem.** Here we describe joint work with M Svensson in [5]. As in the spacelike case, the essential idea is to contrive a situation where the (this time Birkhoff) decomposition (3.1) is trivial along the curve. Looking at their Maurer-Cartan forms, where the leading terms must be non-vanishing for regular surfaces, we can see that $\hat{X}(x)$ will have a pole (in λ) at ∞ while $\hat{Y}(y)$ will have a pole at $\lambda = 0$. Thus, the Birkhoff decomposition is unlikely to be trivial unless we have something like $\hat{X}^{-1}(x)\hat{Y}(y) =$ constant in λ, and indeed we can achieve this along the curve $y = x$ if we can arrange, for example, that

$$\hat{X}(v) = \hat{Y}(v).$$

This leads to a solution to the geometric Cauchy problems along *non-characteristic* curves, i.e. curves which are never tangent to a null curve. For such a curve in $\mathbb{R}^{1,1}$, one can always (with a possible change of orientation) locally choose null coordinates (x,y), such that the curve is given as $u = 0$ in the coordinates $u = (x-y)/2$, $v = (x+y)/2$.

Suppose one can construct, from the geometric Cauchy data, the extended frame $\hat{F}(0,v) = \hat{F}_0(v)$ for a timelike CMC surface along the curve $x = y = v$, i.e. $u = 0$. (It turns out that one can do this). Let us now set

$$\hat{X}(x) = \hat{F}_0(x), \qquad \hat{Y}(y) = \hat{F}_0(y).$$

Then it is easy to check that $\hat{X}$ and $\hat{Y}$ have precisely the required form for the d'Alembert data of a timelike CMC surface, and thus generate such a surface using the scheme outlined above. Along the curve $x = y = v$ we have $\hat{X}(x) = \hat{Y}(y)$, so that, along this curve the Birkhoff decomposition at (3.1) is just $I = I.I$, where I is the identity matrix. Along this curve, the extended frame for the surface so constructed is given by (3.2) as

$$\hat{F}(0,v) = \hat{Y}(v).I = \hat{F}_0(v).$$

Thus the constructed surface solves the geometric Cauchy problem. One can show, again using a normalized Birkhoff decomposition, that the solution is unique.

Finally, the case that the initial curve is characteristic (null), is also treated in [5]. The solution is not unique, but we describe how to construct all solutions. We do not treat initial curves which become characteristic at isolated points.

[2] The notation for the last two groups is not strictly correct here, but in any case $\Lambda^- G$ consists of loops which extend holomorphically to $\{|z| > 1\} \subset \hat{C}$, while $\Lambda^+ G$ consists of loops which extend to the unit disc.

4. The singular geometric Cauchy problem

The solution of the singular geometric Cauchy problem, treated in [2] and [6] for, respectively, spacelike and timelike CMC surfaces is rather complicated to describe. As mentioned in the introduction, the extended frame $\hat{F}$ blows up as such a curve is approached. Therefore, it is clear that the solutions outlined above cannot be applied for such a curve.

We will not attempt to describe the solution here, but only mention a critical idea on which the solution is built. Let us consider the case of spacelike surfaces. As described in Section 2, the extended frame $\hat{F}$ (and hence the surface) is obtained from the holomorphic frame $\hat{\Phi}$ by an Iwasawa decomposition

$$\hat{\Phi} = \hat{F}\hat{B}_+.$$

When the group is non-compact, the Iwasawa decomposition is only written this way on an open dense subset of the loop group, called the *big cell.* The rest of the loop group is a disjoint union $\cup_{i=1}^{\infty} \mathscr{P}_{\pm i}$ of subvarieties, of codimension increasing with $|i|$.

On $\mathscr{P}_i$ the Iwasawa decomposition reads

$$\hat{\Phi} = \hat{U}\omega_i\hat{B}_+, \tag{4.1}$$

where $\hat{U} \in \Lambda G$ and the middle term ω_i is a certain unique special element.

The essential idea, first used in [4], is as follows: suppose that, at some point z_0, we have

$$\hat{\Phi}(z_0) = \omega_i.$$

(The possible terms $\hat{U}(z_0)$ and $\hat{B}_+(z_0)$ in (4.1) do not affect the following). Now set

$$\hat{\Phi}_\omega = \hat{\Phi}\omega_i^{-1}.$$

Then $\hat{\Phi}_\omega(z_0) = I$, which is in the big cell. The big cell is an open set, so $\hat{\Phi}_\omega(z)$ takes values in the big cell on a neighbourhood of z_0. Thus, around z_0, one has an Iwasawa decomposition

$$\hat{\Phi}_\omega = \hat{F}_\omega\hat{B}_+.$$

This was used to analyze the surface as $\mathscr{P}_{\pm 1}$ is encountered in [4], and it turns out that finite singularities occur only (and always) at $\mathscr{P}_1$, and the surface always blows up at $\mathscr{P}_{-1}$. [3]

Now the important, and not obvious, thing is that it turns out that if we let $\hat{F}$ denote the extended frame corresponding to $\hat{\Phi}$, then, on the intersection of the sets where both $\hat{F}$ and $\hat{F}_\omega$ are defined, which is an open dense set in the domain, we have that the Sym-Bobenko formula agrees for both frames:

$$\mathscr{S}(\hat{F}_\omega) = \mathscr{S}(\hat{F}).$$

This means that $\hat{F}_\omega$ can be regarded as another kind of extended frame for the surface, which, however is now *well defined at the singular set* corresponding to $\hat{\Phi}^{-1}(\mathscr{P}_1)$.

[3]The higher order small cells $\mathscr{P}_j$, for $|j| > 1$ have not been analyzed, but would not be relevant to generic singularities as they have higher codimension.

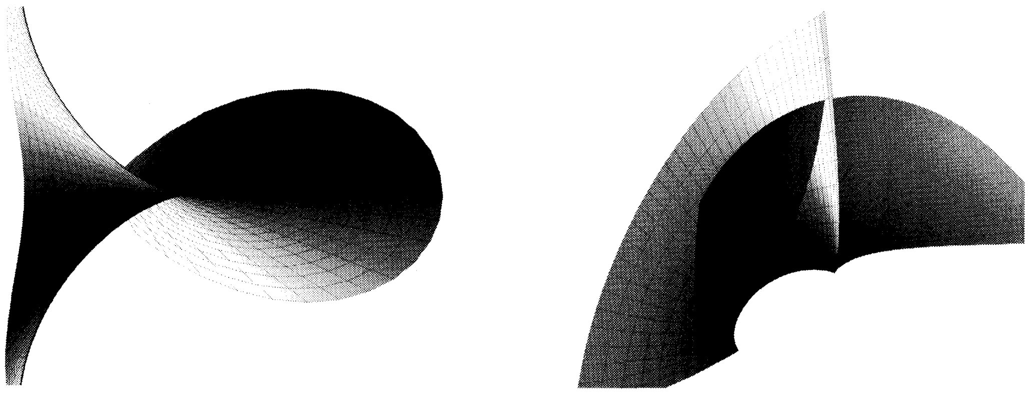

FIGURE 2. Spacelike CMC surfaces with prescribed singularities. Left: swallowtail. Right: cuspidal cross cap

This *singular frame* was used by the author in [2] to solve the singular Björling problem for spacelike CMC surfaces. It is not immediately clear how to define the singular frame from the geometric Cauchy data, because the definition of $\hat{F}_\omega$ is not geometric but rather comes algebraically from the holomorphic frame via an Iwasawa decomposition. Finding a way to do this was a major issue in this work.

The solution of the geometric Cauchy problem for *timelike* CMC surfaces, also depends on the idea of translating the data into the (Birkhoff) big cell and working with a "singular" frame. This will appear in a forthcoming article, joint with M Svensson [6].

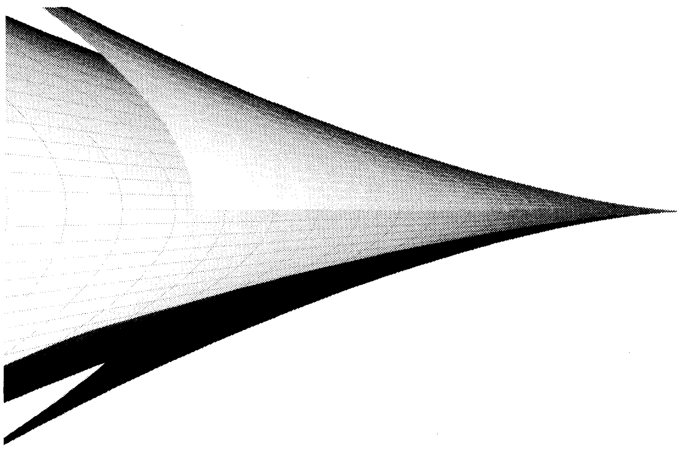

FIGURE 3. A timelike CMC surface with a cuspidal cross cap singularity. Geometric Cauchy data: $s(v) = 2 + 0.2v^2$, $t(v) = v$, θ' constant.

For both the timelike and spacelike cases, the singular geometric Cauchy construction is used to find the generic singularities of the generalized surfaces define there. In both cases, the generic non-degenerate singularities turn out to be cuspidal edges, swallowtails and cuspidal cross caps.

5. Numerical implementations

The solutions to all of these geometric Cauchy problems can be computed using numerical implementations of the DPW method. The DPW method takes the "potentials", $\hat{\Phi}^{-1}\mathrm{d}\hat{\Phi}$ for the spacelike case, and the pair $(\hat{X}^{-1}\mathrm{d}\hat{X}, \hat{Y}^{-1}\mathrm{d}\hat{Y})$ for the timelike case, integrates these, performs a Birkhoff or Iwasawa decomposition and then applies the Sym-Bobenko formula. All of this can be implemented numerically. The essential point now is that our solutions for the geometric Cauchy problem give formulae for these potentials directly from the geometry Cauchy data, and hence can be computed.

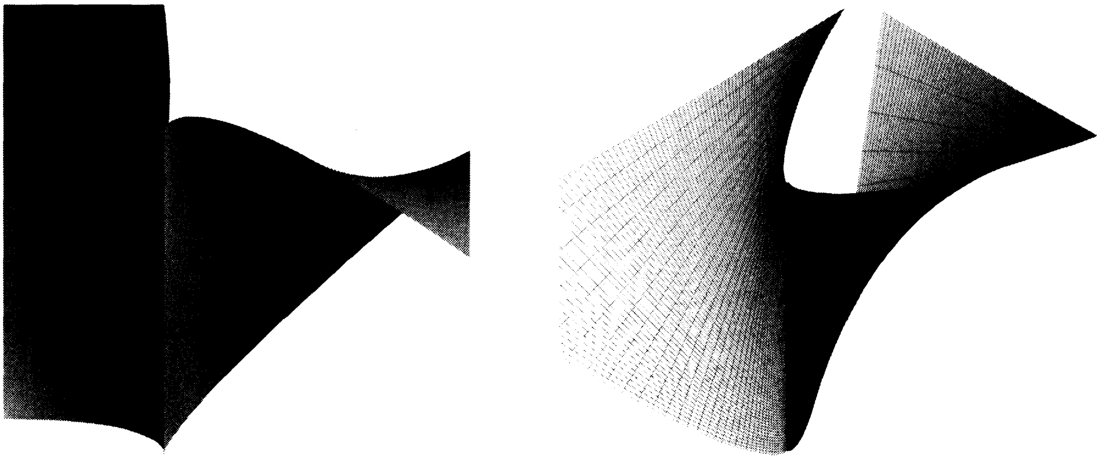

FIGURE 4. The singular curve is always a null curve for both timelike and spacelike CMC surfaces. For *smooth* timelike CMC surfaces, a null curve is characteristic. However this does not apply to singular curves: the curve is not, in general, characteristic. If the singular curve *is* characteristic, then the singularity is always a straight line, and the geometric Cauchy problem has infinitely many solutions. Two examples are shown here.

The images in this article are from the author's own implementation, written in Matlab. The non-characteristic singular geometric Cauchy problem can always be expressed as the problem of finding $f(u,v)$ with

$$f_v = s(-e_0 + \cos(\theta)e_1 + \sin(\theta)e_2),$$
$$f_u = t(-e_0 + \cos(\theta)e_1 + \sin(\theta)e_2).$$

The DPW potentials are given in [6] in terms of the functions $s(v)$, $t(v)$ and $\theta'(v)$. If all functions are non-zero and $s \neq \pm t$ then the surface has a cuspidal edge. If s vanishes to first order, a swallowtail. If

t vanishes to first order, a cuspidal cross cap. See Figures 1 and 3.

Examples of characteristic singular curves are given in Figure 4.

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DEPARTMENT OF MATHEMATICS, MATEMATIKTORVET, BUILDING 303 S, TECHNICAL UNIVERSITY OF DENMARK, DK-2800 KGS. LYNGBY, DENMARK
E-mail address: D.Brander@mat.dtu.dk