数理解析研究所講究録
第 1776 巻 2012 年 75-90

THE STUDY OF MATHEMATICAL HISTORY ON THE EQUATIONS OF NAVIER-STOKES AND BOLTZMANN AS THE MICROSCOPICALLY-DESCRIPTIVE HYDRODYNAMIC EQUATIONS

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ABSTRACT. The microscopically-description of hydromechanics equations are followed by the description of equations of gas theory by Maxwell, Kirchhoff and Boltzmann. Above all, in 1872, Boltzmann formulated the Boltzmann equations, expressed by the following today's formulation :

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f = Q(f,g), \quad t>0, \quad \mathbf{x},\mathbf{v}\in\mathbb{R}^n (n\geq 3), \quad \mathbf{x}=(x,y,z), \quad \mathbf{v}=(\xi,\eta,\zeta), \tag{1}$$

$$Q(f,g)(t,x,v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\{g(v'_*)f(v') - g(v_*)f(v)\}d\sigma dv_*, \quad g(v'_*) = g(t,x,v'_*), \text{ etc.} \tag{2}$$

These equations are able to be reduced for the general form of the hydrodynamic equations, after the formulations by Maxwell and Kirchhoff, and from which the Euler equations and the Navier-Stokes equations are reduced as the special case.

After Stokes' linear equations, the equations of gas theories were deduced by Maxwell in 1865, Kirchhoff in 1868 and Boltzmann in 1872, They contributed to formulate the fluid equations and to fix the Navier-Stokes equations, when Prandtl stated the today's formulation in using the nomenclature as the "so-called Navier-Stokes equations" in 1934, in which Prandtl included the three terms of nonlinear and two linear terms with the ratio of two coefficients as 3 : 1, which arose Poisson in 1831, Saint-Venant in 1843, and Stokes in 1845.

Mathematics Subject Classification 2010 : 76-03, 01.

1. Introduction

[1]

We have studied the original microscopically descriptive Navier-Stokes ($MDNS$) equations as the progenitors, Navier, Cauchy, Poisson, Saint-Venant and Stokes, and endeavor to ascertain their aims and conceptual thoughts in formulations these new equation. "The two-constant theory" was introduced first introduced in 1805 by Laplace [2] in regard to capillary action with constants denoted by H and K.

Thereafter, various pairs of constants have been proposed by their progenitors in formulating $MDNS$ equations or equations describing equilibrium or capillary situations. It is commonly accepted that this theory describes isotropic, linear elasticity. [3] We can find the "two-constant" in the equations of gas theories by Maxwell, Kirchhoff and Boltzmann, which were fixed into the common linear terms, and which originally takes its rise in Poisson and Stokes.

The gas theorists studied also the general equations of hydromechanics, which have the same proportion of coefficients as the equations deduced by Poisson and Stokes with only the linear term and the ratio of the coefficient of Laplacian to that of gradient of divergence term is 3 : 1. (cf Table 1.)

Date: 2011/09/04.

[1](⇓) Throughout this paper, in citation of bibliographical sources, we show our own paragraph or sentences of commentaries by surrounding between (⇓) and (⇑). ((⇑) is used only when not following to next section,). And by =*, we detail the statement by original authors, because we would like to discriminate and to avoid confusion from the descriptions by original authors. The mark : ⇒ means transformation of the statements in brevity by ours. And all the frames surrounding the statements are inserted for important remark of ours. Of course, when the descriptions are explicitly distinct without these marks, these are not the descriptions in citation of bibliographical sources.

[2](⇓) Of capillary action, Laplace[9, V.4, Supplement p.2] achnowledges Clailaut [6, p.22], and Clailaut cites Maupertuis.

[3](⇓) Darrigol [7, p.121].

TABLE 1. The kinetic equations of the hydrodynamics until the "Navier-Stokes equations" was fixed.

no	name/prob	the kinetic equations	Δ	gr.dv	E	F
1 N	Euler (1752-55) [?, p.127] fluid	$\begin{cases} X - \frac{1}{h}\frac{dp}{dx} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}, \\ Y - \frac{1}{h}\frac{dp}{dy} = \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}, \\ Z - \frac{1}{h}\frac{dp}{dz} = \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}, \end{cases} \quad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$				
2	Navier (1827)[14] elastic solid	$(6\text{-}1)_{N^e} \begin{cases} \frac{\Pi}{g}\frac{d^2x}{dt^2} = \varepsilon\left(3\frac{d^2x}{da^2} + \frac{d^2x}{db^2} + \frac{d^2x}{dc^2} + 2\frac{d^2y}{dbda} + 2\frac{d^2z}{dcda}\right), \\ \frac{\Pi}{g}\frac{d^2y}{dt^2} = \varepsilon\left(\frac{d^2y}{da^2} + 3\frac{d^2y}{db^2} + \frac{d^2y}{dc^2} + 2\frac{d^2x}{dadb} + 2\frac{d^2z}{dcdb}\right), \\ \frac{\Pi}{g}\frac{d^2z}{dt^2} = \varepsilon\left(\frac{d^2z}{da^2} + \frac{d^2z}{db^2} + 3\frac{d^2z}{dc^2} + 2\frac{d^2x}{dadc} + 2\frac{d^2y}{dbdc}\right) \end{cases}$ where Π is density of the solid, g is acceleration of gravity.	ε	2ε	$\frac{1}{2}$	
3 N	Navier (1827)[15] fluid	$\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = X + \varepsilon\left(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dxdz}\right) - \frac{du}{dt} - \frac{du}{dx}\cdot u - \frac{du}{dy}\cdot v - \frac{du}{dz}\cdot w\,; \\ \frac{1}{\rho}\frac{dp}{dy} = Y + \varepsilon\left(\frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dydz}\right) - \frac{dv}{dt} - \frac{dv}{dx}\cdot u - \frac{dv}{dy}\cdot v - \frac{dv}{dz}\cdot w\,; \\ \frac{1}{\rho}\frac{dp}{dz} = Z + \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz}\right) - \frac{dw}{dt} - \frac{dw}{dx}\cdot u - \frac{dw}{dy}\cdot v - \frac{dw}{dz}\cdot w\,; \end{cases}$	ε	2ε		$\frac{1}{2}$
4	Cauchy (1828)[5] system of particles in elastic and fluid	$\begin{cases} (L+G)\frac{\partial^2\xi}{\partial x^2} + (R+H)\frac{\partial^2\xi}{\partial y^2} + (Q+I)\frac{\partial^2\xi}{\partial z^2} + 2R\frac{\partial^2\eta}{\partial x\partial y} + 2Q\frac{\partial^2\zeta}{\partial z\partial x} + X = \frac{\partial^2\xi}{\partial t^2}, \\ (R+G)\frac{\partial^2\eta}{\partial x^2} + (M+H)\frac{\partial^2\eta}{\partial y^2} + (P+I)\frac{\partial^2\eta}{\partial z^2} + 2P\frac{\partial^2\zeta}{\partial y\partial z} + 2R\frac{\partial^2\xi}{\partial x\partial y} + Y = \frac{\partial^2\eta}{\partial t^2}, \\ (Q+G)\frac{\partial^2\zeta}{\partial x^2} + (P+H)\frac{\partial^2\zeta}{\partial y^2} + (N+I)\frac{\partial^2\zeta}{\partial z^2} + 2Q\frac{\partial^2\xi}{\partial z\partial x} + 2P\frac{\partial^2\eta}{\partial y\partial z} + Z = \frac{\partial^2\zeta}{\partial t^2}, \end{cases}$ $G = H = I, \quad L = M = N, \quad P = Q = R, \quad L = 3R$	$R+G$	$2R$	if $G=0$ $\frac{1}{2}$	
5	Poisson (1831)[17] elastic solid in general equations	$\begin{cases} X - \frac{d^2u}{dt^2} + a^2\left(\frac{d^2u}{dx^2} + \frac{2}{3}\frac{d^2v}{dydx} + \frac{2}{3}\frac{d^2w}{dzdx} + \frac{1}{3}\frac{d^2u}{dy^2} + \frac{1}{3}\frac{d^2u}{dz^2}\right) = \frac{\Pi}{\rho}\frac{d^2u}{dx^2}, \\ Y - \frac{d^2v}{dt^2} + a^2\left(\frac{d^2v}{dy^2} + \frac{2}{3}\frac{d^2u}{dxdy} + \frac{2}{3}\frac{d^2w}{dzdy} + \frac{1}{3}\frac{d^2v}{dx^2} + \frac{1}{3}\frac{d^2v}{dz^2}\right) = \frac{\Pi}{\rho}\frac{d^2v}{dy^2}, \\ Z - \frac{d^2w}{dt^2} + a^2\left(\frac{d^2w}{dz^2} + \frac{2}{3}\frac{d^2u}{dxdz} + \frac{2}{3}\frac{d^2v}{dydz} + \frac{1}{3}\frac{d^2w}{dx^2} + \frac{1}{3}\frac{d^2w}{dy^2}\right) = \frac{\Pi}{\rho}\frac{d^2w}{dz^2}, \end{cases}$	$\frac{a^2}{3}$	$\frac{2a^2}{3}$	$\frac{1}{2}$	
6	Poisson (1831)[17] fluid in general equations	$\begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} + \alpha(K+k)\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) + \frac{\alpha}{3}(K+k)\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} + \alpha(K+k)\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) + \frac{\alpha}{3}(K+k)\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} + \alpha(K+k)\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) + \frac{\alpha}{3}(K+k)\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \end{cases}$ $\begin{cases} \rho(X - \frac{d^2x}{dt^2}) = \frac{d\varpi}{dx} + \beta(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}), \\ \rho(Y - \frac{d^2y}{dt^2}) = \frac{d\varpi}{dy} + \beta(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}), \\ \rho(Z - \frac{d^2z}{dt^2}) = \frac{d\varpi}{dz} + \beta(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}) \end{cases}$ WHERE $\varpi \equiv p - \alpha\frac{d\psi t}{dt} - \frac{\beta+\beta'}{\chi t}\frac{d\chi t}{dt}, \quad \beta \equiv -\alpha(K+k)$	β	$\frac{\beta}{3}$		3
7	Saint-Venant (1843)[21] fluid	His equations are none in [21], however his tensor is in Table 3 (4).	ε	$\frac{\varepsilon}{3}$		3
8	Stokes (1849)[22] fluid	$(12)_S \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0. \end{cases}$	μ	$\frac{\mu}{3}$		3
9	Maxwell (1865-66) [12] HD	$\begin{cases} \rho\frac{\partial u}{\partial t} + \frac{dp}{dx} - C_M\left[\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + \frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho X, \\ \rho\frac{\partial v}{\partial t} + \frac{dp}{dy} - C_M\left[\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + \frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho Y, \\ \rho\frac{\partial w}{\partial t} + \frac{dp}{dz} - C_M\left[\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2} + \frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho Z \end{cases}$ where, $C_M \equiv \frac{pM}{6k\rho\Theta_2}$	C_M	$\frac{C}{3}$		3
10	Kirchhoff (1876)[8] HD	$\begin{cases} \mu\frac{du}{dt} + \frac{\partial}{\partial x} - C_K\left[\Delta u + \frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu X, \\ \mu\frac{dv}{dt} + \frac{\partial p}{\partial y} - C_K\left[\Delta v + \frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu Y, \\ \mu\frac{dw}{dt} + \frac{\partial p}{\partial z} - C_K\left[\Delta z + \frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu Z, \end{cases} \begin{cases} \frac{1}{\mu}\frac{d\mu}{dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \\ \text{where, } C_K \equiv \frac{1}{3\kappa}\frac{p}{\mu} \end{cases}$	C_K	$\frac{\Delta}{3}$		3
11 N	Rayleigh (1883)[20] HD	$\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = -\frac{du}{dt} + \nu\nabla^2 u - u\frac{du}{dx} - v\frac{du}{dy}, \\ \frac{1}{\rho}\frac{dp}{dy} = -\frac{dv}{dt} + \nu\nabla^2 v - u\frac{dv}{dx} - v\frac{dv}{dy} \end{cases}, \quad \frac{du}{dx} + \frac{dv}{dy} = 0$	ν			
12	Boltzmann (1895)[1] HD	$(221)_B \begin{cases} \rho\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} - \mathcal{R}\left[\Delta u + \frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \rho X, \\ \rho\frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} - \mathcal{R}\left[\Delta v + \frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \rho Y, \\ \rho\frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} - \mathcal{R}\left[\Delta w + \frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \rho Z \end{cases}$	$\mathcal{R}$	$\frac{\mathcal{R}}{3}$		3
13 N	Prandtl (1934)[19] HD	$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),$ FOR INCOMPRESSIBLE, IT IS SIMPLIFIED DIV $\mathbf{w} = 0, \quad \frac{D\mathbf{w}}{dt} = g - \frac{1}{\rho}\text{GRAD } p + \nu\Delta\mathbf{w}$	ν	$\frac{\nu}{3}$		3

TABLE 2. Geneology of tensors in fluid dynamics

no	name	tensor
1	Navier fluid	$t_{ij} = (p - \varepsilon u_{k,k})\delta_{ij} - \varepsilon(u_{i,j} + u_{j,i})$ $\begin{bmatrix} \epsilon' - 2\varepsilon\frac{du}{dx} & -\varepsilon(\frac{du}{dy} + \frac{dv}{dx}) & -\varepsilon(\frac{dw}{dx} + \frac{du}{dz}) \\ -\varepsilon(\frac{du}{dy} + \frac{dv}{dx}) & \epsilon' - 2\varepsilon\frac{dv}{dy} & -\varepsilon(\frac{dv}{dz} + \frac{dw}{dy}) \\ -\varepsilon(\frac{dw}{dx} + \frac{du}{dz}) & -\varepsilon(\frac{dv}{dz} + \frac{dw}{dy}) & \epsilon' - 2\varepsilon\frac{dw}{dz} \end{bmatrix}$, where $\epsilon' = p - \varepsilon(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz})$
2	Cauchy system (contains both elasticity and fluid)	$t_{ij} = \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$ $(60)_C$ $\begin{bmatrix} k\frac{\partial\xi}{\partial a} + K\nu & \frac{k}{2}\left(\frac{\partial\xi}{\partial b} + \frac{\partial\eta}{\partial a}\right) & \frac{k}{2}\left(\frac{\partial\zeta}{\partial a} + \frac{\partial\xi}{\partial c}\right) \\ \frac{k}{2}\left(\frac{\partial\xi}{\partial b} + \frac{\partial\eta}{\partial a}\right) & k\frac{\partial\eta}{\partial b} + K\nu & \frac{k}{2}\left(\frac{\partial\eta}{\partial c} + \frac{\partial\zeta}{\partial b}\right) \\ \frac{k}{2}\left(\frac{\partial\zeta}{\partial a} + \frac{\partial\xi}{\partial c}\right) & \frac{k}{2}\left(\frac{\partial\eta}{\partial c} + \frac{\partial\zeta}{\partial b}\right) & k\frac{\partial\zeta}{\partial c} + K\nu \end{bmatrix}$, where $\nu = \frac{\partial\xi}{\partial a} + \frac{\partial\eta}{\partial b} + \frac{\partial\zeta}{\partial c}$
3	Poisson fluid	$t_{ij} = -p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$ $(7\text{-}7)_{Pf}$ $\begin{bmatrix} \beta\left(\frac{du}{dz} + \frac{dw}{dx}\right) & \beta\left(\frac{du}{dy} + \frac{dv}{dx}\right) & \pi + 2\beta\frac{du}{dx} \\ \beta\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & \pi + 2\beta\frac{dv}{dy} & \beta\left(\frac{du}{dy} + \frac{dv}{dx}\right) \\ \pi + 2\beta\frac{dw}{dz} & \beta\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & \beta\left(\frac{du}{dz} + \frac{dw}{dx}\right) \end{bmatrix}$, where $\pi = p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt}$
4	Saint-Venant fluid	$t_{ij} = (\frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i})$ $= (-p - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i})$ $\begin{bmatrix} \pi + 2\varepsilon\frac{d\xi}{dx}, & \varepsilon\left(\frac{d\xi}{dy} + \frac{d\eta}{dx}\right) & \varepsilon\left(\frac{d\zeta}{dx} + \frac{d\xi}{dz}\right) \\ \varepsilon\left(\frac{d\xi}{dy} + \frac{d\eta}{dx}\right) & \pi + 2\varepsilon\frac{d\eta}{dy} & \varepsilon\left(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\right) \\ \varepsilon\left(\frac{d\zeta}{dx} + \frac{d\xi}{dz}\right) & \varepsilon\left(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\right) & \pi + 2\varepsilon\frac{d\zeta}{dz} \end{bmatrix}$, where $\pi = -p - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right)$
5	Stokes fluid	$t_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i})$, tensor $= -1 \times$ $\begin{bmatrix} p - 2\mu\left(\frac{du}{dx} - \delta\right) & -\mu\left(\frac{du}{dy} + \frac{dv}{dx}\right) & -\mu\left(\frac{dw}{dx} + \frac{du}{dz}\right) \\ -\mu\left(\frac{du}{dy} + \frac{dv}{dx}\right) & p - 2\mu\left(\frac{dv}{dy} - \delta\right) & -\mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right) \\ -\mu\left(\frac{dw}{dx} + \frac{du}{dz}\right) & -\mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & p - 2\mu\left(\frac{dw}{dz} - \delta\right) \end{bmatrix}$, where $3\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$
6	Maxwell fluid	$t_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i})$, $\begin{bmatrix} p - \frac{M}{9k\rho\Theta_2}p\left(2\frac{du}{dx} - \frac{dv}{dy} - \frac{dw}{dz}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{dw}{dx} + \frac{du}{dz}\right) \\ -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & p - \frac{M}{9k\rho\Theta_2}p\left(\frac{du}{dx} - 2\frac{dv}{dy} - \frac{dw}{dz}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{dv}{dz} + \frac{dw}{dy}\right) \\ -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & p - \frac{M}{9k\rho\Theta_2}p\left(\frac{du}{dx} - \frac{dv}{dy} - 2\frac{dw}{dz}\right) \end{bmatrix}$
7	Kirchhoff fluid	$t_{ij} = (-p - 2kv_{l,l})\delta_{ij} + k(v_{i,j} + v_{j,i})$, $\begin{bmatrix} p - 2k\frac{\partial u}{\partial x} & -k\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & -k\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ -k\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & p - 2k\frac{\partial v}{\partial y} & -k\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ -k\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & -k\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & p - 2k\frac{\partial w}{\partial z} \end{bmatrix}$, where $k = \frac{1}{3\kappa}\frac{p}{\mu}$.
8	Boltzmann fluid	$t_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i})$, $\begin{bmatrix} p - 2\mathcal{R}\left\{\frac{\partial u}{\partial x} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\} & -\mathcal{R}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & -\mathcal{R}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ -\mathcal{R}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & p - 2\mathcal{R}\left\{\frac{\partial v}{\partial y} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\} & -\mathcal{R}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ -\mathcal{R}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & -\mathcal{R}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & p - 2\mathcal{R}\left\{\frac{\partial w}{\partial z} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\} \end{bmatrix}$, where $\mathcal{R} = \frac{M}{6k\rho\Theta_2}p$.

2. The succession of the linear equations from Poisson to Stokes

[4] We discuss the linear fluid equations. Poisson's tensor of the pressures in fluid reads as follows :

$(7\text{-}7)_{Pf}$

$$\begin{bmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{bmatrix} = \begin{bmatrix} \beta\left(\frac{du}{dz} + \frac{dw}{dx}\right) & \beta\left(\frac{du}{dy} + \frac{dv}{dx}\right) & p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{du}{dx} \\ \beta\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{dv}{dy} & \beta\left(\frac{du}{dy} + \frac{dv}{dx}\right) \\ p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{dw}{dz} & \beta\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & \beta\left(\frac{du}{dz} + \frac{dw}{dx}\right) \end{bmatrix},$$

[4](⇓) In Poisson [17], the title of the chaper 7 is "*Calcul des Pressions dans les Fluides en mouvement ; équations defferentielles de ce mouvement.*"

$$(k+K)\alpha=\beta,\quad (k-K)\alpha=\beta',\quad p=\psi t=K,\quad \text{then}\quad \beta+\beta'=2k\alpha, \tag{3}$$

where χt is the density of the fluid around the point M, and ψt is the pressure. We put ϖ as following :

$$\varpi \equiv p-\alpha\frac{d\psi t}{dt}-\frac{\beta+\beta'}{\chi t}\frac{d\chi t}{dt}, \tag{4}$$

[5] then we get the linear equation by Poisson as following :

$$(7\text{-}9)_{Pf}\qquad \begin{cases} \rho(X-\frac{d^2x}{dt^2})=\frac{d\varpi}{dx}+\beta(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}), \\ \rho(Y-\frac{d^2y}{dt^2})=\frac{d\varpi}{dy}+\beta(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}), \\ \rho(Z-\frac{d^2z}{dt^2})=\frac{d\varpi}{dz}+\beta(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}). \end{cases} \tag{5}$$

Stokes comments on Poisson's $(7\text{-}9)_{Pf}$ as follows :

> On this supposition we shall get the value of $\frac{d\Psi t}{dt}$ from that of $R_1'-K$ in the equations of page 140 by putting
>
> $$\frac{du}{dx}=\frac{dv}{dy}=\frac{dw}{dz}=-\frac{1}{3\chi t}\frac{d\chi t}{dt}.$$
>
> We have therefore
>
> $$\alpha\frac{d\chi t}{dt}=\frac{\alpha}{3}(K-5k)\frac{d\chi t}{\chi t dt}.$$
>
> $$(7\text{-}2)_{Pf}\qquad \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=-\frac{1}{\chi t}\frac{d\chi t}{dt}. \tag{6}$$
>
> Putting now for $\beta+\beta'$ its value $2\alpha k$, and for $\frac{1}{\chi t}\frac{d\chi t}{dt}$ its value given by equation (6) [6], the expression for ϖ, page 152, [7] becomes
>
> $$\varpi=p-\alpha\frac{d\psi t}{dt}-\frac{\beta+\beta'}{\chi t}\frac{d\chi t}{dt}=p-\Big(\frac{\alpha}{3}(K-5k)+2\alpha k\Big)\frac{d\chi t}{\chi t dt}=p+\frac{\alpha}{3}(K+k)(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}).$$
>
> Observing that $\alpha(K+k)=\beta$, this value of ϖ reduces Poisson's equation $(7\text{-}9)_{Pf}$[=(5)] to the equation $(12)_S$ of this paper. ([22, p.119]).

Namely, by using $\alpha(K+k)=\beta$ in (3), we get the following :

$$\begin{cases} \frac{d\varpi}{dx}=\frac{dp}{dx}+\frac{\beta}{3}\frac{d}{dx}(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}), \\ \frac{d\varpi}{dy}=\frac{dp}{dy}+\frac{\beta}{3}\frac{d}{dy}(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}), \\ \frac{d\varpi}{dz}=\frac{dp}{dz}+\frac{\beta}{3}\frac{d}{dz}(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}), \end{cases}$$

then $(5)(=(7\text{-}9)_{Pf})$ turns out :

$$\begin{cases} \rho(\frac{Du}{Dt}-X)+\frac{dp}{dx}+\alpha(K+k)\Big(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\Big)+\frac{\alpha}{3}(K+k)\frac{d}{dx}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0, \\ \rho(\frac{Dv}{Dt}-Y)+\frac{dp}{dy}+\alpha(K+k)\Big(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\Big)+\frac{\alpha}{3}(K+k)\frac{d}{dy}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0, \\ \rho(\frac{Dw}{Dt}-Z)+\frac{dp}{dz}+\alpha(K+k)\Big(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\Big)+\frac{\alpha}{3}(K+k)\frac{d}{dz}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0, \end{cases}$$

$$\Rightarrow (12)_S\qquad \begin{cases} \rho(\frac{Du}{Dt}-X)+\frac{dp}{dx}-\mu\Big(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\Big)-\frac{\mu}{3}\frac{d}{dx}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0, \\ \rho(\frac{Dv}{Dt}-Y)+\frac{dp}{dy}-\mu\Big(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\Big)-\frac{\mu}{3}\frac{d}{dy}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0, \\ \rho(\frac{Dw}{Dt}-Z)+\frac{dp}{dz}-\mu\Big(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\Big)-\frac{\mu}{3}\frac{d}{dz}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0. \end{cases}$$

Here, we remark that the succession from $(7\text{-}9)_{Pf}$ to $(12)_S$ means that the Stokes' equations comes from Poisson's linear equations, however, Poisson's proper equations contain both compressible and incompressible fluid, taking no notice of the Navier's equations including both linear and non-linear terms until Rayleigh [20] in 1883. (cf. Table 1.)

---

[5](⇓) $(7\text{-}9)_{Pf}$ means the equation number with chapter of Poisson [17]

[6](⇓) Poisson[17, p.141]

[7](⇓) cf. (4)

3. *Drafts of 'On the dynamical theory of Gases'* **by Maxwell**

3.1. A progenitor of gas theory after Poisson and Stokes.

Even after Poisson, Saint-Venent and Stokes, we can cite the progenitors of microscopically descriptive, hydromechanical equations, which are specializes in gas theories, in which they describe the hydrodynamic equations, and they contribute to fix the tensor and equations of NS, so we have to trace them. cf. Table 1, 2.

Maxwell [12] had presented between late 1865 and early 1866, the original equations calculating his original coefficient, with which his tensor coincides with Poisson and Stokes, and his gas theory prior to Kirchhoff [8] in 1876 and Boltzmann [1] in 1895. Maxwell says as follows:

if the motion is not very violent we may also neglect $\frac{\partial}{\partial t}(\rho\xi^2 - p)$ and then we have

$$\xi^2\rho = p - \frac{M}{9k\rho\Theta_2}p\Big(2\frac{du}{dx} - \frac{dv}{dy} - \frac{dw}{dz}\Big) \tag{7}$$

which similar expressions for $\eta^2\rho$ and $\zeta^2\rho$. By transformation of coordinates we can easily obtain the expressions for $\xi\eta\rho$, $\eta\zeta\rho$ and $\zeta\xi\rho$. They are of the form

$$\zeta\xi\rho = -\frac{M}{6k\rho\Theta_2}p\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big) \tag{8}$$

$$\begin{bmatrix} \rho\overline{\xi_0^2} & \rho\overline{\xi_0\eta_0} & \rho\overline{\xi_0\zeta_0} \\ \rho\overline{\xi_0\eta_0} & \rho\overline{\eta_0^2} & \rho\overline{\eta_0\zeta_0} \\ \rho\overline{\xi_0\zeta_0} & \rho\overline{\zeta_0\eta_0} & \rho\overline{\zeta_0^2} \end{bmatrix} = \begin{bmatrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ Z_x & Z_y & Z_z \end{bmatrix} = \begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix},$$

Having thus obtained the values of the pressures in different directions we may substitute them in the equation of motion

$$\begin{cases} \rho\frac{\partial u}{\partial t} + \frac{d}{dx}(\rho\xi^2) + \frac{d}{dy}(\rho\xi\eta) + \frac{d}{dz}(\rho\xi\zeta) = X\rho, \\ \rho\frac{\partial v}{\partial t} + \frac{d}{dx}(\rho\xi\eta) + \frac{d}{dy}(\rho\eta^2) + \frac{d}{dz}(\rho\eta\zeta) = Y\rho, \\ \rho\frac{\partial w}{\partial t} + \frac{d}{dx}(\rho\xi\zeta) + \frac{d}{dy}(\rho\zeta\eta) + \frac{d}{dz}(\rho\zeta^2) = Z\rho. \end{cases} \tag{9}$$

This becomes as follows :

$$\begin{cases} \rho\frac{\partial u}{\partial t} + \frac{dp}{dx} - \frac{pM}{6k\rho\Theta_2}\Big[\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + \frac{1}{3}\frac{d}{dx}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big)\Big] = \rho X, \\ \rho\frac{\partial v}{\partial t} + \frac{dp}{dy} - \frac{pM}{6k\rho\Theta_2}\Big[\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + \frac{1}{3}\frac{d}{dy}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big)\Big] = \rho Y, \\ \rho\frac{\partial w}{\partial t} + \frac{dp}{dz} - \frac{pM}{6k\rho\Theta_2}\Big[\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2} + \frac{1}{3}\frac{d}{dz}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big)\Big] = \rho Z. \end{cases} \tag{10}$$

Maxwell states as follows:

> This is the equation of motion in the direction of x. The other equations may be written down by symmetry. The form of the equations is identical
> - with that deduced by Poisson [8] from the theory of elasticity by supposing the strain to be constantly relaxed at the given rate
> - and the ratio of the coefficients of ∇^2 to $\frac{d}{dx}\frac{1}{\rho}\frac{\partial\rho}{\partial t}$ agrees with that given by Professor Stokes, [9] which means (10) equals $(12)_S$.
>
> The quantity $\frac{pM}{6k\rho\Theta_2}$ is the coefficient of viscosity or of internal friction and is denoted by μ in the writings of Professor Stokes and in my paper on the Viscosity of Air and other Gases. [13, pp.261-262].

3.2. Law of Volumes.

In late 1865 or early 1866, Maxwell proposed this paper. It was likely that Boltzmann[10] had got his idea from this paper.

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[8](⇓) The Equation(9) in [17, p.139], which we cite as (5) $(7\text{-}9)_{Pf}$ above.
[9](⇓) Stokes [22]
[10](⇓) 1844-1906.

u, v, w are the components of the mean velocity of all the molecules which are at a given instant in a given element of volume, hence there is no motion of translation. ξ, η, ζ are the components of the relative velocity of one of these molecules with respect to the mean velocity, the 'velocity of agitation of molecules'.

In the case of a single gas in motion let Q be the total energy of a single molecule then

$$Q = \frac{1}{2}M\Big\{(u+\xi)^2 + (v+\eta)^2 + (w+\zeta)^2 + \beta(\xi^2+\eta^2+\zeta^2)\Big\}$$

and

$$\frac{\delta Q}{\delta t} = M(uX + vY + wZ).$$

The general equation becomes

$$\begin{aligned}
&\frac{1}{2}\rho\frac{\partial}{\partial t}\Big\{u^2+v^2+w^2+(1+\beta)(\xi^2+\eta^2+\zeta^2)\Big\} \\
+\;& \frac{d}{dx}(u\rho\xi^2 + v\rho\xi\eta + w\rho\xi\zeta) + \frac{d}{dy}(u\rho\xi\eta + v\rho\eta^2 + w\rho\eta\zeta) + \frac{d}{dz}(u\rho\xi\zeta + v\rho\eta\zeta + w\rho\zeta^2) \\
+\;& \frac{1}{2}\frac{d}{dx}(1+\beta)\rho\xi(\xi^2+\eta^2+\zeta^2) + \frac{1}{2}\frac{d}{dy}(1+\beta)\rho\eta(\xi^2+\eta^2+\zeta^2) + \frac{1}{2}\frac{d}{dz}(1+\beta)\rho\zeta(\xi^2+\eta^2+\zeta^2) \\
=\;& \rho(uX+vY+wZ).
\end{aligned}$$

Substituting the values of ρX, ρY and ρZ with $\frac{d\xi}{dx}$, $\frac{d\eta}{dy}$ and $\frac{d\zeta}{dz}$ and dividing by ρ of both hand-side, then

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}(1+\beta)(\xi^2+\eta^2+\zeta^2) \\
+\;& \xi^2\frac{du}{dx} + \eta^2\frac{dv}{dy} + \zeta^2\frac{dw}{dz} + \eta\zeta\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big) + \zeta\xi\Big(\frac{dw}{dx}+\frac{du}{dz}\Big) + \xi\eta\Big(\frac{du}{dy}+\frac{dv}{dx}\Big) \\
+\;& \frac{1}{2}(1+\beta)(\xi^2+\eta^2+\zeta^2)\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big) = 0.
\end{aligned}$$

If we set $\mathcal{R} \equiv \frac{2}{(1+\beta)}$, then we get the second, linear term of the left hand-side by Maxwell is written by tensor

$$\begin{bmatrix} \rho\xi^2 & \rho\xi\eta & \rho\xi\zeta \\ \rho\xi\eta & \rho\eta^2 & \rho\eta\zeta \\ \rho\xi\zeta & \rho\zeta\eta & \rho\zeta^2 \end{bmatrix} = -\mathcal{R}\begin{bmatrix} \frac{\partial u}{\partial x} & \Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big) & \Big(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\Big) \\ \Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big) & \frac{\partial v}{\partial y} & \Big(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\Big) \\ \Big(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\Big) & \Big(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\Big) & \frac{\partial w}{\partial z} \end{bmatrix}$$

which is Maxwell called it 'general tensor'.

3.3. Determination of the inequality of pressure in a medium.

Maxwell constructs the tensor with his viscosity coefficient as follows :

$$\begin{bmatrix} \rho\xi^2 & \rho\xi\eta & \rho\xi\zeta \\ \rho\xi\eta & \rho\eta^2 & \rho\eta\zeta \\ \rho\xi\zeta & \rho\zeta\eta & \rho\zeta^2 \end{bmatrix} = \begin{bmatrix} p - \frac{M}{9k\rho\Theta_2}p\Big(2\frac{du}{dx}-\frac{dv}{dy}-\frac{dw}{dz}\Big) & -\frac{M}{6k\rho\Theta_2}p\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big) & -\frac{M}{6k\rho\Theta_2}p\Big(\frac{dw}{dx}+\frac{du}{dz}\Big) \\ -\frac{M}{6k\rho\Theta_2}p\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big) & p - \frac{M}{9k\rho\Theta_2}p\Big(\frac{du}{dx}-2\frac{dv}{dy}-\frac{dw}{dz}\Big) & -\frac{M}{6k\rho\Theta_2}p\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big) \\ -\frac{M}{6k\rho\Theta_2}p\Big(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\Big) & -\frac{M}{6k\rho\Theta_2}p\Big(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\Big) & p - \frac{M}{9k\rho\Theta_2}p\Big(\frac{du}{dx}-\frac{dv}{dy}-2\frac{dw}{dz}\Big) \end{bmatrix} \quad (11)$$

Here, it tells of the equivalent in the structure between (11) and (8). If we set $\mathcal{R} \equiv \frac{Mp}{6k\rho\Theta_2}$, then these equations are completely equal to $(221)_B$ (=(24)) by Boltzmann. These facts state that Boltzmann had got his idea of special form of hydromechanics from Maxwell.

4. *'Lectures on Gas theory'* by Boltzmann

In general, according to Ukai [23], we can state the Boltzmann equations as follows: [11]

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f = Q(f,g), \quad t>0, \quad \mathbf{x},\mathbf{v}\in\mathbb{R}^n (n\ge 3), \quad \mathbf{x}=(x,y,z), \quad \mathbf{v}=(\xi,\eta,\zeta), \tag{12}$$

$$Q(f,g)(t,x,v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\{g(v'_*)f(v') - g(v_*)f(v)\}d\sigma dv_*, \quad g(v'_*) = g(t,x,v'_*), \tag{13}$$

$$v' = \frac{v+v_*}{2} + \frac{|v+v_*|}{2}\sigma, \quad v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad \sigma\in\mathbb{S}^{n-1} \tag{14}$$

where,

- $f = f(t,\ x,\ v)$ is interpretable as many meanings such as
  - density distribution of a molecule
  - number density of a molecule
  - probability density of a molecule

  at time : t, place : $\mathbf{x}$ and velocity : $\mathbf{v}$.
- $f(v)$ means $f(t,x,v)$ as abbreviating t and x in the same time and place with $f(v')$
- $Q(f,g)$ of the right-hand-side of (12) is the Boltzmann bilinear *collision operator.*
- $\mathbf{v}\cdot\nabla_{\mathbf{x}} f$ is the *transport operator,*
- $B(z,\sigma)$ of the right-hand-side in (13) is the non-negative function of *collision cross-section.*
- $Q(f,g)(t,x,v)$ is expressed in brief as $Q(f)$.
- $(v,\ v_*)$ and $(v',\ v'_*)$ are the velocities of a molecule before and after collision.
- According to Ukai [24], the *transport operators* are expressed with two sort of terms like Boltzmann's descriptions : $(114)_B$ and $(115)_B$ including the collision term $\nabla_{\mathbf{v}}\cdot(\mathbf{F}f)$ by exterior force $\mathbf{F}$ as follow : [12]

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}}\cdot(\mathbf{F}f) = Q(f) \tag{15}$$

$$Q(f) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\{f(v'_*)f(v') - f(v_*)f(v)\}d\sigma dv_* \tag{16}$$

where, $\mathbf{v}\cdot\nabla_{\mathbf{x}} f + \nabla_v\cdot(\mathbf{F}f)$ are *transport operators* operating under the exterior force : $\mathbf{F}(t,x,v) = (F_1,F_2,F_3)$. The right-hand side of (15) is expressed in brief as $Q(f)$ meaning $Q(f)(t,x,v)$.

4.1. Reduction of the partial differential equations for f and F.

We show the Figure 6 in the last page of our paper, which defines the model of the *collision* between the molecule m_1 calling the point of it and the molecule m wich we call the point m. The instant when the molecule m passes vertically throught the disc of m_1 molecule, is defined as *collision.* We show Boltzmann's definitions as follow :

> We fix our attention on the parallelepiped representing all space points whose coordinates lie between the limits [13]
>
> $$(97)_B \quad [x,\ x+dx], \quad [y,\ y+dy], \quad [z,\ z+dz], \quad do = dxdydz$$
>
> We now construct a second rectangular parallelepiped, which include all points whose coordinates lie between the limits
>
> $$(98)_B \quad [\xi,\ \xi+d\xi], \quad [\eta,\ \eta+d\eta], \quad [\zeta,\ \zeta+d\zeta]$$
>
> We set its volume equal to
>
> $$d\xi d\eta d\zeta = d\omega \tag{17}$$
>
> and we call it the parallelepiped $d\omega$. The molecules that are in do at the time t and whose velocity points lie in $d\omega$ at the same time will again be called the specified molecules, or the "dn molecules." Their number is clearly proportional to the product $do\cdot d\omega$. Then all volume elements immediately adjacent to do find themselves subject to similar conditions,

---

[11](⇓) We refer the *Lecture Note* by S.Ukai: *Boltzmann equations: New evolution of theory, Lecture Note of the Winter School in Kyushu of Non-linear Partial Differential Equations,* Kyushu University, 6-7, November, 2009.

[12](⇓) In the Boltzmann' original equations, they are used with two terms like $(114)_B$, $(115)_B$. We can refer the *General lecture in the autumn meeting of MSJ* by S.Ukai [24] : *The study of Boltzmann equations: past and future,* MSJ, 23, September, 2010.

[13](⇓) $(\ \cdot\ )_B$ in the top of the equation or expression means the number cited in Boltzmann[2] in below of our paper.

TABLE 3. The symbols and definitions

no	symbol	defined	content of conformation in modeling of collision. cf. Figure 6 in the last page.	cf.	m	m_1
1	$X,\ Y,\ Z$	(18)	The component of accelarating force of a molecule in a coödinate direction.			
2	mX, mY, mZ		The component of the external force acting on any m-molecule.		m	
3	$\xi,\ \eta,\ \zeta$	$(98)_B$	The component of velocity of any m-molecule in a coödinate direction.		m	
4	f	$(99)_B$	$f = f(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)$		m	
5	f_1	$(99)_B$	$f_1 = f(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1,\ t)$, different only with velocity of f.		m	
6	F	$(100)_B$	$F = F(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)$			m_1
7	F_1	$(103)_B$	$F_1 = F(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1,\ t)$, different only with velocity of F.			m_1
8	$\xi_1,\ \eta_1,\ \zeta_1$	$(102)_B$	The component of velocity of any m_1-molecule in a coödinate direction.			m_1
9	g	p.116	The moving direction (or velocity) of an m-molecule to an m_1-molecule.	Fig. 6	m	
10	gdt	p.116	The moving distance of an m-molecule to an m_1-molecule during dt.	Fig. 6	m	
11	b	$(104)_B$	The length of a line originated from m_1-molecule, where, b is the smallest possible distance of the two colliding molecules that could be attained if they moved without interaction in straight lines with the velocities they had before the collision. In other words, b is the line P_1P, where P_1 and P are the two points at which m_1 and m would be found at the moment of their closest approach if there were no interraction.	Fig. 6		m_1
12	σ		The limit of the length of a line. $[0,\ \sigma]$.	Fig. 6		m_1
13	ϵ	$(104)_B$	An angle formed between a line b and a line m_1H, where, ϵ is the angle between the two planes through the direction of relative motion, one parallel to P_1P along b, and the other to the abscissa axis.	Fig. 6		m_1
14	$\xi',\ \eta',\ \zeta'$	$(108)_B$	The component of velocity of a molecule after the *collision*.		m	
15	b'	$(109)_B$	The length of a line after the *collision*.	Fig. 6		m_1
16	ϵ'	$(109)_B$	An angle formed between a line b and a line m_1H after the *collision* .	Fig. 6		m_1
17	do : parallelepiped	$(97)_B$	We set $do = dxdydz$ in which the m-molecules lie, and we always call this parallelepiped the parallelepiped do.		m	
18	$d\omega$: parallelepiped of velocity point	(98_B) (17)	We set $d\omega = d\xi d\eta d\zeta$ in which velocity point of the m-molecules lie, and we always call this parallelepiped the parallelepiped $d\omega$.		m	
19	$d\omega_1$	(102_B) (21)	We set $d\omega_1 = d\xi_1 d\eta_1 d\zeta_1$ as well as $d\omega$, in which velocity point of the m_1-molecules lie, and we always call this parallelepiped the parallelepiped $d\omega_1$.			m_1
20	dn	$(99)_B$	The m-molecules that are in do at time t and whose velocity points lie in $d\omega$ at the same time will again be called the specified molecules, or the "dn molecules." $dn = f(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)dod\omega = fdod\omega$		m	
21	dn'	$(99)'_B$	The number of m-molecules that satisfy the conditions $(97)_B$ and $(98)_B$ at time $t + dt$. $dn' = f(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t + dt)dod\omega$		m	
22	dN	$(100)_B$	The number of m_1-molecules that satisfy the conditions $(97)_B$ and $(98)_B$ at time t. $dN = F(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)dod\omega = Fdod\omega$			m_1
23	dN_1	$(103)_B$	$dN_1 = F(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1,\ t)dod\omega = F_1 dod\omega_1$			m_1
24	ν_1	$(107)_B$	The number of all *collisions* of our dn molecules during dt with m_1-molecules.		m	m_1
25	ν_2	$(106)_B$	The number of m-points that pass an m_1-point at any distance less than σ during dt.		m	m_1
26	ν_3	$(105)_B$	The number of *collisions* between m-molecules and m_1-molecules.		m	m_1
27	V_1	(19)	The increase which dn experiences as a result of motion of the molecules during time dt, where all m-molecules whose velocity points lie in $d\omega$ move in the x-direction with velocity ξ, in the y-direction with velocity η, and in the z-direction with velocity ζ.	$A_2(\varphi)$	m	
28	V_2	(20)	As a result of the action of *external forces*, the velocity components of all the molecules change with time, and hence the velocity points of the molecules in do will move.	$A_3(\varphi)$	m	
28	i_1	$(111)_B$	The *total increase* experienced by dn as a result of *collisions* of m-molecules with m_1-molecules.		m	m_1
30	V_3	$(112)_B$	The *net increase* experienced by dn as a result of *collisions* of m-molecules with m_1-molecules. $V_3 = i_1 - \nu_1$.	$A_4(\varphi)$	m	m_1
31	V_4	$(113)_B$	The increment experienced by dn as a result of *collisions* of m or m_1-molecules with *each other*.	$A_5(\varphi)$	m	m_1
32	$\varphi,\ \sum_{d\omega,do}\varphi$	$(116)_B$	$\varphi = \varphi(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)$, $\sum_{d\omega,do}\varphi \equiv \varphi fdod\omega$, multipling the number $fdod\omega$ by φ		m	
33	$\Phi,\ \sum_{d\omega,do}\Phi$	$(117)_B$	$\Phi = \Phi(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)$, $\sum_{d\omega,do}\Phi \equiv \Phi Fdod\omega$, multipling the number $Fdod\omega$ by Φ		m	
34	$\Phi_1,\ \sum_{d\omega,do}\Phi_1$	$(117)_B$	$\Phi_1 = \Phi(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1,\ t)$, $\sum_{d\omega,do}\Phi_1 \equiv \Phi_1 F_1 dod\omega_1$, multipling the number $F_1 dod\omega$ by Φ_1			m_1
35	$A_1(\varphi)$	$(121)_B$	The effect of explicit dependance of φ on t.			
36	$A_2(\varphi)$	$(122)_B$	The effect of *the motion* of the molecules.	V_1	m	
37	$A_3(\varphi)$	$(123)_B$	The effect of *external forces*.	V_2	m	
38	$A_4(\varphi)$	$(124)_B$	The effect of collisions of m-molecules with m_1-molecules.	V_3	m	m_1
39	$A_5(\varphi)$	$(125)_B$	The effect of collisions of m-molecules with *each other*.	V_4	m	
40	$B_1(\varphi)$	$(127)_B$	The total effect in ω of explicit dependance of φ on t.			
41	$B_2(\varphi)$	$(128)_B$	The effect in ω of *the motion* of the molecules.	V_1	m	
42	$B_3(\varphi)$	$(129)_B$	The effect in ω of *external forces*.	V_2	m	
43	$B_4(\varphi)$	$(134)_B$	The effect in ω of collisions of m-molecules with m_1-molecules.	V_3	m	m_1
44	$B_5(\varphi)$	$(139)_B$	The effect in ω of collisions of m-molecules with *each other*.	V_4	m	
44	$\{C_n(\varphi)\}_1^5$	$(125)_B$	The effect in ω and o as the same as $\{A_n(\varphi)\}_1^5$ or $\{B_n(\varphi)\}_1^5$			

so that in a parallelepiped twice as large there will be twice as many molecules. We can therefore set this number equal to

$$(99)_B \quad dn = f(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)dod\omega = fdod\omega$$

Similarly the number of m_1-molecules that satisfy the conditions $(97)_B$ and $(98)_B$ at time t will be :

$$(100)_B \quad dN = F(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)dod\omega = Fdod\omega$$

The two functions f and F completely characterize the state of motion, the mixing ratio, and the velocity distribution at all places in the gas mixture. We shall allow a very short time dt to elapse, and during this time we keep the size and position of do and $d\omega$ completely unchanged. The number of m-molecules that satisfy the conditions $(97)_B$ and $(98)_B$ at time $t+dt$ is, according to Equation $(99)_B$,

$$dn' = f(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t+dt)dod\omega = fdod\omega$$

and the total increase experienced by dn during time dt is

$$(101)_B \quad dn' - dn = \frac{\partial f}{\partial t} do\ d\omega\ dt.$$

$\xi,\ \eta,\ \zeta$ are the rectangular coödinates of the velocity point. Although this is only an imaginary point, still it moves like the molecule itself in space. Since $X,\ Y,\ Z$ are the components of the accelerating force,[14] we have:

$$\frac{d\xi}{dt} = X, \quad \frac{d\eta}{dt} = Y, \quad \frac{d\zeta}{dt} = Z \tag{18}$$

4.2. **Four different causes bringing up increase of dn.**

Boltzmann explains an increase of dn as a result of the following *four different causes* of V_1, V_2, V_3 and V_4 :

- V_1 : increment by *transport* through do
- V_2 : increment by *transport* of external force
- V_3 : increment as a result of *collisions* of m-molecules with m_1-molecules
- V_4 : increment by *collision* of molecules with each other

We extract an outline by the Boltzmann [1] as follows :

The number dn experiences an increase as a result of *four different causes.*

(1) (V_1 : increase going out through do ;) All m-molecules whose velocity points lie in $d\omega$ move in the x-direction with velocity ξ, in the y-direction with velocity η, and in the z-direction with velocity ζ.

Hence through the left of the side of the parallelepiped do facing the negative abscissa direction there will enter during time dt as many molecules satisfying the condition (98_B) as may be found, at the beginning of dt, in a parallelepiped of base $dydz$ and height ξdt,[15] viz. $\xi \cdot f(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)dydzd\omega dt$ molecules. Likewise, for the number of $m-$molecules that satisfying (98_B) and go out through the opposite face of do during time dt, the value:

$$\xi \cdot f(x+dx,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)dydzd\omega dt$$

By similar arguments for the four other sides of the parallelepiped, one finds that during time dt,

$$-\Big(\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z}\Big)do \cdot d\omega\ dt$$

more molecules satisfying (98_B) enter do than leave it. This is therefore the increase V_1 which dn experiences as a result of motion of the molecules during time dt.

$$V_1 = -\Big(\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z}\Big)do\ d\omega\ dt \tag{19}$$

---

[14](⇓) Da $X,\ Y,\ Z$ die Componenten der beschleunigenden Kraft sind, so ist: ··· Boltzmann [2, p.103].

[15](⇓) ξ : the x-direction with velocity multiplied by dt becomes the length of a edge of which consists a parallelepiped with a base $dydz$.

(2) (V_2 : increase by external force ;) As a result of the action of external forces, the velocity components of all the molecules change with time, and hence the velocity points of the molecules in do will move. Some velocity points will leave $d\omega$, others will come in, and since we always include in the number dn only those molecules whose velocity points lie in $d\omega$, dn likewise be changed for this reason.

$$V_2 = -\Big(X\frac{\partial f}{d\xi} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z}\Big) do\ d\omega\ dt \tag{20}$$

Boltzmann defines the effects of *collisions* as follows :

(3) (V_3 : increase as a result of *collisions* of m-molecules with m_1-molecules ;) Those of our dn molecules that undergo a *collision* during the time dt will clearly have in general different velocity components after the *collision*.

- (Decrease :) Their velocity points will therefore be expected, as it were, from the parallelepiped by the *collision*, and thrown into a completely different parallelepiped. The number dn will thereby be *decreased*.
- (Increase :) On the other hand, the velocity points of m-molecules in other parallelepipeds will be throne into $d\omega$ by *collisions*, and dn will thereby *increase*.
- (Total increase by *collision* between m-molecules and m_1-molecules :) It is now a question of finding this total increase V_3 experienced by dn during time dt as a result of the *collisions* taking place between any m-molecules and any m_1-molecules.

For this purpose we shall fix our attention on a very small fraction of the total number ν_1 of *collisions* undergone by our dn molecules during time dt with m_1-molecules. We construct a third parallelepiped which includes all points whose coordinates lie between the limits

$$(102)_B \quad [\xi_1,\ \xi_1 + d\xi_1], \quad [\eta_1,\ \eta_1 + d\eta_1], \quad [\zeta_1,\ \zeta_1 + d\zeta_1]$$

Its volume is

$$d\omega_1 = d\xi_1 d\eta_1 d\zeta_1 \tag{21}$$

It constitutes the parallelepiped $d\omega_1$. By analogy with Equation $(100)_B$, the number of m_1-molecules in do whose velocity points lie in $d\omega_1$ at time t is :

$$(103)_B \quad dN_1 = F_1 do d\omega_1,$$

where F_1 is an abbreviation for $F(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1)$.

Boltzmann defines a *passage* of an m-point by an m_1-point as follows :

(a) (How to pass :) We define a passage of an m-point by an m_1-point as that instant of time when distance between the points has its smallest value ; thus m would pass through the plane through m_1 perpendicular to the direction g, if no interaction took place between the two molecules.

(b) (ν_2 : the number of passages of an m-point by an m_1-point :) Hence, ν_2 is equal to the number of passages of an m-point by an m_1-point that occurs during time dt, such that the smallest distance between the two molecules is less than σ.

(c) (A plane E :) In order to find this number, we draw through each m_1-point a plane E moving with m_1, perpendicular to the direction of g, and a line G, which parallel to this direction.

(d) (When a passage ends :) As soon as an m-point crosses E, a passage take place between it and the m_1-point.

(e) (A line m_1X :) We draw through each m_1-point a line m_1X parallel to the positive abscissa direction and similarly directed.

(f) (Half-plane :) The half-plane bounded by G, which contains the latter line, cuts E in the line m_1H, which of course again contains each m_1-point.

(g) (b and ϵ :) Furthermore, we draw from each m_1-point in each of the plane E a line of length b, which forms an angle ϵ with the line m_1H.

(h) (Rectangles of surface area R formed by b and ϵ :) All points of the plane E for which b and ϵ lie between the limits

$$(104)_B \quad [b,\ b+db], \quad [\epsilon,\ \epsilon+d\epsilon]$$

form a rectangle of surface area $R = bdbd\epsilon$.

In his Figure 6, [16] the intersections of all these lines with a sphere circumscribed about m_1 are shown. The large circle (shown as an ellipse) lies in the plane E ; the circular arc GXH lies in the half-plane defined above. In each of planes E, an equal and identically situated rectangle will be found. We consider for the moment only those passages of an m-point by an m_1-point in which the first point penetrates one of the rectangles R.

Below, Boltzmann calculates V_3 in order of $\Pi \to \nu_3 \to \nu_2 \to \nu_1 \to i_1 \to V_3$.

At first,

$$\Pi = Rgdt = \underbrace{bdbd\epsilon}_{R} gdt, \quad \sum \Pi = dN_1\Pi = \underbrace{F_1 dod\omega_1}_{dN_1\ (103)_B} \underbrace{gbdbd\epsilon dt}_{\Pi}$$

Since these volumes are infinitesimal, and lie infinitely close to the point with coordinates $x,\ y,\ x$, then by analogy with Equation $(99)_B$ the number of m-points (i.e., m-molecules whose velocity points lie in $d\omega$) that are initially in the volumes $\sum \Pi$ is equal to :

$$(105)_B \quad \nu_3 = fd\omega \sum \Pi = fF_1 dod\omega d\omega_1 gbdbd\epsilon dt$$

This is at the same time the number of m-points that pass an m_1-point during time dt at a distance between b and $b+db$, in such a way that the angle ϵ lie between ϵ and $\epsilon + d\epsilon$.

By ν_2 we mean the number of m-points that pass an m_1-point at any distance less than σ during dt. We find ν_2 by integrating the differential expression ν_3 over ϵ from 0 to 2π, and over b from 0 to σ.

$$(106)_B \quad \nu_2 = \int_0^{\sigma} db \int_0^{2\pi} \nu_3 d\epsilon = dod\omega d\omega_1 dt \int_0^{\sigma} db \int_0^{2\pi} d\epsilon\ g \cdot b \cdot f \cdot F_1 \cdot$$

The number denoted by ν_1 of all *collisions* of our dn molecules during dt with m_1-molecules is therefore found by integrating over the three variable $\xi_1,\ \eta_1,\ \zeta_1$ whose differentials occur in $d\omega_1$, from $-\infty$ to $+\infty$; we indicate this a single integral sign :

$$(107)_B \quad \nu_1 = \int_{-\infty}^{\infty} \nu_2 d\omega_1 = do \cdot d\omega \cdot dt \int_{-\infty}^{\infty} d\omega_1 \int_0^{\sigma} db \int_0^{2\pi} fF_1 gbd\epsilon$$

We shall consider again those *collisions* between m-molecules and m_1-molecules, whose number was denoted by ν_3 and is given by Equation $(105)_B$.

These are the *collisions* that occur in unit time in the volume element do in such a way the following conditions are satisfied :

• The velocity components of the m-molecules and the m_1-molecules lie between the limits $(98)_B$ and $(102)_B$, respectively, before the interaction begins.

• We denote by b the closest distance of approach that would be attained if the molecules did not interact but retained the velocities they had before the *collision*.

The *total increment* i_1 experienced by dn as a result of *collisions* of m-molecules with m_1-molecules is founded by integrating over ϵ from 0 to 2π, over b from 0 to σ,

[16]($\Downarrow$) We show this Figure 6 in the last page of our paper citing [2, p.107], which is equal to [1, p.117], however, we must correct the symbol R by H of [1, p.117].

and over ξ_1, η_1, ζ_1 from $-\infty$ to $+\infty$. We shall write the result of this integration in the form :

$$(111)_B \quad i_1 = do d\omega dt \iint_0^\sigma \int_0^{2\pi} f' F_1' g b d\omega_1 db d\epsilon$$

Of course we cannot perform explicitly the integration with respect to b and ϵ since the variable ξ', η', ζ' and ξ_1', η_1', ζ_1' occurring in f' and F_1' are functions of (ξ, η, ζ, ξ_1', η_1', ζ_1', b and ϵ), which cannot be computed until the force law is given.[17]

The difference $i_1 - \nu_1$ expresses the *net increase* of dn during time dt as a result of *collisions* of m-molecules with m_1-molecules. It is therefore the *total increase* V_3 experienced by dn as a result of these *collisions*, and one has

$$(112)_B \quad V_3 = i_1 - \nu_1 = do d\omega dt \iint_0^\sigma \int_0^{2\pi} (f' F_1' - f F_1) d\omega_1 db d\epsilon$$

(4) (V_4 : increment by collision of molecules with each other ;) The increment V_4 experienced by dn as a result of *collisions* of m-molecules with each other is found from Equation $(112)_B$ by a simple permutation. One now uses ξ_1, η_1, ζ_1 and ξ_1', η_1', ζ_1' for the velocity components of the other m-molecule *before and after the collision,* respectively, and one writes f_1 and f_1' for

$$f_1 = f(x, y, z, \xi_1, \eta_1, \zeta_1, t) \quad \text{and} \quad f_1' = f(x, y, z, \xi_1', \eta_1', \zeta_1', t)$$

Then : $(113)_B \quad V_4 = do d\omega dt \iint_0^\infty \int_0^{2\pi} (f' f_1' - f f_1) g b d\omega_1 db d\epsilon.$

4.3. Formulation of Boltzmann's transport equations.

According to Boltzmann[2, pp.110-115], [18] his equations (so-called *transport equations*) are the following : [19]

> Since now $V_1 + V_2 + V_3 + V_4$ is equal to the increment $dn' - dn$ of dn during time dt, and this according to Equation $(101)_B$ must be equal to $\frac{\partial f}{\partial t} do d\omega dt$, one obtains on substituting all the appropriate value and deviding by $do d\omega dt$ the following partial differential equation for the function f :
>
> $$(114)_B \quad \frac{\partial f}{\partial t} + \underbrace{\xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z}}_{V_1} + \underbrace{X \frac{\partial f}{\partial x} + Y \frac{\partial f}{\partial y} + Z \frac{\partial f}{\partial z}}_{V_2}$$
> $$= \underbrace{\iint_0^\infty \int_0^{2\pi} \left[(f' F_1' - f F_1) + (f' f_1' - f f_1) \right] g b \, d\omega_1 \, db \, d\epsilon}_{V_3 + V_4}$$

Similarly we obtain the equation of F :

$$(115)_B \quad \frac{\partial F_1}{\partial t} + \underbrace{\xi_1 \frac{\partial F_1}{\partial x} + \eta_1 \frac{\partial F_1}{\partial y} + \zeta_1 \frac{\partial F_1}{\partial z}}_{V_1} + \underbrace{X_1 \frac{\partial F_1}{\partial x} + Y_1 \frac{\partial F_1}{\partial y} + Z_1 \frac{\partial F_1}{\partial z}}_{V_2}$$
$$= \underbrace{\iint_0^\infty \int_0^{2\pi} \left[(f' F_1' - f F_1) + (F' F_1' - F F_1) \right] g b \, d\omega_1 \, db \, d\epsilon}_{V_3 + V_4}$$

where,

$$\begin{cases} f = f(x, y, z, \xi, \eta, \zeta, t), & f_1 = f(x, y, z, \xi_1, \eta_1, \zeta_1, t), & f_1' = f(x, y, z, \xi_1', \eta_1', \zeta_1', t), \\ F = F(x, y, z, \xi, \eta, \zeta, t), & F_1 = F(x, y, z, \xi_1, \eta_1, \zeta_1, t), & F_1' = F(x, y, z, \xi_1', \eta_1', \zeta_1', t) \end{cases} \tag{22}$$

Namely, we can verify $(114)_B$ for f :

---

[17]($\Downarrow$) Hier kann die Integration nach b und ϵ natürlich nicht mehr sofort aus geführt werden, da die in f' and F_1' vorkommen den Variabeln ξ', η', ζ' und ξ_1', η_1', ζ_1' Function von ξ, η, ζ, ξ_1', η_1', ζ_1', b und ϵsind, welche nur berechnet werden können, wenn Virkungsgesetz der während eines Zusammenstosses wirksamen Kräfte gegeben ist. [2, p.112].

[18]($\Downarrow$) Boltzmann(1844-1906) had put the date in the foreword to part I as September in 1895, part II as August in 1898.

[19]($\Downarrow$) We mean the equation number in the left-hand side with $(\cdot)_B$ the citations from the Boltzmann[2] or [1]. We state only the symbol $\int$ instead of $\int_{-\infty}^{\infty}$. cf. $(107)_B$.

TABLE 4. Combination of function before and after collision

no	item	V_3 before	V_3 after	f of V_4 before	f of V_4 after	F of V_4 before	F of V_4 after
1	function of m_1	f	f'	f	f'	F	F'
2	function of m	F_1	F_1'	f_1	f_1'	F_1	F_1'
3	increment		$f'F_1' - fF_1$		$f'f_1' - ff_1$		$F'F_1' - FF_1$

$$\begin{aligned}\frac{V_1+V_2+V_3+V_4}{dod\omega dt} &= \frac{\partial f}{\partial t} = -\underbrace{\left(\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial z}\right)}_{V_1} - \underbrace{\left(X\frac{\partial f}{d\xi}+Y\frac{\partial f}{\partial y}+Z\frac{\partial f}{\partial z}\right)}_{V_2}\\ &+ \underbrace{\iint_0^\infty\int_0^{2\pi}(f'F_1'-fF_1)gb\cdot d\omega_1 dbd\epsilon}_{V_3} + \underbrace{\iint_0^\infty\int_0^{2\pi}(f'f_1'-ff_1)gb\cdot d\omega_1 dbd\epsilon.}_{V_4}\end{aligned}$$

Similarly we obtain $(115)_B$ for F.

$$\begin{aligned}\frac{V_1+V_2+V_3+V_4}{dod\omega dt} &= \frac{\partial F_1}{\partial t} = -\left(\xi\frac{\partial F_1}{\partial x}+\eta\frac{\partial F_1}{\partial y}+\zeta\frac{\partial F_1}{\partial z}\right) - \left(X\frac{\partial F_1}{d\xi}+Y\frac{\partial F_1}{\partial y}+Z\frac{\partial F_1}{\partial z}\right)\\ &+ \iint_0^\infty\int_0^{2\pi}(f'F_1'-fF_1)gb\cdot d\omega_1 dbd\epsilon + \iint_0^\infty\int_0^{2\pi}(F'F_1'-FF_1)gb\cdot d\omega_1 dbd\epsilon.\end{aligned}$$

($\Downarrow$) Here, we can confirm the identity with the today's description of the Boltzmann equations (12) and (13) :

$$\partial_t f + \underbrace{\mathbf{v}\cdot\nabla_{\mathbf{x}} f}_{V_1} + \underbrace{\mathbf{w}\cdot\nabla_{\mathbf{v}} f}_{V_2} = \underbrace{Q(f,g)}_{V3,\ V4},\quad \partial_t F + \underbrace{\mathbf{v}\cdot\nabla_{\mathbf{x}} F}_{V_1} + \underbrace{\mathbf{w}\cdot\nabla_{\mathbf{v}} F}_{V_2} = \underbrace{Q(F,G)}_{V3,\ V4},$$

$$Q(f,g)(t,x,v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\{g(v_*')f(v') - g(v_*)f(v)\}d\sigma dv_*,\quad g(v_*') = g(t,x,v_*'),\ \text{etc.}$$

$$t>0,\quad \mathbf{x},\mathbf{v},\mathbf{w}\in\mathbb{R}^n(n\ge 3),\quad \mathbf{x}=(x,y,z),\quad \mathbf{v}=(\xi,\eta,\zeta),\quad \mathbf{w}=(X,\ Y,\ Z).$$

In the case of (15) and (16)

$$\partial_t f + \underbrace{\mathbf{v}\cdot\nabla_{\mathbf{x}} f}_{V_1} + \underbrace{\nabla_{\mathbf{v}}\cdot(\vec{F}f)}_{V_2} = \underbrace{Q(f)}_{V_3,V_4}$$

$$Q(f) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\{f(v_*')f(v') - f(v_*)f(v)\}d\sigma dv_*$$

4.4. General form of the hydrodynamic equations.

As the general expressions for fluid mechanics, he states that when we substitute for $\frac{\partial f}{\partial t}$ its value from Equation $(114)_B$, it turns into $(120)_B$, $(126)_B$, $(140)_B$, a sum of five terms, each of which has its own physical meaning, as follows:

$$\begin{cases}(116)_B\ \sum_{d\omega,do}\varphi \equiv \varphi f dod\omega, & (120)_B\ \frac{\partial}{\partial t}\sum_{d\omega,do}\varphi = \left(f\frac{\partial\varphi}{\partial t}+\varphi\frac{\partial f}{\partial t}\right)dod\omega = \left[\sum_{n=1}^5 A_n(\varphi)\right]dod\omega,\\ (117)_B\ \sum_{d\omega,do}\Phi \equiv \Phi F dod\omega_1, & \sum_{d\omega,do}\Phi_1 = \Phi_1 F_1 dod\omega_1,\\ (118)_B\ \sum_{\omega,do}\varphi \equiv do\int\varphi f d\omega, & (126)_B\ \frac{\partial}{\partial t}\sum_{\omega,do}\varphi = do\int\left(f\frac{\partial\varphi}{\partial t}+\varphi\frac{\partial f}{\partial t}\right)d\omega = \left[\sum_{n=1}^5 B_n(\varphi)\right]do,\\ (119)_B\ \sum_{\omega,o}\varphi \equiv \iint\varphi f dod\omega, & (140)_B\ \frac{d}{dt}\sum_{\omega,o}\varphi = \iint\left(f\frac{\partial\varphi}{\partial t}+\varphi\frac{\partial f}{\partial t}\right)dod\omega = \sum_{n=1}^5 C_n(\varphi)\end{cases}$$

4.5. Special form of the incompressible, hydrodynamic equations.

$$(171)_B\ \frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$$

$$(173)_B\ \begin{cases}\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right) = \rho X - \frac{\partial(\rho\overline{\xi_0^2})}{\partial x} - \frac{\partial(\rho\overline{\xi_0\eta_0})}{\partial y} - \frac{\partial(\rho\overline{\xi_0\zeta_0})}{\partial z},\\ \rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right) = \rho Y - \frac{\partial(\rho\overline{\xi_0\eta_0})}{\partial x} - \frac{\partial(\rho\overline{\eta_0^2})}{\partial y} - \frac{\partial(\rho\overline{\zeta_0\eta_0})}{\partial z},\\ \rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) = \rho Z - \frac{\partial(\rho\overline{\xi_0\zeta_0})}{\partial x} - \frac{\partial(\rho\overline{\eta_0\zeta_0})}{\partial y} - \frac{\partial(\rho\overline{\zeta_0^2})}{\partial z}\end{cases}$$

Boltzmann says "these equations as well as Equation $(171)_B$, are *only special cases of the general equation* $(126)_B$ and were derived from it by Maxwell and (following him) by Kirchhoff." Boltzmann concludes that if one collects all these terms, then Equation (126) reduces in this special case to:

$$(177)_B \quad \frac{\partial(\rho\overline{\varphi})}{\partial t}+\frac{\partial(\rho\overline{\xi\varphi})}{\partial x}+\frac{\partial(\rho\overline{\eta\varphi})}{\partial y}+\frac{\partial(\rho\overline{\zeta\varphi})}{\partial z}-\rho\Big[X\frac{\overline{\partial\varphi}}{\partial\xi}+Y\frac{\overline{\partial\varphi}}{\partial\eta}+Z\frac{\overline{\partial\varphi}}{\partial\zeta}\Big]=\underbrace{m\Big[B_4(\varphi)+B_5(\varphi)\Big]}_{collision\ terms}$$

Boltzmann states about $(177)_B$:

> From this equation Maxwell calculated the viscosity, diffusion, and heat conduction and Kirchhoff therefore calls it the basic equation of the theory. If one sets $\varphi = 1$, he obtains at once the continuity equation (171); for it follows from Equations (134) and (137) that $B_4(1) = B_5(1) = 0$. Subtraction of the continuity equation, multiplied by φ, from (177) gives (using the substitution [158]): [1, p.152].

where, (158) : $\xi = \xi_0 + u, \quad \eta = \eta_0 + v, \quad \zeta = \zeta_0 + w.$

$$(178)_B \quad \rho\Big(\frac{\partial\overline{\varphi}}{\partial t}+u\frac{\partial\overline{\varphi}}{\partial x}+v\frac{\partial\overline{\varphi}}{\partial y}+w\frac{\partial\overline{\varphi}}{\partial z}\Big)+\frac{\partial(\rho\overline{\xi_0\varphi})}{\partial x}+\frac{\partial(\rho\overline{\eta_0\varphi})}{\partial y}+\frac{\partial(\rho\overline{\zeta_0\varphi})}{\partial z}-\rho\Big[X\frac{\overline{\partial\varphi}}{\partial\xi}+Y\frac{\overline{\partial\varphi}}{\partial\eta}+Z\frac{\overline{\partial\varphi}}{\partial\zeta}\Big]$$
$$=\underbrace{m\Big[B_4(\varphi)+B_5(\varphi)\Big]}_{collision\ terms}$$

If one denotes the six quantities $(179)_B : \rho\overline{\xi_0^2}, \rho\overline{\eta_0^2}, \rho\overline{\zeta_0^2}, \rho\overline{\eta_0\zeta_0}, \rho\overline{\xi_0\zeta_0}, \rho\overline{\xi_0\eta_0}$ by X_x, Y_y, Z_z, $Y_z = Z_y$, $Z_x = X_z$, $X_y = Y_x$, namely, when we use the symmetric tensor, then we get the following :

$$\begin{bmatrix} \rho\overline{\xi_0^2} & \rho\overline{\xi_0\eta_0} & \rho\overline{\xi_0\zeta_0} \\ \rho\overline{\xi_0\eta_0} & \rho\overline{\eta_0^2} & \rho\overline{\eta_0\zeta_0} \\ \rho\overline{\xi_0\zeta_0} & \rho\overline{\zeta_0\eta_0} & \rho\overline{\zeta_0^2} \end{bmatrix} = \begin{bmatrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ Z_x & Z_y & Z_z \end{bmatrix} = \begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix}, \qquad (23)$$

$$(180)_B \quad \begin{cases} \rho\Big(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\Big)+\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}=\rho X, \\ \rho\Big(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\Big)+\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}=\rho Y, \\ \rho\Big(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\Big)+\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}=\rho Z \end{cases}$$

These are not NS equations for lack of the pressure term. Moreover $(181)_B$: $p = \rho\overline{\xi_0^2} = \rho\overline{\eta_0^2} = \rho\overline{\zeta_0^2}$, $\overline{\xi_0\eta_0} = \overline{\xi_0\zeta_0} = \overline{\eta_0\zeta_0} = 0$. Here, he assumes that from the supposition of isotropy and homogeneity, $p = \frac{1}{3}(X_x + Y_y + Z_z)$, which is the same as the principle by Saint-Venant or Stokes.

He deduces a special case of the hydrodynamic equations as follows:

> For the present, we assume as a fact of experience that in gases the normal pressure is always nearly equal in all directions, and that tangential elastic forces are very small, so that Equations (181) are approximately true. Substitution of the values given by this equation into Equation (173) yields:

$$(183)_B \quad \begin{cases} \rho\Big(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\Big)+\frac{\partial p}{\partial x}-\rho X=0, \\ \rho\Big(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\Big)+\frac{\partial p}{\partial y}-\rho Y=0, \\ \rho\Big(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\Big)+\frac{\partial p}{\partial z}-\rho Z=0 \end{cases}$$

which are the so-called Euler equations in incompressible condition of $(171)_B$.

$$(185)_B \quad \begin{cases} \rho\frac{\partial u}{\partial t}+\frac{\partial(\rho\overline{\xi_0^2})}{\partial x}+\frac{\partial(\rho\overline{\xi_0\eta_0})}{\partial y}+\frac{\partial(\rho\overline{\xi_0\zeta_0})}{\partial z}-\rho X=0, \\ \rho\frac{\partial v}{\partial t}+\frac{\partial(\rho\overline{\xi_0\eta_0})}{\partial x}+\frac{\partial(\rho\overline{\eta_0^2})}{\partial y}+\frac{\partial(\rho\overline{\eta_0\zeta_0})}{\partial z}-\rho Y=0, \\ \rho\frac{\partial w}{\partial t}+\frac{\partial(\rho\overline{\xi_0\zeta_0})}{\partial x}+\frac{\partial(\rho\overline{\zeta_0\eta_0})}{\partial y}+\frac{\partial(\rho\overline{\zeta_0^2})}{\partial z}-\rho Z=0 \end{cases}$$

We set the values of (23) as follows, which is the same tensor as Stokes :

$$(220)_B \quad \begin{bmatrix} \rho\overline{\xi_0^2} & \rho\overline{\xi_0\eta_0} & \rho\overline{\xi_0\zeta_0} \\ \rho\overline{\xi_0\eta_0} & \rho\overline{\eta_0^2} & \rho\overline{\eta_0\zeta_0} \\ \rho\overline{\xi_0\zeta_0} & \rho\overline{\zeta_0\eta_0} & \rho\overline{\zeta_0^2} \end{bmatrix} = \begin{bmatrix} p-2\mathcal{R}\Big\{\frac{\partial u}{\partial x}-\frac{1}{3}\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\Big)\Big\} & -\mathcal{R}\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big) & -\mathcal{R}\Big(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\Big) \\ -\mathcal{R}\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big) & p-2\mathcal{R}\Big\{\frac{\partial v}{\partial y}-\frac{1}{3}\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\Big)\Big\} & -\mathcal{R}\Big(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\Big) \\ -\mathcal{R}\Big(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\Big) & -\mathcal{R}\Big(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\Big) & p-2\mathcal{R}\Big\{\frac{\partial w}{\partial z}-\frac{1}{3}\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\Big)\Big\} \end{bmatrix}$$

From $(220)_B$, we calculate the components of $(185)_B$ as follows:

$$\begin{bmatrix} \frac{\partial(\rho\overline{\xi_0^2})}{\partial x} & \frac{\partial(\rho\overline{\xi_0\eta_0})}{\partial y} & \frac{\partial(\rho\overline{\xi_0\zeta_0})}{\partial z} \\ \frac{\partial(\rho\overline{\xi_0\eta_0})}{\partial x} & \frac{\partial(\rho\overline{\eta_0^2})}{\partial y} & \frac{\partial(\rho\overline{\eta_0\zeta_0})}{\partial z} \\ \frac{\partial(\rho\overline{\xi_0\zeta_0})}{\partial x} & \frac{\partial(\rho\overline{\zeta_0\eta_0})}{\partial y} & \frac{\partial(\rho\overline{\zeta_0^2})}{\partial z} \end{bmatrix} = \begin{bmatrix} p-\mathcal{R}\left\{2\frac{\partial u}{\partial x}-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right\} & -\mathcal{R}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) & -\mathcal{R}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) \\ -\mathcal{R}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) & p-\mathcal{R}\left\{2\frac{\partial v}{\partial y}-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right\} & -\mathcal{R}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ -\mathcal{R}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) & -\mathcal{R}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) & p-\mathcal{R}\left\{2\frac{\partial w}{\partial z}-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right\} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

Then, substitution of these values into the equations of motion $(185)_B$ yields:

$$(221)_B \quad \begin{cases} \rho\frac{\partial u}{\partial t}+\frac{\partial p}{\partial x}-\mathcal{R}\left[\Delta u+\frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]-\rho X=0, \\ \rho\frac{\partial v}{\partial t}+\frac{\partial p}{\partial y}-\mathcal{R}\left[\Delta v+\frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]-\rho Y=0, \\ \rho\frac{\partial u}{\partial t}+\frac{\partial p}{\partial z}-\mathcal{R}\left[\Delta w+\frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]-\rho Z=0 \end{cases} \qquad (24)$$

We can interpret that as the special cases, Boltzmann have deduced the NS equations after substituting the tensor $(220)_B$ to $(173)_B$, for lack of pressure terms. Here, we remark that from Maxwell's viscosity coefficient : $\mathcal{R} \equiv \frac{M}{6k\rho\Theta_2}p$, we get the tensor $(220)_B$, which equals to (11). The equations (9) equals $(185)_B$ and (10) equals $(221)_B$ (=(24)) except for the symbol of viscosity coefficient.

5. Conclusions. Contributions to the NS equations

Basically, the NS equations were deduced from Newton's kinetic equation (the second law of motion) : $\mathbf{F} = m\mathbf{r}$, [20] however Boltzmann's gas equations were not deduced from it, but he extended the ideas of gas theory including the problem of gas collision by its progenitors Maxwell and Kirchhoff. In fact, Boltzmann had confessed his fear the authority in the preface of the Part II of his book (cf. Appendix).

When we consider the contributions by Boltzmann to the NS equations, Boltzmann shows the Euler equations and the NS equation as the special case of his general hydrodynamic equations. He verified the validity of the Euler equations and the NS equations, which were recognized in 1934 at latest by Prandtl [19, p.259], and at the epoch about one hundred years after Navier's paper [15], read by the referees in 1822 and published in *Mémoires de L'Academie des Sience de l'Institute de France* in 1827.

Maxwell in 1865, Boltzmann in 1895 and Prandtl[18, 19] in 1904 both used the "well-known hydrodynamic equations" and at latest in 1929, used the nomenclature of "Navier-Stokes equations", using the two-constant not of Navier, but of Saint-Venant, Stokes, and expanded by Maxwell, Kirchhoff and Boltzmann. These three persons verified the hydrodynamic equations without the name as Navier-Stokes equations.

In short, we can state that after formulating by Navier (1827) [15], Cauchy (1828) [5], Poisson (1831) [17], Saint-Venant (1843) [21] and Stokes (1849) [22], the topics of hydrodynamic history are rebuilt by Maxwell (1865) [12], Boltzmann (1895) [1] and Prandtl (1927) [19] in the cyclic interval of about 30 years or so.

As the two constants, Saint-Venant had used ε and $\frac{\varepsilon}{3}$, and Stokes μ and $\frac{\mu}{3}$, while Boltzmann used $\mathcal{R}$ and $\frac{\mathcal{R}}{3}$ after tracing Maxwell.

Boltzmann states hydrodynamic equations as well as the Euler equations of $(183)_B$. According to Boltzmann's description, we can suppose the fact that the then academic society had not fixed yet the name of this equations, up to 1895 or 1898.

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[20]($\Downarrow$) By d'Alembert's principle in 1758, from the Newton's kinetic equation (the second law of motion) : $\mathbf{F} = m\mathbf{r}$, d'Alembert proposed $\mathbf{F} - m\mathbf{r} = 0$, where, $\mathbf{F}$: the force, m : the gravity, $\mathbf{r}$: the acceleration. According to his assertion, the problem of kinetic dynamics turns into that of the static dynamics.

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(The inside cover of this book reads : the present work is a reprint, in four volumes, of Nathaniel Bowditch's English translation of volumes I,II,III and IV of the French-language treaties *Traité de Méchanique Céleste*, by P.S. Laplace. The translation was originally published in Boston in 1829, 1832, 1834 and 1839, under the French title, *"Méchanique Céleste"*, which has now been changed to its English-language form, *"Celestial Mechanics."*)

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Remark: we use *Lu* (: in French) in the bibliography meaning "read" date by the referees of the journals, for example MAS. In citing the original paragraphs in our paper, the underscoring are of ours.

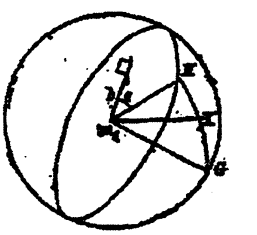

Fig. 6.