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Generalizations of the continued fraction transformation and the Selberg zeta functions

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1. Introduction

Let $\mathbb{H}$ be the upper half-plane in $\mathbb{C}$ endowed with the Poncaré metric $ds^2 = (dx^2 + dy^2)/y^2$ and let $\Gamma_1 = PSL(2,\mathbb{Z})$ be the modular group

$$PSL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}, ad-bc = 1 \right\} / \{I, -I\}$$

identified with the set of linear fractional transformations $z \mapsto (az+b)/(cz+d)$ as usual. We consider the so-called continued fraction transformation (Gauss transformation) T_G defined by $T_G : (0,1) \to (0,1), : x \mapsto 1/x - [1/x]$. It is well-known that the ergodic theory of T_G is closely related to the dynamical theory of the geodesic flow on the modular surface $M_1 = \mathbb{H}/\Gamma_1$ as well as the metric theory of continued fractions ([14], [15], [16], [25], [32]). Now we pay attention to the fact that the modular surface M_1 has two different aspects. The first aspect is rather classical. We consider M_1 as an example of Riemann surface of finite area (possibly with cusps and elliptic singularities). The second one is slightly fancy. We consider M_1 as the moduli space of complex structures of surface of genus 1. So there are at least two directions to generalize M_1 according to which aspect we look at. The aim of the article is to explain about the generalizations focusing the role of the continued fraction transformation.

In the case of the modular surface the crucial point is the existence of a natural one-to-one correspondence among the following three sets.

- $CG(M_1)$: the totality of oriented prime closed geodesics γ in M_1.
- $HC(\Gamma_1)$: the totality of primitive hyperbolic conjugacy classes c in Γ_1, i.e. c can be written as $c = \langle h \rangle = \{g^{-1}hg \ : \ g \in \Gamma_1\}$, where h is a primitive hyperbolic element in Γ_1.
- $PO(T_G^2)$: the totality of prime periodic orbits τ of T_G^2, i.e. τ can be regarded as the set of the distinct points $\tau = \{x,\ T_G^2 x,\ \ldots,\ T_G^{2(p-1)}x\}$, where x is a periodic point of T_G^2 and p is the least period of x.

To be more precise, assume that $\gamma \in CG(M_1)$, $c \in HC(\Gamma_1)$, and $\tau \in PO(T_G^2)$ are corresponding one another. Let $l(\gamma)$ be the least period of γ, $\lambda(c)$, the maximal eigenvalue of $h \in c$ as a matrix and $N(\tau) = J(T_G^{2p})(x) = |DT_G^{2p}(x)|$. Note that for each periodic point $x \in \tau$ there exists a neighborhood U_x of x and an hyperbolic element $h_x \in \Gamma_1$ such that $T_G^{2p} = h_x$ in U_x. In addition we can easily show that $h_x, \dots, h_{T_G^{2(p-1)}x}$ belong to the same conjugacy class. Under the notation above, the corresponding elements satisfy

$$\exp(l(\gamma)) = \lambda(c) = N(\tau). \tag{1.1}$$

The one-to-one correspondence enables us to express the Selberg zeta function in terms of periodic orbits of T_G^2. In order to see this, we recall the definition of the Selberg zeta function. Let $U(N)$ be the unitary group of degree N and $\rho : \Gamma_1 \to U(N)$ a unitary representation. The Selberg zeta function for Γ_1 with representation ρ is formally defined as

$$Z(s,\rho) = \prod_{k=0}^{\infty} \prod_{c \in HC(\Gamma_1)} \det(I_N - \rho(c)e^{-(s+k)l(c)}),$$

where s is a complex variable and $\det(I_N - \rho(c)e^{-(s+k)l(c)})$ is regarded as $\det(I_N - \rho(h)e^{-(s+k)l(c)})$ for some $h \in c$ since the determinant is a conjugacy invariant. The Selberg L function for Γ_1 with ρ is also formally defined as

$$L(s,\rho) = \frac{Z(s+1,\rho)}{Z(s,\rho)} = \prod_{c \in HC(\Gamma_1)} \det(I_N - \rho(c)e^{-sl(c)})^{-1}.$$

By virtue of (1.1), a formal calculation leads us to the identity:

$$Z(s,\rho) = \exp\left(-\sum_{k=0}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n} \sum_{x : T_G^{2n}x = x} \operatorname{trace}(\rho(T_G^{2n}|_x) J(T_G^{2n})(x)^{-(s+k)} \right), \tag{1.2}$$

where $T_G^{2n}|_x$ denotes the hyperbolic element in Γ_1 which coincides with T_G^{2n} in a neighborhood of x. Note that the formal products and formal series in the above are absolutely convergent in the half-plane $\operatorname{Re} s > 1$ and determine analytic functions. Classically the Selberg zeta function is studied via the Selberg trace formula (see [7], [8]). The equation (1.2) suggests us an alternative approach to the Selberg zeta function via thermodynamic formalism for T_G^2 (see [31]). In fact, Pollicott [25] obtained some results on the distribution of closed geodesics on the modular surface and quadratic irrationals and Mayer [16] gave a determinant representation of the Selberg zeta function in terms of transfer operator with complex parameter associated to T_G. Speaking of ergodic theoretic approach to the distribution of closed geodesics, there is a monumental work by Margulis in 1970 concerned with asymptotics of number of closed trajectories of Anosov systems on a compact manifold (see [12]). Note that the methods in [12] is different from ours but it is also remarked in [12] that those methods are applicable to the geodesic flows over non-compact

Riemannian manifolds of finite volume with negative sectional curvature (M_1 is of finite area but not compact and has an elliptic singularity). On the other hand, Parry [22] and Parry and Pollicott [23] (see also [24]) employ another approach via zeta functions in analytic number theory to obtain the prime number type theorem for closed orbits of Axiom A flows on compact manifolds. They represents Axiom A flows as suspension flows over appropriately chosen subshifts of finite type and studied the analytic properties of some zeta functions by thermodynamic formalism.

Next let us consider the two way of generalization of M_1 in view of the equation (1.1). In the first generalization, we want to extend the results to the case when Γ_1 is replaced by any co-finite Fuchsian group Γ. If we have a map playing the same role as T_G^2 for a general co-finite Fuchsian group, we can follow the arguments in [15]. Fortunately we know that Bowen and Series construct in [3] a one-dimensional Markov map whose action on an appropriately chosen subset in $\mathbb{R}$ is orbit equivalent to that of Γ on $\mathbb{R} \cup \{\infty\}$. Inspired by [15], [16], and [23], Morita [19] considered some renormalized Boewn-Series map instead of subshift of finite type and studied a unified approach to the determinant representation of the Selberg zeta function and the prime number type theorem for general co-finite Fuchsian groups via thermodynamic formalism. We have to note that there is a work of Pollicott [26] in the case when Γ is co-compact and a similar problem for co-compact Keinian groups are treated in [30].

In the second generalization, we consider the moduli space M_g of genus $g \geq 2$ instead of M_1. It is known that M_g has a similar structure to M_1. For example, the Teichmüller space of genus g, the Teichmüller modular group, and the Teichmüller geodesic flow play the roles of the upper-half plane $\mathbb{H}$, the modular group $PSL(2, \mathbb{Z})$, and the geodesic flow of M_1, respectively. Note that Masur [13] and Veech [34] solved the Keane Conjecture (see [9], [33]) independently by showing the ergodicity of the Teichmüller geodesic flow with respect to a canonical invariant measure. To the question "what plays the role of T_G^2 in the case of M_g?" we do not have a satisfactory answer at present while the Rauzy induction and its renormalizations are expected to play the role of T_G^2 for Teichmüller modular group (see [29], [34], [35], [36], [37], [38], and [40]). Selberg

Keeping the above mentioned facts in mind, we explain about the first generalization in Section 2. We mainly take up the determinant representation of Selberg zeta functions for co-finite Fuchsian groups with representation ρ. Since we already obtained such a result in [19] in the case when ρ is trivial, we just give some ideas to handle nontrivial ρ. The section include some unpublished results but they might be folklore in nowadays. Section 3 is devoted to the second generalization. We discuss about some results in [20] and [21] on distribution of periodic orbits of a class of renormalized Ruazy-Veech-Zorich inductions. Comparing with the first generalization, these results are not satisfactory and it seems there are many problems left unsolved.

2. Generalization I : Markov systems for co-finite Fuchsian groups

Following [3] we work on the Poincaré disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ endowed with the metric $ds^2 = 4dzd\bar{z}/(1-|z|^2)^2$ in this section. Let Γ be a co-finite subgroup of $SU(1,1)$. Starting with a Markov map assocated to Γ in [3], we can construct a triplet $\mathcal{T}_\Gamma = (\mathcal{R}, \mathcal{P}, T)$ called a Markov system associated to Γ. The Markov system plays the role of T_G^2 although it is not a single transformation. We summarize the fundamental facts on our Markov system below (for the detailed construction see [19]).

(M.1) (finiteness) $\mathcal{R} = \{A(1), \dots, A(q)\}$ is a set of a finite number of closed arcs on the unit circle S^1 with mutually disjoint nonempty interior.

(M.2) $\mathcal{P} = \{J\}$ is a set of closed arcs with mutually disjoint nonempty interior such that for each J there exists $A \in \mathcal{R}$ such that $\mathrm{int}J \subset \mathrm{int}A$ and $|\bigcup_{A\in\mathcal{R}} A \setminus \bigcup_{J\in\mathcal{P}} J| = 0$, where $|\cdot|$ denotes the Haar measure on the unit disc. In particular, there exists a map $\iota : \mathcal{P} \to \{1, \dots, q\}$ such that $\mathrm{int}J \subset \mathrm{int}A(\iota(J))$.

From the construction, we see that Γ is co-compact if and only if $\sharp\mathcal{P} < \infty$.

(M.3) To each $J \in \mathcal{P}$, an arc $A \in \mathcal{R}$ and an element $T_J \in \Gamma$ satisfying $T_J J = A$ are assigned. Namely, there exists a map $\tau : \mathcal{P} \to \{1, \dots, q\}$ such that $T_J J = A(\tau(J))$.

(M.4) T is an a.e. defined transformation on $X = \bigsqcup_{A\in\mathcal{R}} A$ such that $T|_{\mathrm{int}J} = T_J|_{\mathrm{int}J}$ for each $J \in \mathcal{P}$.

We need some notation. For closed arc A, we denote by $[A)$ (resp. $(A]$) the half-open arc obtained by taking the right (resp. left) endpoint away from A. T_R (resp. T_L) is a right continuous modification (resp. a left continuous modification) of T.

(M.5) (orbit equivalence) For for any x there exists $g \in \Gamma$ (resp. $g' \in \Gamma$) such that $gx \in \bigcup_{A\in\mathcal{R}}[A)$ (resp. $g'x \in \bigcup_{A\in\mathcal{R}}(A]$) and for any x, $y \in \bigcup_{A\in\mathcal{R}}[A)$ (resp. x, $y \in \bigcup_{A\in\mathcal{R}}(A]$) satisfying $y = gx$ for some $g \in \Gamma$, we can fined m, $n \geq 0$ such that $T_R^m x = T_R^n y$ (resp. $T_L^m x = T_L^n y$).

Next we introduce the notion of n-fold iteration $\mathcal{T}_\Gamma^n$ of $\mathcal{T}_\Gamma$. Let $\mathcal{P}_1 = \mathcal{P}$. For $n \geq 2$, let $\mathcal{P}_n$ be the family of closed arcs J with nonempty interior having the form

$$(2.1)\quad J = J(i_0) \cap T_{J(i_0)}^{-1} J(i_1) \cap T_{J(i_0)}^{-1} \circ T_{J(i_1)}^{-1} J(i_2) \cap \cdots \cap T_{J(i_0)}^{-1} \circ T_{J(i_1)}^{-1} \circ \cdots \circ T_{J(i_{n-2})}^{-1} J(i_{n-1})$$

for $J(i_0)$, $\dots$, $J(i_{n-1}) \in \mathcal{P}$. Such a J is often denoted by $J(i_0, \cdots i_{n-1})$. To each $J = J(i_0, \cdots i_{n-1}) \in \mathcal{P}_n$ we assign an element $T_J^n \in \Gamma$ defined by $T_J^n = T_{J(i_{n-1})} \circ \cdots \circ T_{J(i_0)}$.

Then n-fold iteration $T^n : X \to X$ of T is defined a.e. so that $T^n|_{\mathrm{int}J} = T^n_J|_{\mathrm{int}J}$ for each $J \in \mathcal{P}_n$. Then we can easily see that the triplet $(\mathcal{R}, \mathcal{P}_n, T^n)$ satisfies $(M.1)$, $(M.2)$, $(M.3)$, and $(M.4)$. We call it the n-fold iteration of $\mathcal{T}_\Gamma$ and denote it by $\mathcal{T}_\Gamma^n$. Note that ι and $\tau : \mathcal{P}_n \to \{1, \dots, q\}$ are defined in the same way as in $(M.2)$ and $(M.3)$, respectively.

By virtue of Proposition 3.2 in [19], we may assume without loss of generality that the Markov system $\mathcal{T}_\Gamma$ satisfies the following mixing condition.

(M.6) (mixing) There exists a positive integer n_0 such that for each $J \in \mathcal{P}$ we can find $J_1, \dots, J_q \in \mathcal{P}_{n_0}$ such that $J_j \subset J$ and $T^{n_0}_{J_j} J_j = A(j)$ hold for each $j = 1. \dots, q$.

The following are needed to investigate the analytic properties of the Selberg zeta function for Γ.

(M.7) (expanding) There exists a positive integer k such that

$$\inf_{J \in \mathcal{P}_k} \inf_{x \in J} |DT^k_J(x)| > 1.$$

(M.8) (Rényi condition) For any positive integer n we have

$$R(\mathcal{T}_\Gamma^n) = \sup_{J \in \mathcal{P}_n} \sup_{x \in J} \frac{|D^2 T^n_J(x)|}{|DT^n_J(x)|^2} < \infty.$$

Moreover, $\sup_n R(\mathcal{T}_\Gamma^n)/n < \infty$.

(M.9) For each $j = 1, 2, \dots, q$, there exist simply connected domains $B(j)$ and $D(j)$ in complex plane such that

(M.9,i) $A(j) \subset B(j) \subset\subset D(j)$;

(M.9.ii) $T_J^{-1} D(\tau(J)) \subset B(\iota(J))$ for any $J \in \mathcal{P}$;

(M.9.iii) for any $\delta > 1/2$, we have

$$\sum_{J \in \mathcal{P}} \sup_{z \in D(\tau(J))} |DT_J(T_J^{-1} z)|^{-\delta} < \infty.$$

Now we introduce some notions concerned with $\mathcal{T}_\Gamma$-periodic orbit. We write as $\mathcal{P} = \{J(j)\}_{j=1}^{\sharp\mathcal{P}}$ for the sake of convenience. A sequence $\xi = (\xi_0 \xi_1 \cdots)$ is called admissible if $T_{J(\xi_i)} J(\xi_i) \supset J(\xi_{i+1})$ for any i. To a point $x \in X = \bigsqcup_{I=1}^q A(i)$, one can assign an admissible sequence $\xi = (\xi_0 \xi_1 \cdots)$ called $\mathcal{T}_\Gamma$-itinerary of x such that $x \in J(\xi_0)$ and $T_{J(\xi_k)} \circ \cdots \circ T_{J(\xi_0)} x \in J(\xi_{k+1})$ for each $k \geq 0$. It is denoted as $\xi(x) = (\xi_0(x)\xi_1(x)\cdots)$. A point x having a periodic $\mathcal{T}_\Gamma$-itinerary is called a $\mathcal{T}_\Gamma$-periodic point. A set of p distinct points $\tau = \{x_0, x_1, \dots, x_{p-1}\}$ is a $\mathcal{T}_\Gamma$-periodic orbit of period p if there exists a permutation π of the set $\{0, 1, \cdots, p-1\}$ such that $\mathcal{T}_\Gamma$-itineraries $\xi(x_{\pi(i)})$ of $x_{\pi(i)}$ satisfy $\sigma\xi(x_{\pi(i)}) = \xi(x_{\pi(i)+1})$ mod p, where σ is the shift transformation on the sequence space such that $(\sigma\xi)_k = \xi_{k+1}$

for any $k \geq 0$. Note for any admissible sequence ξ with least period p, there exists a unique $\mathcal{T}_\Gamma$-periodic point $x \in X$ such that $\xi(x) = \xi$ by virtue of the expanding condition (M.7). In particular, the set $\tau(x) = \{x, T_{J(\xi_0(x))}x, \cdots, T_{J(\xi_{p-1}(x))} \circ \cdots \circ T_{J(\xi_0(x))}x\}$ forms a $\mathcal{T}_\Gamma$-periodic point of period p. We denote by $PO(\mathcal{T}_\Gamma)$ the totality of $\mathcal{T}_\Gamma$-periodic orbits. Then we have the following.

LEMMA 2.1. *(1) For* $\tau = \{x, \ldots, x_{p-1}\} \in PO(\mathcal{T}_\Gamma)$ *with* $\xi(x) = (\dot{j}_0, \ldots, \dot{j}_{p-1})$, $T^p_{J(i_0,\ldots,i_{p-1})}$ *is a primitive hyperbolic element in* Γ.

(2) If we define a map $\Phi : PO(\mathcal{T}_\Gamma) \to HC(\Gamma)$ *by*

$$\Phi(\tau) = (T^p_{J(i_0,\ldots,i_{p-1})}),$$

then we have

$$\lambda(\Phi(\tau)) = |DT^p_{J(i_0,\ldots,i_{p-1})}(x_0)|,$$

where τ *is written as in (1) above and* (γ) *denotes the conjugacy class of* $\gamma \in \Gamma$.

(3) The set $PO(\mathcal{T}_\Gamma)$ *is divided into four subsets as* $PO(\mathcal{T}_\Gamma) = \bigcup_{i=0}^{3} PO(\mathcal{T}_\Gamma)_i$ *the following hold.*

(3.i) $\Phi(PO(\mathcal{T}_\Gamma)_0) \cup \Phi(PO(\mathcal{T}_\Gamma)_1) = HC(\Gamma)$ *(disjpoint union).*

(3.ii) The restrictions $\Phi|_{PO(\mathcal{T}_\Gamma)_i}$ *are injective.*

(3.iii) $PO(\mathcal{T}_\Gamma)_i$ $(i = 1, 2, 3)$ *are finite sets having the common image under* Φ. *In particular, these are empty if the corresponding surface to*Γ *has genus* 0.

For the proof see Lemma 4.2 in [19].

We need some function spaces. For a bounded domain D in $\mathbb{C}$ and a positive integer N, $\mathcal{A}(D;\mathbb{C}^N)$ and $\mathcal{A}_b(D;\mathbb{C}^N)$ denote the totality of $\mathbb{C}^N$ valued holomorphic functions on D and its subspace of $\mathbb{C}^N$ valued bounded holomorphic functions on D, respectively. $\mathcal{A}(D;\mathbb{C}^N)$ is a nuclear Fréchet space with semi-norms $p_K(f) = \sup_{x\in K} |f(z)|_{\mathbb{C}^N}$, where K is any compact subset of D and $|\cdot|_{\mathbb{C}^N}$ is the usual norm on $\mathbb{C}^N$. $\mathcal{A}_b(D;\mathbb{C}^N)$ is a Banach space with the norm $\|f\|_\infty = \sup_{z\in D} |f(z)|_{\mathbb{C}^N}$. For a compact set K, $C(K;\mathbb{C}^N)$ denotes the space of all $\mathbb{C}^N$ valued continuous functions with the norm $\|f\|_\infty = \sup_{x\in K} |f(x)|_{\mathbb{C}^N}$. For a finite number of domains $D_1, \ldots, D_k$ in $\mathbb{C}$ and a positive integer N, a $\mathbb{C}^N$ valued function f on $\bigsqcup_{j=1}^k D_j$ is said to be an element in $\mathcal{A}(\bigsqcup_{j=1}^k D_j;\mathbb{C}^N)$ (resp. $\mathcal{A}_b(\bigsqcup_{j=1}^k D_j;\mathbb{C}^N)$) if $f|_{D_j}$ is in $\mathcal{A}(D_j;\mathbb{C}^N)$ (resp. $\mathcal{A}_b(D_j;\mathbb{C}^N)$) for each j. The space $\mathcal{A}(\bigsqcup_{j=1}^k D_j;\mathbb{C}^N)$ (resp. $\mathcal{A}_b(\bigsqcup_{j=1}^k D_j;\mathbb{C}^N)$) is naturally identified with the space $\bigoplus_{j=1}^k \mathcal{A}(D_j;\mathbb{C}^N)$ (resp. $\bigoplus_{j=1}^k \mathcal{A}_b(D_j;\mathbb{C}^N)$) by $\mathcal{A}(\bigsqcup_{j=1}^k D_j;\mathbb{C}^N) \ni f \mapsto \bigoplus_{j=1}^k f_j \in \bigoplus_{j=1}^k \mathcal{A}(D_j;\mathbb{C}^N)$, where $f_j = f|_{D_j}$ for each j. Finally a finite number of compact sets $K_1, \ldots, K_k$, $C(\bigsqcup_{j=1}^k K_j;\mathbb{C}^N)$

denotes the totality of functions f defined on $\bigsqcup_{j=1}^k K_j$ with $f|_{K_j} \in C(K_j;\mathbb{C}^N)$ for each j and $C(\bigsqcup_{j=1}^k K_j;\mathbb{C}^N)$ is identified with $\bigoplus_{j=1}^k C(K_j;\mathbb{C}^N)$.

Put $B(X) = \bigsqcup_{j=1}^q B(j)$ and $D(X) = \bigsqcup_{j=1}^q D(j)$, where $B(j)$'s and $D(j)$'s are the domains appearing in $(M.9)$. For $J \in \mathcal{P}$, consider a holomorphic function $B(\iota(J)) \times \mathbb{C} \ni (w,s) \to G_J(s)(w) \in \mathbb{C}$ defined by

$$G_J(s)(w) = \left(DT_J(w) \frac{w}{T_J w} \right)^{-s}$$

Note that if $w \in S^1$, we have $G_J(-1)(w) = (DT_J(w))(w/T_J(w)) = |DT_J(w)|$.

Now we are in a position to define a family of twisted transfer operators with complex parameter. It is shown that the family determines a meromorphic function which takes values in the space of nuclear operators acting on an appropriate chosen Banach space. The main purpose of this section is to represent the Selberg zeta function by using the Fredholm determinants of these operators. Let $\rho : \Gamma \to U(N)$ be a unitary representation. For an element $f = \bigoplus_{j=1}^q f_j \in \mathcal{A}(D(X);\mathbb{C}^N)$ or $C(X;\mathbb{C}^N)$, we define an operator f, ρ formally by

$$(\mathcal{L}(s,\rho)f)_i(z) = \sum_{J\in\mathcal{P},\tau(J)=i} G_J(s)(T_J^{-1}z)\rho(T_J)f_i(T_J^{-1}z). \tag{2.2}$$

By virtue of the condition $(M.9)$, for Re $s > 1/2$ the formally defined operator $\mathcal{L}(s,\rho)$ can be realized as an element in $\mathcal{L}(\mathcal{A}_b(B(X);\mathbb{C}^N);\mathcal{A}_b(D(X);\mathbb{C}^N))$, $\mathcal{L}(\mathcal{A}_b(D(X);\mathbb{C}^N))$, and $\mathcal{L}(C(X;\mathbb{C}^N))$, where $\mathcal{L}(\mathcal{X};\mathcal{Y})$ denotes the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ for topological linear spaces $\mathcal{X}$ and $\mathcal{Y}$ and $\mathcal{L}(\mathcal{X}) = \mathcal{L}(\mathcal{X};\mathcal{X})$. We see that for $f = \bigoplus_{j=1}^q f_j$ as above, the n-fold iteration $\mathcal{L}(s,\rho)^n$ of the operator $\mathcal{L}(s,\rho)$ can be written as

$$(\mathcal{L}(s,\rho)^n f)_i(z) = \sum_{J\in\mathcal{P}_\backslash,\tau(J)=i} G_{n,J}(s)(T_J^{-n}z)\rho(T_J^n)f_i(T_J^{-n}z),$$

where $T_J^{-n} = (T_J^n)^{-1}$ and $G_{n,J}(s,w)$ is given by

$$G_{n,J}(s)(w) = G(s)(w)G(s)(T_{J(i_0)}w)G(s)(T^2_{J(i_0i_1)}w)\cdot \cdots \cdot G(s)(T^{n-1}_{J(i_0i_1,\dots,i_{n-2})}w)$$

if $J \in \mathcal{P}_n$ has the form as in (2.1).
The following theorem is easily proved in the similar way to Theorem 5.1 and its corollary in [19].

THEOREM 2.2. *An analytic function* $\{s \in \mathbb{C} : \text{Re } s > 1/2\} \ni s \mapsto \mathcal{L}(s,\rho) \in \mathcal{A}_b(D(X);\mathbb{C}^N)$ *has a meromorphic extension to the entire s-plane which takes values in the space of* *nuclear operators of order* 0. *The candidates of poles are the points* $s = -k/2,\ k = -1,\ 0,\ 1,\ 2, \cdots$. *In particular, the Fredholm determinant* $\mathrm{Det}(I - \mathcal{L}(s,\rho))$ *extends to a meromorphic function to the entire s-plane and the candidates of its poles are the same as those of* $\mathcal{L}(s,\rho)$, *possibly with different order.*

Now we can state the following.

THEOREM 2.3. *Let* $\mathcal{T}_\Gamma$ *is the Markov system associated to co-finite Fuchsian group* Γ. *Consider a representation* $\rho : \Gamma \to U_N$. *For s with* $\mathrm{Re}\, s > 1$, *the Fredholm determinant* $\mathrm{Det}(I - \mathcal{L}(s,\rho))$ *of the twisted transfer operator* $\mathcal{L}(s,\rho)$ *with respect to* $\mathcal{T}_\Gamma$ *is represented by an absolutely convergent series as*

$$\begin{aligned}\mathrm{Det}(I - \mathcal{L}(s,\rho)) &= \exp\left(-\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{J\in\mathcal{P}_n : T_J^n J\supset J}\frac{1}{n}\mathrm{trace}(\rho(T_J^n))|DT_J^n(x_J)|^{-(s+k)}\right)\\ &= \exp\left(-\sum_{\tau\in PO(\mathcal{T}_\Gamma)}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{n}\mathrm{trace}(\rho(\Phi(\tau))^n)\lambda(\Phi(\tau))^{-(s+k)n}\right),\end{aligned}$$

where x_J *is the unique fixed point of* T_J^n *for* $J \in \mathcal{P}_n$ *with* $T_J^n J \supset J$.

Proof. By Theorem 2.2 and the estimate of the norm of $\mathcal{L}(s,1)$ with trivial representation in [19], we see that $\mathcal{L}(s,\rho)$ is a nuclear operator of order 0 with operator norm less than 1 if $\mathrm{Re}\, s > 1$. Therefore we have

$$\mathrm{Det}(I - \mathcal{L}(s,\rho)) = \exp\left(-\sum_{n=0}^{\infty}\frac{1}{n}\mathrm{trace}\mathcal{L}(s,\rho)^n\right).$$

On the other hand we have

$$\mathrm{trace}\mathcal{L}(s,\rho)^n = \sum_{J\mathcal{P}_n, T_J^n\supset J}\mathrm{trace}\mathcal{L}(n,s,J,\rho),$$

where $\mathcal{L}(n,s,J,\rho)$ is the nuclear operator on $\mathcal{A}_b(D(\iota(J)); \mathbb{C}^N)$ of order 0 defined by

$$\mathcal{L}(n,s,J,\rho)f(z) = G_{n,J}(s)(T_J^{-n}z)\rho(T_J^n)f(T_J^{-n}z) \quad (z \in D(\iota(J)))$$

for $f \in \mathcal{A}_b(D(\iota(J)); \mathbb{C}^N)$. Therefore we can obtain the theorem in the same way as Theorem 7.2 in [19] if we verify the validity of the following lemma. □

LEMMA 2.4. *Let* φ *be a univalent function on a domain* $D \subset \mathbb{C}$ *into itself with* $\sup_{z\in D}|D\varphi(z)| < 1$. *Let* $a \in D$ *be a unique fixed point of* φ. *Given an analytic function* F *on* D *with* $F(a) \neq 0$ *and a unitary matrix* U *of order* N, *consider the operator* $L : \mathcal{A}_b(D; \mathbb{C}^N) \to \mathcal{A}_b(D; \mathbb{C}^N)$

defined by

$$Lf(z) = F(\varphi(z))Uf(\varphi(z)).$$

Then we have $\mathrm{Spec}(L) \setminus \{0\} = \{F(a)(D\varphi)(a)^k\lambda_j \,:\, j = 1, 2, \ldots, N \text{ and } k = 0, 1, \ldots\}$, *where* $\lambda_1, \ldots \lambda_N$ *are the eigenvalues of* U.

Proof. We write $f \in \mathcal{A}_b(D; \mathbb{C}^N)$ as $f = {}^t(f_1, \ldots, f_N)$, Since U is a unitary matrix, there exists a unitary matrix V such that $VUV^{-1t}(e_1, \ldots, e_N) = {}^t(\lambda_1 e_1, \ldots, \lambda_N e_N)$, where $\{e_1, \ldots, e_N\}$ is a orthonormal basis of $\mathbb{C}^N$ consisting of eigenvectors of U. Consider the operator $V^{-1}LV$ which is spectrally equivalent to L. Since $V^{-1}LV^t(f_1, \ldots, f_N) = {}^t(\lambda_1 F(\varphi(z))f_1(\varphi(z)), \ldots, \lambda_N F(\varphi(z))f_N(\varphi(z)))$, we reduce our problem to the cases when $N = 1$. Thus we obtain the desired result from Lemma 7.1 in [19]. □

In order to rewrite the Selberg zeta function $Z(s, \rho)$ in terms of the Fredholm determinant of $\mathcal{L}(s, \rho)$, we introduce the following.

$$\Xi(s, \rho) = \prod_{k=0}^{\infty} \prod_{c \in \Phi(PO_1(\Gamma))} \det(I_N - \rho(c)e^{-(s+k)l(c)}). \tag{2.3}$$

Since $\Phi(PO_1(\Gamma))$ is a finite set, the analytic properties of $\Xi(s, \rho)$ are easily investigated.

Combining Theorem 2.2 with Theorem 2.3 we obtain our main result.

THEOREM 2.5. *For* $\mathrm{Re}\, s > 1$, *we have*

$$Z(s, \rho)\Xi(s, \rho)^2 = \mathrm{Det}(I - \mathcal{L}(s, \rho)).$$

In particular, $Z(s, \rho)$ *is an analytic function in the half-plane* $\mathrm{Re}\, s > 1$ *without zero having a meromorphic extension to the entire* s*-plane. Moreover, the candidates of poles are located on the* $\mathrm{Re}\, s = -k/2$, $k = -1, 0, 1, \ldots$.

Furthermore we can show the following theorem for the L-functions without consulting the Selberg trace formula.

THEOREM 2.6 (cf.[1]). *The Selberg L-function has the following properties.*

(L.1) *In the half-plane* $\mathrm{Re}\, s > 1$, $L(s, \rho)$ *is absolutely convergent and analytic.*

(L.2) $L(s, \rho)$ *has a meromorphic extension to the entire* s*-plane.*

(L.3) *In the closed half-plane* $\mathrm{Re}\, s \geq 1$, $L(s, \rho)$ *has no zeros.*

(L.4) *Let* $\varphi : \Gamma \to G$ *be a group homomorphism such that the image of those elements whose conjugacy classes contains elements of* $HC(\Gamma)$ *generates* G. *Let* $\psi : G \to U(N)$ *be any nontrivial irreducible representation. Consider a representation given by* $\rho = \psi\varphi$. *Then* $L(s, \rho)$ *is analytic in the half-plane* $\mathrm{Re}\, s > 1/2$.

(L.5) $L(s, 1)$ *has a simple zero at* $s = 1$ *and is analytic in the half-plane* $\mathrm{Re}\, s > 1/2$.

As a corollary to Theorem 2.6, we can show the following Chebotarev type density theorem (cf, [1], [28]).

THEOREM 2.7. *Let G be a normal subgroup of Γ with finite index. For any conjugacy class $[g] \in [\Gamma/G]$, Then we have*

$$\sharp\{c \in HC(\Gamma) : \pi_G c \in [g],\, l(c) \leq t\} \sim \frac{\sharp[g]}{[\Gamma, G]} \frac{\exp(t)}{t} \qquad (t \to \infty),$$

where $\pi_G : \Gamma \to \Gamma/G$ is the natural projection and $A(t) \sim B(t)$ $(t \to \infty)$ means $\lim_{t\to\infty} A(t)/B(t) = 1$.

Sketch of Proof of Theorem 2.6. If $\mathrm{Re}\, s > 1$ we see that

$$L(s,\rho) = \frac{Z(s+1,\rho)}{Z(s,\rho)}$$

holds. Thus the assertions (L.1), (L.2), and (L.3) follow from Theorem 2.5. The validity of the assertion (L.5) is verified in Theorem 7.4 in [19]. It remains to show the assertion (L.4). By virtue of Theorem 2.5 and the fact that $\Xi(s,\rho)$ given by (2.3) is an analytic function without zeros in the half-plane $\mathrm{Re}\, s > 0$, we see that each pole of $L(s,\rho)$ in the half-plane $\mathrm{Re}\, s > 1/2$ is located on the axis $\mathrm{Re}\, s = 1$ and for s with $\mathrm{Re}\, s = 1$ is a pole if and only if 1 is an eigenvalue of $\mathcal{L}(s,\rho)$. Therefore we have only to show that the following lemma. If $N = 2$

LEMMA 2.8. *Let $\rho : \Gamma \to U_N$ be a unitary representation appearing in (L.4). For s with $\mathrm{Re}\, s = 1$, we denote by $\mathcal{L}_X(s,\rho) : C(X;\mathbb{C}^N) \to C(X;\mathbb{C}^N)$ the twisted transfer operator defined by (2.2). Then $\mathcal{L}_X(s,\rho)$ has no eigenvalues of modulus* 1.

Idea of Proof We do not have enough space to give the proof. So we just explain about how to prove it. First of all, we note that Lemma 6. 2 in [19] is strong enough that we can show the lemma in the case when $N = 1$ i.e. ρ is a character. In the case when $N = 2$, we can show that if $\mathcal{L}_X(s,\rho)$ has an eigenvalue of modulus 1 and f is the corresponding eigenvector, then there exists a function $\alpha : X \times X \to S^1$ such that $f(y) = \alpha(x,y)f(x)$ for any $(x,y) \in X \times X$. This contradicts the irreducibility of ρ. Note that the proof of this step is carried out following the method proving Proposition 4.4 in[1].

3. GENERALIZATION II : RENORMALIZED RAUZY-VEECH-ZORICH INDUCTIONS

In this section, we consider renormalized Rauzy-Veech-Zorich inductions as generalizations of the continued fraction transformation. The definition of these transformations is based on the ergodic property of Rauzy induction established by Veech [34] and Masur [13] in their way to solve the Keane conjecture on interval exchange transformations. Therefore we start with the definition of interval exchange transformations ([9], [33]). Let $d \geq 2$ be an integer. Consider the cone $\Lambda_d = \{\lambda = (\lambda_1, \ldots, \lambda_d)^t \in \mathbb{R}^d : \lambda_j > 0 \text{ for each } j\}$ and the symmetric group $\mathfrak{S}_d$ of degree d. For $(\lambda, \pi) \in \Lambda_d \times \mathfrak{S}_d$, we define $\beta(\lambda) \in \{0\} \times \Lambda_d$ so that $\beta_j(\lambda) = \sum_{i=1}^{j} \lambda_j$ for $0 \leq j \leq d$. Consider a partition $\alpha(\lambda)$ of the interval $X(\lambda) = [0, |\lambda|_1)$ into subintervals $X_j(\lambda) = [\beta_{j-1}(\lambda), \beta_j(\lambda))$ $(1 \leq j \leq d)$. Let $\lambda^\pi = (\lambda_{\pi^{-1}1}, \ldots, \lambda_{\pi^{-1}d})^t$. Then the interval exchange transformation $T_{(\lambda,\pi)} : X(\lambda) \to X(\lambda)$ is defined by

$$T_{(\lambda,\pi)}x = x + \sum_{j=1}^{d} (\beta_{\pi j-1}(\lambda^\pi) - \beta_{j-1}(\lambda)) I_{X_j(\lambda)}(x).$$

By definition $T_{(\lambda,\pi)}$ maps the j-th interval $X_j(\lambda)$ in $\alpha(\lambda)$ onto πj-th interval $X_{\pi j}(\lambda^\pi)$ in $\alpha(\lambda^\pi)$ isometrically preserving the orientation. Thus the Lebesgue measure m restricted to $X(\lambda)$ is an invariant measure for $T_{(\lambda,\pi)}$. Keane conjectured that for fixed irreducible $\pi \in \mathfrak{S}_d$, $T_{(\lambda,\pi)}$ is uniquely ergodic Lebesgue almost every $\lambda \in \Lambda_d$.

Next we recall the definition of Rauzy induction $\mathcal{T}_0 : \Lambda_d \times \mathfrak{S}_d \to \Lambda_d \times \mathfrak{S}_d$ for our convenience. Consider the following $d \times d$ matrices $L(\pi)$ and $R(\pi)$

$$L(\pi) = \begin{pmatrix} I_{d-1} & \mathbf{0}_{d-1} \\ \mathbf{e}_{d-1}(\pi^{-1}j)^t & 1 \end{pmatrix}, \quad R(\pi) = \begin{pmatrix} I_{\pi^{-1}d} & K_{\pi^{-1}d, d-\pi^{-1}d} \\ O_{d-\pi^{-1}d, \pi^{-1}d} & J_{d-\pi^{-1}d} \end{pmatrix},$$

where I_k is the $k \times k$ identity matrix, $\mathbf{0}_{d-1}$ is $d-1$-dimensional zero column vector, $\mathbf{e}_{d-1}(\pi^{-1}j)$ is the $d-1$-dimensional unit vector whose $\pi^{-1}j$-th component is 1, $O_{k,l}$ is the $k \times l$ zero matrix and $K_{\pi^{-1}d, d-\pi^{-1}d}$ and $J_{d-\pi^{-1}d}$ are $\pi^{-1}d \times (d - \pi^{-1}d)$ matrix and $(d - \pi^{-1}d) \times (d - \pi^{-1}d)$ matrix given by

$$K_{\pi^{-1}d, d-\pi^{-1}d} = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \end{pmatrix}, \quad J_{d-\pi^{-1}d} = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ 1 & 0 & \ldots & 0 \end{pmatrix},$$

respectively. In addition we consider two transformations $L, R : \mathfrak{S}_d \to \mathfrak{S}_d$ defined by

$$(L\sigma)j = \begin{cases} \sigma j & (\sigma j \leq \sigma d) \\ \sigma d + 1 & (\sigma j = d) \\ \sigma j + 1 & \text{otherwise} \end{cases}, \quad (R\sigma)j = \begin{cases} \sigma j & (j \leq \sigma^{-1}d) \\ \sigma d & (j = \sigma^{-1}d + 1) \\ \sigma(j-1) & \text{otherwise} \end{cases}$$

For $(\lambda, \pi) \in \Lambda_d \times \mathfrak{S}_d$ with $\lambda_{\pi^{-1}d} \neq \lambda_d$, we put

$$A(\lambda, \pi) = \begin{cases} L(\pi) & (\text{ if } \lambda_d > \lambda_{\pi^{-1}d}) \\ R(\pi) & (\text{ if } \lambda_d < \lambda_{\pi^{-1}d}) \end{cases}, \quad D(\lambda) = \begin{cases} L & (\text{ if } \lambda_d > \lambda_{\pi^{-1}d}) \\ R & (\text{ if } \lambda_d < \lambda_{\pi^{-1}d}). \end{cases}$$

Then the Rauzy inductions $\mathcal{T}_0 : \Lambda_d \times \mathfrak{S}_d \to \Lambda_d \times \mathfrak{S}_d$ and $\mathcal{T} : \Delta_{d-1} \times \mathfrak{S}_d \to \Delta_{d-1} \times \mathfrak{S}_d$ are defined for (λ, π) with $\lambda_{\pi^{-1}d} \neq \lambda_d$ by

$$(3.1) \qquad \mathcal{T}_0(\lambda,\pi) = (A(\lambda,\pi)^{-1}\lambda, D(\lambda)\pi), \quad \mathcal{T}(\lambda,\pi) = \left(\frac{A(\lambda,\pi)^{-1}\lambda}{|A(\lambda,\pi)^{-1}\lambda|_1}, D(\lambda)\pi\right).$$

A permutation $\pi \in \mathfrak{S}_d$ is called irreducible if $\pi\{1, \dots, k\} = \{1, \dots, k\}$ yields $k = d$. Fix a irreducible element $\pi_0 \in \mathfrak{S}_d$. Consider the Rauzy class $\mathcal{R} = \mathcal{R}(\pi_0)$ introduced in [29]. ω_{d-1} and $\sharp_{\mathfrak{R}}$ below denote the volume measure on Δ_{d-1} and the counting measure on $\mathcal{R}$, respectively. We need the following result on $\mathcal{T} : \Delta_{d-1} \times \mathcal{R} \to \Delta_{d-1} \times \mathcal{R}$.

THEOREM 3.1 (Veech [34]). *There exists a $\mathcal{T}$ invariant measure μ equivalent to $\omega_{d-1} \times \sharp_{\mathcal{R}}$ on $\Delta_{d-1} \times \mathcal{R}$ which makes $\mathcal{T}$ both conservative and ergodic. For each $\pi \in \mathcal{R}$, the density μ on $\Delta_{d-1}(= \Delta_{d-1} \times \{\pi\})$ with respect to ω_{d-1} is given by the restriction of a function on Λ_d which is rational, positive, and homogeneous of degree $-d$.*

Theorem 3.1 implies that $\mathcal{T}$ satisfies the Poincaré recurrence. Thus we can define jump transformations and induced transformations for $\mathcal{T}$. Recall these notions briefly. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and $T : X \to X$ a μ-nonsingular transformation satisfying the Poincaré recurrence i.e. μ almost every $x \in X$ has the property that for any $E \in \mathcal{B}$ with $\mu(E) > 0$, $T^n x \in E$ holds for infinitely many $n \geq 0$. Then for any E, $F \in \mathcal{B}$ with $\mu(E) > 0$ and $\mu(F) > 0$, we put for $x \in E$

$$n(E, F; x) = \inf\{n \geq 1 : T^n x \in F\}.$$

In the case when $E = F$ we just write as $n(E; x) = n(E, E : x)$. From our assumption $n(E, F; x) < \infty$ μ-a.e. Thus we obtain almost everywhere defined transformation $T_{E,F} : E \to F$ called the jump transformation of T from E to F by

$$T_{E,F}x = T^{n(E,F;x)}x.$$

In the case $E = F$, $T_{E,F}$ is denoted by T_E and called the induced transformation of T to E or the first return map of T to E. Roughly speaking, 'renormalization of the transformation T' means the procedure of constructing a new transformation by producing jump transformations and their compositions.

We now consider the renormalization of the Rauzy induction $\mathcal{T} : \Delta_{d-1} \times \mathcal{R} \to \Delta_{d-1} \times \mathcal{R}$ given by (3.1). Set

$$\Delta(L, \pi) = \{\lambda \in \Delta_{d-1} : \lambda_d > \lambda_{\pi^{-1}d}\} \times \{\pi\} \quad \Delta(R, \pi) = \{\lambda \in \Delta_{d-1} : \lambda_d < \lambda_{\pi^{-1}d}\} \times \{\pi\}$$

$$\Delta(L) = \bigcup_{\pi \in \mathcal{R}} \Delta(L, \pi), \quad \Delta(R) = \bigcup_{\pi \in \mathcal{R}} \Delta(R, \pi).$$

Note that the sets $\Delta(L,\pi)$ and $\Delta(R,\pi)$ are expressed by

$$\Delta(L,\pi) = (L(\pi)\Lambda_{d-1} \cap \Delta_{d-1}) \times \{\pi\}, \quad \Delta(R,\pi) = (R(\pi)\Lambda_{d-1} \cap \Delta_{d-1}) \times \{\pi\}.$$

We consider the jump transformations $\mathcal{T}_{\Delta(L),\Delta(R)} : \Delta(L) \to \Delta(R)$ and $\mathcal{T}_{\Delta(R),\Delta(L)} : \Delta(R) \to \Delta(L)$. The Rauzy-Veech-Zorich induction $\mathcal{G} : \Delta(L) \cup \Delta(R) \to \Delta(L) \cup \Delta(R)$ is the transformation such that $\mathcal{G}|_{\Delta(L)} = \mathcal{T}_{\Delta(L),\Delta(R)}$ and $\mathcal{G}|_{\Delta(R)} = \mathcal{T}_{\Delta(R),\Delta(L)}$. The transformation $\mathcal{S} = \mathcal{T}_{\Delta(L),\Delta(R)} \circ \mathcal{T}_{\Delta(R),\Delta(L)} : \Delta(L) \to \Delta(L)$ is a typical example of the renormalized Rauzy-Veech-Zorich induction. Note that if $d = 2$ and $\pi = (21)$, then $\mathcal{S}$ is conjugate with T_G^2. Therefore we would like to consider the renormalizations of the Rauzy induction as generalizations of the continued fraction transformation.

The rest of the section is devoted to the study of a special class of renormalized Rauzy-Veech-Zorich inductions whose members play the same role as T_G^2 in our argument. Let $(\hat{\lambda}, \pi_0) \in \Delta(L, \pi_0)$ be such that $\hat{\lambda}$ is irrational, i.e. the entries of λ are linearly independent over $\mathbb{Q}$. Then the corresponding interval exchange transformation $T_{(\hat{\lambda},\pi_0)}$ is minimal by the result in [9]. Therefore we can find $N \geq 2$ such that $A_N(\hat{\lambda}, \pi_0) > 0$ by virtue of the remark after Proposition 3.30 in [33]. We denote $A_N(\hat{\lambda}, \pi_0)$ by B for the sake of simplicity. Consider the set $\Delta_B = B\Lambda_d \cap \Delta_{d-1}$ and $\Delta(B,\pi) = \Delta_B \times \{\pi\}$. We are interested in the induced transformation $\mathcal{S}_B$ of $\mathcal{S}$ to the set $\Delta(B,\pi)$. We regard $\mathcal{S}_B$ as a transformation on Δ_B in a natural way. Note that since $\mathcal{S}_B$ is an renormalization of $\mathcal{S}$, it is also a renormalization of $\mathcal{G}$. In particular, $\mathcal{S}_B$ and $\mathcal{G}_B$ coincides in this case.

For a nonnegative invertible matrix A, let $\Delta_A = A\Lambda_d \cap \Delta_{d-1}$ and define the map $\bar{A} : \Delta_{d-1} \to \Delta_{d-1}$ by $\bar{A}x = Ax/|Ax|_1$ for $x \in \Delta_{d-1}$. Then we have the following.

Lemma 3.2 (Lemma 3.1 in [21]). *Let $\mathcal{S}_B$ be as above. There exist sequences of distinct nonnegative integral matrices $\mathcal{A} = \{A^{(k)}\}$ and $\mathcal{C} = \{C^{(k)}\}$ satisfying the following:*

(1) $A^{(k)}B = BC^{(k)}$ *and* $\det A^{(k)} = \det C^{(k)} = \pm 1$.

(2) $\mathcal{S}_B|_{\Delta_{AB}} = \overline{A^{-1}}$, *i.e.* $\mathcal{S}_B x = \dfrac{A^{-1}x}{|A^{-1}x|_1}$ *for* $A \in \mathcal{A}$. *In particular,* $\mathcal{S}_B \Delta_{AB} = \Delta_B$ *for each* $A \in \mathcal{A}$.

(3) *The family of the set* $\mathcal{P} = \{\Delta_{AB} : A \in \mathcal{A}\}$ *forms a measurable partition of* Δ_B, *i.e.* $\omega_B(\Delta_{AB} \cap \Delta_{A'B}) = 0$ *for* A, $A' \in \mathcal{A}$ *with* $A \neq A'$ *and* $\omega_B(\Delta_B \setminus \bigcup_{A\in\mathcal{A}} \Delta_{AB}) = 0$, *where* $\omega_B = \omega_{d-1}(\Delta_B)^{-1}\omega_{d-1}|_{\Delta_B}$

Next we introduce the Hilbert projective metric on Δ_{d-1}. Note that the results on the Hilbert projective metrics that we need as well as their application to the study of ergodic behavior of dynamical systems are summarized in [11]. For x, $y \in \Lambda_d$, we write $x \leq y$ if each entry of $y-x$ is nonnegative. Put

$$\alpha(x,y) = \sup\{a \geq 0 \,:\, ax \leq y\}, \quad \beta(x,y) = \inf\{b \geq 0 \,:\, y \leq bx\},$$
$$\Theta(x,y) = \log \frac{\beta(x,y)}{\alpha(x,y)}.$$

Θ is called the Hilbert projective metric on Λ_d. Θ is a pseudo-metric on Λ_d such that $\Theta(x,y)=0$ if and only if $x=cy$ holds for some $c>0$. Thus Θ is a metric on the projective space Δ_{d-1}. We summarize the basic properties of the renormalized Rauzy-Veech-Zorich induction $\mathcal{S}_B$ as the following lemma.

LEMMA 3.3 (Lemma 3.4 in [21]). *Let $\mathcal{S}_B$ be as above. Then we have the following.*

(1) (Markov property) *For any $n \geq 1$ we have*

$$\mathcal{P}_n = \bigvee_{k=0}^{n-1} \mathcal{S}_B^{-k}\mathcal{P} \text{ and } \mathcal{S}_B^n \Delta_{AB} = \Delta_B$$

for any $\Delta_{AB} \in \mathcal{P}_n$. In particular $\mathcal{S}_B^n : \Delta_{AB} \to \Delta_B$ is a homeomorphism.

(2) (expanding) *There exist $C_1 > 0$ and $\theta \in (0,1)$ such that for any $n \geq 1$*

$$\Theta(\mathcal{S}_B^n x, \mathcal{S}_B^n y) \geq C_1^{-1}\theta^{-n}\Theta(x,y)$$

holds for any x, $y \in \Delta_{AB} \in \mathcal{P}_n$.

(3) (finite distortion) *There exists $C_2 > 0$ such that for any $n \geq 1$*

$$\left|\log \frac{J(\mathcal{S}_B^n)(x)}{J(\mathcal{S}_B^n)(y)}\right| \leq C_2 \Theta(\mathcal{S}_B^n x, \mathcal{S}_B^n y)$$

holds for any x, $y \in \Delta_{AB} \in \mathcal{P}_n$, where $J(\mathcal{S}_B^n)$ denotes the Jacobian of $\mathcal{S}_B^n$ with respect to ω_B.

(4) *There exist $\delta \in (0,1)$ and $C_3 > 0$ such that*

$$\sum_{A \in \mathcal{A}} \sup_{x \in \Delta_B} \frac{1}{|Ax|_1^{d(1-\delta)}} < C_3.$$

Note that for the proof of the assertion (4) in Lemma 3.3, we need some results in Bufetov [4].

Now we introduce a family of transfer operators. Let $\mathcal{S}_B$ be the renormalized Rauzy-Veech-Zorich induction defined just before Lemma 3.2 above. For $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 1-\delta$ and a complex-valued function on Δ_B, we put

$$\mathcal{L}(s)f(x) = \sum_{A \in \mathcal{A}} \frac{1}{|Ax|_1^{ds}} f(\bar{A}x),$$

where $\delta \in (0,1)$ is as in Lemma 3.3. Let $C(\Delta_B)$ be the Banach space of complex-valued continuous functions on Δ_B endowed with the supremum norm $\|\cdot\|_\infty$ and let $F_\Theta(\Delta_B)$ be the Banach space of complex-valued Lipschitz continuous functions on Δ_B with respect to the projective metric Θ endowed with the norm

$$\|g\|_\Theta = [g]_\Theta + \|g\|_\infty,$$

where $\|g\|_\infty = \sup_{x\in\Delta_B} |g(x)|$ and $[g]_\Theta = \sup_{x,y\in\Delta_B : x\neq y} |g(x) - g(y)|/\Theta(x,y)$ i.e. the Lipschitz constant of g with respect to Θ. $C(\Delta_B \to \mathbb{R})$ and $F_\Theta(\Delta_B \to \mathbb{R})$ denote the subspaces of real-valued elements of $C(\Delta_B)$ and $F_\Theta(\Delta_B)$, respectively. In [20] we proved a weak version of local central limit theorem for the partial sum $\sum_{k=0}^{n-1} f \circ \mathcal{S}_B^k$ with $f \in F_\Theta(\Delta_B)$ using the methods in [18]. If the perturbed Perron-Frobenius operators given by

$$\mathcal{L}(1)(e^{\sqrt{-1}tf} g)(x) = \sum_{A\in\mathcal{A}} \frac{\exp(\sqrt{-1}tf(\bar{A}x))}{|Ax|_1^d} g(\bar{A}x)$$

form an analytic family of bounded linear operators on $F_\Theta(\Delta_B)$, it seems that the same technique does work.. The assertion (4) in Lemma 3.3 guarantees that if we choose $f(x) = \log|A^{-1}x|_1$, the family $\mathcal{L}(1)(e^{\sqrt{-1}tf}\cdot)$ becomes an analytic family of bounded linear operators on $F_\Theta(\Delta_B)$ although f is not an element in $F_\Theta(\Delta_B)$. Anyway we can show the following;

PROPOSITION 3.4 (Proposition 4.4 in [21]). *There exist a neighborhood U of the half-plane* Re $s \geq 1$ *and the open disc $r_0 \in (0,\delta) \subset U$ with radius $r_0 < \delta$ centered at* 1 *such that the analytic family $\{\mathcal{L}(s) : s \in D(1,r_0)\}$ of bounded linear operators on $F_\Theta(\Delta_B)$ satisfies the following,.*

(1) *For $s \in D(1,r_0)$, $\mathcal{L}(s)$ has the spectral decomposition*

$$\mathcal{L}(s)^n = \lambda(s)^n E(s) + R(s)^n$$

for each $n \in \mathbb{N}$, where $\lambda(s)$ is a simple eigenvalue of $\mathcal{L}(s)$ with maximal modulus, $E(s)$ is the projection onto the one-dimensional eigenspace corresponding to $\lambda(s)$, and $R(s)$ is a bounded linear operator with spectral radius less than r_1 for some $r_1 \in (0,1)$ independent of $s \in D(1,r_0)$.

(2) *For $s \in U \setminus D(1,r_0)$, the spectral radius of $\mathcal{L}(s)$ is less than* 1.

(3) *$\lambda(s)$ in the assertion (1) is a analytic function on $D(1,r_0)$ such that $\lambda(1) = 1$ and is decreasing on $[1-r_0, 1]$.*

(4) *$E(s)$ and $R(s)$ in the assertion (1) are analytic functions on $D(1, r_0)$ with values in bounded linear operators on $F_\Theta(\Delta_B)$ given by the Dunford integrals*

$$E(s) = \frac{1}{2\pi\sqrt{-1}} \int_{|z-1|=r_2} R(\mathcal{L}(s), z)\, dz,$$

$$R(s)^n = \frac{1}{2\pi\sqrt{-1}} \int_{|z|=r_1} z^n R(\mathcal{L}(s), z)\, dz$$

for each $n \in \mathbb{N}$, where $0 < r_1, r_2 < 1$ are independent of $s \in D(1, r_0)$ satisfying $r_1 + r_2 < 1$ and $R(\mathcal{L}(s), z) = (zI - \mathcal{L}(s))^{-1}$ denotes the resolvent operator of $\mathcal{L}(s)$.

Moreover, we can show that there exists $\delta_1 > 0$ such that $\mathcal{L}(s)$ is quasicompact for s with Re $s > 1 - \delta_1$ as follows. For each $A \in \mathcal{A}_n$, x_A denotes the unique fixed point of $\bar{A}$ in Δ_B. Defined an operator $\mathcal{K}_n$ on $F_\Theta(\Delta_B)$ by

$$\mathcal{K}_n g(x) = \sum_{A \in \mathcal{A}_n} \frac{1}{|Ax|_1^{ds}} g(x_A)$$

for $f \in F_\Theta(\Delta_B)$. Let r_0 and $\lambda(s)$ be as in Proposition 3.4, we show the following.

PROPOSITION 3.5 (Proposition 4.5 in [21]). *There exists positive constants C_4 and C_5 such that for any s with Re $s > 1 - r_0$ and $g \in F_\Theta(\Delta_B)$ we have*

$$\|(\mathcal{L}(s)^n - \mathcal{K}_n)g\|_\infty \le C_4 \lambda(\mathrm{Re}\ s)^n \theta^n [g]_\Theta,$$

$$[(\mathcal{L}(s)^n - \mathcal{K}_n)g]_\Theta \le C_5(|s| + 1)\lambda(\mathrm{Re}\ s)^n \theta^n [g]_\Theta.$$

In particular $\mathcal{L}(s)$ is quasicompact as far as $\lambda(\mathrm{Re}\ s)\theta < 1$ holds.

Let $PO(\mathcal{S}_B)$ denote the totality of prime periodic orbits τ of $\mathcal{S}_B$. For $\tau = \{\lambda, \mathcal{S}_B\lambda, \ldots, \mathcal{S}_B^{p-1}\lambda\} \in PO(\mathcal{S}_B)$, put

$$N(\tau) = J(\mathcal{S}_B^p)(\lambda)^{\frac{1}{d}}$$

Let us consider the following zeta function given by the formal Euler product

$$\zeta(s) = \prod_{\tau \in PO(\mathcal{S}_B)} (1 - N(\tau)^{-ds})^{-1}. \tag{3.2}$$

A formal calculation leads us to the equation

$$\begin{aligned} \zeta(s) &= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{x : \mathcal{S}_B^n x = x} J(\mathcal{S}_B^n)(x)^{-s}\right) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{A \in \mathcal{A}_n} |Ax_A|_1^{-ds}\right) \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{A \in \mathcal{A}_n} \lambda_A^{-ds}\right). \end{aligned} \tag{3.3}$$

The main theorem in this section is the following.

THEOREM 3.6 (cf. Theorem 5.1 in [21]). *The infinite product in the right hand side of (3.2) is absolutely convergent for s with* Re $s > 1$ *and defines an analytic function without zero. In addition, the series in (3.3) are absolutely convergent and the equations are all justified. Moreover there exists $\delta_1 > 0$ such that $\zeta(s)$ has the meromorphic extension to the half-plane* Re s $> 1 - \delta_1$ *satisfying the following:*

(1) $s = 1$ *is the unique pole on the axis* Re s $= 1$ *and it is simple.*

(2) In the half-plane Re s $> 1 - \delta_1$, *$\zeta(s)$ does not have zeros.*

(3) There exists $\delta_2 \in (0, \delta_1)$ such that in the half-plane Re s $> 1 - \delta_2$, *$s = 1$ remains to be the unique pole of $\zeta(s)$. i.e. $\{s : \mathrm{Re}\, s > 1 - \delta_2\} \setminus \{1\}$ turns out to be a pole free region for $\zeta(s)$.*

Sketch of Proof The reader familiar with the transfer operator approach to dynamical zeta function will notice that Proposition 3.4 and Proposition 3.5 imply the validity of the assertions (1) and (2). In order to prove the assertion (3) we consult the results in [2] concerned with exponential decay of correlations of $\mathcal{S}_B$ (see also [27]).

It is well known that Theorem 3.6 provides us with enough information in order to prove the following.

THEOREM 3.7 (cf. [27]). *Thee exists $\alpha \in (0, d)$ such that*

$$\sharp\{\tau \in PO(\mathcal{S}_B) \ : \ \log N(\tau) \leq t\} = \frac{e^{dt}}{dt} + O(e^{\alpha t}) \quad (t \to +\infty).$$

Finally we explain about an geometric interpretation of Theorem 3.7 following Veech [34] and Mosher [17]. Let $g \geq 2$ be an integer. T_g and Mod_g denote the Teichmüller space and the mapping class group of genus g, respectively. Consider the following sets.

- $CG(T_g)$: the totality of oriented prime closed geodesics γ with respect to the Teichmüller metric in T_g.
- $HC(Mod_g)$: the totality of primitive hyperbolic conjugacy classes c in Mod_g, i.e. c can be written as $c = \langle h \rangle = \{g^{-1}hg \ : \ g \in Mod_g\}$, where h is a primitive hyperbolic element in in Mod_g whose representative is a pseudo-Anosov diffeomorphism .

For $\gamma \in CG(T_g)$ and $c = \langle h \rangle$, we put

- $l(\gamma)$: the least period of γ.
- $\lambda(c)$: the dilatation of h.

Then there exists a natural one-to-one correspondence between these sets such that $\exp(l(\gamma)) = \lambda(c)$ holds if $\gamma \in CG(T_g)$ and $c \in HC(Mod_g)$ are corresponding each other. If we consider an analogue of the prime number theorem for length spectrum of Teichmüller space of genus greater than 1, we arrive at a difficulty that there is no results for the zeta function which plays the role of the Selberg zeta function for the modular surface. On the other hand, if we look at the renormalized Rauzy-Veech-Zorich induction which is a sort of generalization of the continued fraction transformation, we notice the following facts. For any periodic point x of $\mathcal{S}_B$ there exists $A \in \bigcup_{n=1}^{\infty} \mathcal{A}_n$ such that the eigenvector x_A corresponding to the Perron-Frobenius root λ_A coincides with x. By way of zippered rectangles in [34] (see also [39], and [10]), there exist an positive integer g depending only on the irreducible permutation π_0, a closed Riemann surface R of genus g, a holomorphic 1-form ω, and a pseudo-Anosov diffeomorphism φ on R such that λ_A is the dilatation of φ and the interval exchange transformation $T_{(x_A,\pi_0)}$ is obtained by choosing an appropriate transversal to the measured foliation determined by ω. Therefore we see that for each $\tau \in P(\mathcal{S}_B)$, we can find a Teichmüller closed geodesic γ and a hyperbolic conjugacy class c of Mod_g such that $\exp(l(\gamma)) = \lambda(c) = N(\tau)$. Note that the number d of intervals turns out to be the dimension of the corresponding moduli space of Abelian differentials. Although τ is a primitive periodic orbit, γ and c are not necessarily so. We have to note that recently Eskin and Mirzakhani [6] establish remarkable result. They prove an analogue of the prime number theorem for closed Teichmüller geodesics on the principal stratum of the moduli space of quadratic differentials using the method in Margulis [12]. As a trial Theorem 3.7 may be still interesting, but it seems that we need more ideas to establish an analogue of the prime number theorem for closed Teichmüller geodesics on any stratum of the moduli space of quadratic differentials.

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