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A SURVEY ON THE EXTREMAL LENGTH GEOMETRY ON TEICHMÜLLER SPACE

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1. Introduction

By the *extremal length geometry*, we naively mean the geometry on the Teichmüller space studied via the extremal length on measured foliations. From the Kerckhoff's formula on the Teichmüller distance, the geometry on the Teichmüller distance is naturally in the category of the extremal length geometry.

In [12], S. Kerckhoff developed the study of the "end" of the Teichmüller space by using the extremal length. In [6], F. Gardiner and H. Masur formulated the extremal geometry of Teichmüller space and defined the compactification, which we recently call the *Gardiner-Masur boundary*, in terms of the extremal length geometry.

The aim of this paper is to give a survey of the author's resent progress in the extremal length geometry on Teichmüller space.

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2. Teichmüller theory

2.1. Teichmüller space and Measured foliations. Let X be a Riemann surface of analytically finite type (g,n) with $2g-2+2>0$. The *Teichmüller space* $T(X)$ is the set of equivalence classes of pairs (Y,f) of Riemann surfaces Y and quasiconformal mapping $f:X\to Y$. Two pairs (Y_1,f_1) and (Y_2,f_2) are equivalent if $f_2\circ f_1^{-1}$ is homotopic to a conformal mapping from Y_1 to Y_2. Let $x_0=(X,id)$ be the base point.

For $y_1=(Y_1,f_1)$, $y_2=(Y_2,f_2)\in T(X)$, the *Teichmüller distance* d_T between y_1 and y_2 is defined by

$$d_T(y_1,y_2)=\frac{1}{2}\log\inf_h K(h)$$

where h runs over all quasiconformal mappings $Y_1\to Y_2$ homotopic to $f_2\circ f_1^{-1}$ and $K(h)$ is the maximal dilatation of h. A metric space $(T(X),d_T)$ is known to be complete and a uniquely geodesic space (cf. [9]). However, to the author's knowledge, there is no *nice* characterization of the metric space $(T(X),d_T)$, and several sad news are known. For instance, it is known that $(T(X),d_T)$ is neither a CAT(0)-space or a Gromov hyperbolic space (cf. [17], [25], [18], and [19]).

Let $\mathcal{S}$ be the set of homotopy classes of non-peripheral and non-trivial simple closed curves on X. Let $\mathbb{R}_+^{\mathcal{S}}$ be the space of non-negative functions on $\mathcal{S}$ which equipped with the topology of the pointwise convergence, and set $P\mathbb{R}_+^{\mathcal{S}}:=(\mathbb{R}_+^{\mathcal{S}}-$

$\{0\})/\mathbb{R}_{>0}$ the projective space. The *space $\mathcal{MF}$ of measured foliations* is the closure of the embedded image of the mapping

$$\mathbb{R}_+ \otimes \mathcal{S} \ni t\alpha \mapsto [\mathcal{S} \ni \beta \mapsto t\, i(\beta, \alpha)] \in \mathbb{R}_+^{\mathcal{S}}$$

$\mathbb{R}_+ \otimes \mathcal{S}$ is the set of formal products $t\alpha$ of $t \in \mathbb{R}_+$ and $\alpha \in \mathcal{S}$, and $i(\cdot, \cdot)$ is the *geometric intersection number* between simple closed curves. It is known that the intersection number

$$(\mathbb{R}_+ \otimes \mathcal{S}) \times (\mathbb{R}_+ \otimes \mathcal{S}) \ni (t\alpha, s\beta) \mapsto i(t\alpha, s\beta) := ts\, i(\alpha, \beta)$$

extends continuously on $\mathcal{MF} \times \mathcal{MF}$ (cf. [1] and [26]). The projective space

$$\mathcal{PMF} = (\mathcal{MF} - \{0\})/\mathbb{R}_{>0} \subset P\mathbb{R}_+^{\mathcal{S}}$$

is called the *space of projective measured foliations*.

2.2. **Extremal length.** For $\alpha \in \mathcal{S}$ and $y = (Y, f) \in T(X)$, the *extremal length* $\mathrm{Ext}_y(\alpha)$ of α on y is the reciprocal of the supremum of the modulus of annuli whose cores are homotopic to $f(\alpha)$ in Y. S. Kerckhoff showed that when we set $\mathrm{Ext}_y(t\alpha) = t^2 \mathrm{Ext}_y(\alpha)$ for $t\alpha \in \mathbb{R}_+ \otimes \mathcal{S}$, the extremal length $\mathrm{Ext}_y(\cdot)\colon \mathbb{R}_+ \otimes \mathcal{S} \to \mathbb{R}$ extends continuously on $\mathcal{MF}$ (cf. [12]).

It is known that the Teichmüller distance has a geometric description

$$d_T(y_1, y_2) = \frac{1}{2} \log \sup_{\alpha \in \mathcal{S}} \frac{\mathrm{Ext}_{y_1}(\alpha)}{\mathrm{Ext}_{y_2}(\alpha)}$$

for $y_1, y_2 \in T(X)$, which we call *Kerckhoff's formula* (cf. [12]). We define

$$\mathcal{MF}_1 = \{F \in \mathcal{MF} \mid \mathrm{Ext}_{x_0}(F) = 1\}.$$

2.3. **Gardiner-Masur closure.** In [6], F. Gardiner and H. Masur observe that the mapping

$$\Phi_{GM}\colon T(X) \ni y \mapsto [\mathcal{S} \ni \alpha \mapsto \mathrm{Ext}_y(\alpha)^{1/2}] \in P\mathbb{R}_+^{\mathcal{S}}$$

is embedding and the image is relatively compact. The mapping Φ_{GM} is called the *Gardiner-Masur embedding.* The closure $\mathrm{cl}_{GM}(T(X))$ is said to be the *Gardiner-Masur compactification* and the complement $\partial_{GM}T(X)$ of the image from the closure is called the *Gardiner-Masur boundary.* In [6], Gardiner and Masur observed the following (see also [20] and [21]).

Theorem 2.1 (Gardiner and Masur). *We have $\mathcal{PMF} \subset \partial_{GM}T(X)$ in general. If X is neither a four punctured sphere or a once punctured torus, $\mathcal{PMF}$ is a proper subset of $\partial_{GM}T(X)$.*

Hence, we have the following topological observation.

Corollary 2.1. *If X is neither a four punctured sphere or a once punctured torus, the Gardiner-Masur boundary is not homeomorphic to the sphere of dimension $6g-7+2n$.*

Proof. Otherwise, from Borsuk-Ulam theorem (cf. [16]), the inclusion $\mathcal{PMF} \hookrightarrow \partial_{GM}T(X)$ should be surjective, because $\mathcal{PMF}$ is homeomorphic to the sphere of dimension $6g-7+2n$ (cf. [3]). $\square$

On the other hand, if X is either a four punctured sphere or a once punctured torus, $\partial_{GM}T(X)$ coincides with $\mathcal{PMF}$, and hence $\partial_{GM}T(X)$ is homeomorphic to a circle (i.e. the one-dimensional sphere). For instance, see [20] for the proof.

3. THE INTERSECTION NUMBER

3.1. **Motivation.** In [25], we develop the extremal length geometry on Teichmüller space via intersection number (cf. Theorem 3.2). This study is motivated from the comparison with the Thurston compactification. Namely, to define the Thurston compactification, the hyperbolic length of $\alpha \in \mathcal{S}$ and $y \in T(X)$ is recognized as the "intersection number" between a marked Riemann surface y and a simple closed curve $\alpha \in \mathcal{S}$ (cf. [3]). With this recognition, any point of $T(X)$ is thought of an element of the space $\mathbb{R}_+^{\mathcal{S}}$ of functions on the set $\mathcal{S}$ of simple closed curves. The Thurston compactification is defined by taking the closure of the image of $T(X)$ in the projective space $P\mathbb{R}_+^{\mathcal{S}}$ of $\mathbb{R}_+^{\mathcal{S}}$. Thurston's celebrated theorem tells us that the boundary defined by this closure coincides with $\mathcal{PMF}$. This setting is also well-understood from the Bonahon's work on geodesic currents (cf. [1]).

The main goal here is to *unify* the geometric structures (or geometric quantities) on a surface via "intersection number". From our observation (Theorem 3.2), we can define the intersection number in the category of the extremal length geometry. Indeed, in this category, we observe that the intersection number between $y, z \in T(X)$ (with respect to the base point) is equal to $\exp(-2\langle y \,|\, z\rangle_{x_0})$, where $\langle y \,|\, z\rangle_{x_0}$ is the Gromov product between y and z with the base point x_0 with respect to the Teichmüller distance. This observation links the geometry of the Teichmüller distance (an analytic aspect in Teichmüller theory) and the geometry of measured foliations via intersection number (an topological aspect in Teichmüller theory).

3.2. **Thurston theory for the extremal length geometry.** For $y \in T(X)$, we define a continuous function $\mathcal{E}_y$ on $\mathcal{MF}$

$$\mathcal{E}_y(F) = \left\{ \frac{\mathrm{Ext}_y(F)}{K_y} \right\}^{1/2} \tag{3.1}$$

where $K_y = \exp(2d_T(x_0, y))$. We will think $\mathcal{E}_y(F)$ the *intersection number* between $y \in T(X)$ and $F \in \mathcal{MF}$. Notice in the following theorem, the function $\mathcal{E}_y$ depends on the choice of the base point x_0 since so does K_y.

Theorem 3.1 (cf. [21] and [25]). *For any $p \in \mathrm{cl}_{GM}(T(X))$, there is a unique continuous function $\mathcal{E}_p$ on $\mathcal{MF}$ with the following properties.*

(1) *The function $[\mathcal{S} \ni \alpha \mapsto \mathcal{E}_p(\alpha)] \in \mathbb{R}_+^{\mathcal{S}}$ represents p.*
(2) *For a sequence $\{y_n\}_n \subset T(X)$ tends to $p \in \mathrm{cl}_{GM}(T(X))$, the functions $\mathcal{E}_{y_n}$ converges to $\mathcal{E}_p$ uniformly on any compact set of $\mathcal{MF}$.*
(3) $\max_{F \in \mathcal{MF}_1} \mathcal{E}_p(F) = 1$.
(4) *For $[G] \in \mathcal{PMF}$,*

$$\mathcal{E}_{[G]}(F) = \frac{i(F,G)}{\mathrm{Ext}_{x_0}(G)^{1/2}}$$

for $F \in \mathcal{MF}$.

Consider the mapping

$$\Psi_{GM} \colon \mathrm{cl}_{GM}(T(X)) \ni p \mapsto [\mathcal{S} \ni \alpha \mapsto \mathcal{E}_p(\alpha)] \in \mathbb{R}_+^{\mathcal{S}}.$$

From (3.1), the mapping Ψ_{GM} is a lift of the Gardiner-Masur embedding Φ_{GM}. Namely,

$$\Phi_{GM}(y) = \mathrm{proj} \circ \Psi_{GM}(y)$$

for $y \in T(X)$, where $\mathrm{proj}\colon \mathbb{R}_+^{\mathcal{S}} \to P\mathbb{R}_+^{\mathcal{S}}$ is the projection. Let

$$\mathcal{C}_{GM} = \mathrm{proj}^{-1}(\mathrm{cl}_{GM}(T(X))) \cup \{0\} \subset \mathbb{R}_+^{\mathcal{S}}.$$

Notice from $\mathcal{PMF} \subset \partial_{GM}T(X)$ that $\mathcal{MF} \subset \mathcal{C}_{GM}$. Furthermore, $\Psi_{GM}(\mathrm{cl}_{GM}(T(X))) \subset \mathcal{C}_{GM}$ because Ψ_{GM} is a lift of Φ_{GM}.

Theorem 3.2 ($\mathcal{E}_p$ is an intersection number (cf. [25])). *There is a unique continuous function*

$$i(\,\cdot\,,\,\cdot\,)\colon \mathcal{C}_{GM} \times \mathcal{C}_{GM} \to \mathbb{R}$$

with the following properties.

(i) *For any $y \in T(X)$, the projective class of the function $\mathcal{S} \ni \alpha \mapsto i(\Psi_{x_0}(y), \alpha)$ is exactly the image of y under the Gardiner-Masur embedding. In addition,*

$$i(\Psi_{GM}(p), F) = \mathcal{E}_p(F)$$

for $p \in \mathrm{cl}_{GM}(T(X))$ and $F \in \mathcal{MF}$.

(ii) *For $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}_{GM}$, $i(\mathfrak{a}, \mathfrak{b}) = i(\mathfrak{b}, \mathfrak{a})$.*

(iii) *For $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}_{GM}$ and $t, s \geq 0$, $i(t\mathfrak{a}, s\mathfrak{b}) = ts\, i(\mathfrak{a}, \mathfrak{b})$.*

(iv) *For any $y, z \in T(X)$,*

$$i(\Psi_{x_0}(y), \Psi_{x_0}(z)) = \exp(-2\langle y \,|\, z\rangle_{x_0}).$$

where $\langle y \,|\, z\rangle_{x_0}$ is the Gromov product of y and z with base point x_0 with respect to the Teichmüller distance d_T, that is:

$$\langle y \,|\, z\rangle_{x_0} = \frac{1}{2}(d_T(x_0, y) + d_T(x_0, z) - d_T(y, z)).$$

(v) *For $F, G \in \mathcal{MF} \subset \mathcal{C}_{GM}$, the value $i(F, G)$ is equal to the original geometric intersection number between F and G.*

As a corollary, we obtain an alternate approach to the characterization of the isometry group of $(T(X), d_T)$ (cf. [25]). Namely, we can see that with few exception, the isometry group of $(T(X), d_T)$ is canonically isomorphic to the extended mapping class group. This type of the characterization was already given by Royden [28], Earle-Kra [4], Earle-Markovic [5], and Ivanov [11].

4. Busemann points

Let T be an unbounded set in $[0, \infty)$ with $0 \in T$. A mapping $\gamma : T \to T(X)$ is said to be an *almost geodesic ray* if for any $\epsilon > 0$ there is an $N > 0$ such that $\gamma(0) = x_0$ and

$$|d_T(\gamma(t), \gamma(s)) + d_T(\gamma(s), \gamma(0)) - t| < \epsilon$$

for all $t > s > N$. By definition, any geodesic ray emanating x_0 is an almost geodesic ray.

In [14], L. Liu and W. Su observed that the Gardiner-Masur compactification is canonically identified with the horofunction compactification of $T(X)$ with respect to d_T (cf. Gromov [7]). Combining Rieffel's result in [27], they showed that any almost geodesic ray has the limit in the Gardiner-Masur boundary (see also [24] for a proof from Teichmüller theory).

The boundary point $p \in \partial_{GM}T(X)$ is called a *Busemann point* if it is the limit point of some almost geodesic ray.

Theorem 4.1 (cf. [24]). *The Gardiner-Masur boundary contains a point which is not a Busemann point.*

Since the horofunction boundary of any CAT(0)-space consists of Busemann points (cf. [2]), we deduce the following corollary, which was first observed by H. Masur in [17].

Corollary 4.1. *Teichmüller space equipped with the Teichmüller distance is not a* CAT(0)*-space.*

5. Lipschitz algebra

Lipschitz functions on a metric space are basic functions for investigating the geometry of the metric space. In [22], we develop an algebraic structure of the Lipschitz algebra on $(T(X), d_T)$ and give a relation between the Gardiner-Masur compactification and the compactification, which we call Q-compactification, defined with a subset Q of the Lipschitz algebra.

5.1. **Lipschitz algebra.** Let $[F]$ be a projective measured foliation. Consider the function

$$\ell_F(y) = \frac{1}{4}(\log \mathrm{Ext}_y(F) - \log \mathrm{Ext}_{x_0}(F) - 2d_T(x_0, y)).$$

Notice that $\ell_F(x_0) = 0$ for all F, and ℓ_F depends only on the projective class of F. The function ℓ_F is a non-positive 1-Lipschitz function on $T(X)$ with respect to Teichmüller distance. Since ℓ_F is not bounded below, we consider a truncation

$$\ell_{F:a} = \ell_F \vee a = \sup\{\ell_F, a\}$$

for $a < 0$ to obtain a bounded Lipschitz function.

For a subset Σ in the space $\mathcal{PMF}$ of projective measured foliations and a set T_0 in $(-\infty, 0]$, we define a family

$$\mathcal{L}_0(\Sigma, T_0) = \{\ell_{F:a} \mid [F] \in \Sigma,\ a \in T_0\},$$

We first study the algebraic structure of the Lipschitz algebra. Indeed, in [22], we show a version of the Stone-Weierstrass theorem for the space $\mathrm{BL}_0(T(X), \mathbb{F})$ of bounded $\mathbb{F}$-valued Lipschitz functions on $T(X)$ which vanish at x_0, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.

Theorem 5.1 (Stone-Weierstrass theorem for $\mathrm{BL}_0(T(X), \mathbb{F})$ (cf. [22])). *Let $\mathcal{A}$ be a self-adjoint, norm-closed and order-complete subalgebra in $\mathrm{BL}_0(T(X), \mathbb{F})$. If there are a dense subset Σ in $\mathcal{PMF}$ and an unbounded set $T_0 \subset (-\infty, 0]$ such that $\mathcal{L}_0(\Sigma, T_0) \subset \mathcal{A}$, then $\mathcal{A} = \mathrm{BL}_0(T(X), \mathbb{F})$.*

Let $\mathcal{A}$ be a subspace of either $\mathrm{Lip}(T(X), \mathbb{F})$ or $\mathrm{BL}_0(T(X), \mathbb{F})$. $\mathcal{A}$ is said to be *self-adjoint* if the complex conjugate $\overline{f}$ is in $\mathcal{A}$ for any $f \in \mathcal{A}$. A self-adjoint subspace $\mathcal{A}$ is, by definition, *order-complete* if every norm-bounded directed net of real valued functions in $\mathcal{A}$ has a least upper bound in $\mathcal{A}$, to which it converges pointwise. Finally, $\mathcal{A}$ is said to be *norm-closed* if whenever a sequence $\{f_n\}_n$ in $\mathcal{A}$ converges to g in norm, then $g \in \mathcal{A}$ (cf. e.g. [29] and [30]).

5.2. **Q-compactification.** A *Hausdorff compactification* of a Hausdorff space M is a Hausdorff compact space Y which contains, as a dense subset, the image of M under a fixed homeomorphism $f : M \hookrightarrow Y$. We always identify M with its image $f(M)$, and we say that Y contains M as a dense subset. We denote by ΔY the closure of $Y - M$ (cf. [15]).

Let M be a non-compact Hausdorff space, and let Q be a nonvoid set of continuous functions on M with each $f \in Q$ having its range contained in a compact

Hausdorff space S_f. Let $S_Q = \prod_{f \in Q} S_f$ be a product space. The evaluation map $e: M \to S_Q$ is defined by $e(x)(f) = f(x)$ for all $f \in Q$. Set

$$\Delta^Q M = \cap\{\overline{e(X-K)} \mid K \text{ compact}, K \subset M\}$$

and let $\mathrm{cl}_{GM}(M)^Q$ be the disjoint union $M \cup \Delta$. Given an open set U in S_Q and a compact set $K \subset M$, we set

$$U_K = (U \cap \Delta) \cup (e^{-1}(U) - K).$$

If $\mathfrak{T}$ is the topology on $\mathrm{cl}_{GM}(M)^Q$ generated by the base consisting of all open sets in M and all the sets U_K, then $(\mathrm{cl}_{GM}(M)^Q, \mathfrak{T})$ is called the *Q-compactification* of M. By definition, M is open in $\mathrm{cl}_{GM}(M)^Q$ since $\mathfrak{T}$ contains the topology of M.

Theorem 5.2 (Gardiner-Masur compactification revisited). *Let Σ be a dense subset of $\mathcal{PMF}$ and T_0 an unbounded set in $(-\infty, 0]$. Set $Q = \mathcal{L}_0(\Sigma, T_0)$. Then, the identity mapping $\mathcal{T}(X) \to \mathcal{T}(X)$ extends to a homeomorphism from the Q-compactification to the Gardiner-Masur compactification.*

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