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THE BIRMAN EXACT SEQUENCE AND A VISUALIZATION OF THE LINEARITY FOR MAPPING CLASS GROUP

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ABSTRACT. This is a preliminary report which gives a necessary and sufficient condition for the mapping class group of a once-punctured orientable surface be linear, in terms of the action of the mapping class group on the deformation space of linear representations of the corresponding surface group. Detailed account will appear elsewhere [2].

1. NOTATION

1.1. The linearity of a group. A linear representation of a group is said to be *faithful* if it is injective as a homomorphism of the group into the corresponding linear transformation group. A group is said to be linear if it admits a *finite dimensional* faithful linear representation over some field. If the group admits a faithful finite dimensional liner representation over a field K, then the group is said to be K*-linear.*

1.2. Mapping class groups of surfaces. Let Σ_g be an closed orientable surface of genus $g > 1$. The mapping class group of Σ_g, denoted by $\mathcal{M}_g$, is defined as the group of the isotopy classes of orientation preserving homeomorphisms of Σ_g.

Let $\Sigma_{g,*}$ be the pair of Σ_g and a fixed base point $* \in \Sigma_g$. The mapping class group of $\Sigma_{g,*}$, denoted by $\mathcal{M}_{g,*}$, is defined as the isotopy classes of the orientation preserving homeomorphisms of Σ_g keeping the base point fixed where all the isotopies are assumed to keep the base point.

1.3. The Birman exact sequence. Forgetting the base point induces a surjective homomorphism $\mathcal{M}_{g,*} \to \mathcal{M}_g$, which implies the following exact sequence which is called the Birman exact sequence

$$1 \to \pi_1(\Sigma_g, *) \to \mathcal{M}_{g,*} \to \mathcal{M}_g \to 1$$

1.4. The deformation space of representations of the surface group. The surface group is nothing but the fundamental group $\pi_1(\Sigma_g, *)$ of Σ_g. Let G be an arbitrary group. We denote $R_G = \mathrm{Hom}(\pi_1(\Sigma_g, *), G)$ the set of the homomorphisms of $\pi_1(\Sigma_g, *)$ into G. Let $\mathrm{Aut}\,(\pi_1(\Sigma_g, *))$ denote the group of the automorphisms of the surface group $\pi_1(\Sigma_g, *)$. As usual, the direct product group $\mathrm{Aut}\,(\pi_1(\Sigma_g, *)) \times G$ acts on R_G by

$$(h, g) \cdot \rho(\gamma) := g \cdot \rho(h^{-1}(\gamma)) \cdot g^{-1}$$

where $h \in \mathrm{Aut}\,(\pi_1(\Sigma_g, *))$, $g \in G$, $\rho \in R_G$, and $\gamma \in \pi_1(\Sigma_g, *)$. We take the quotient of R_G by the action of G and denote it by $X_G = R_G/G$.

1.5. The action of $\mathcal{M}_g$ on X_G. As is well-known, the mapping class group $\mathcal{M}_{g,*}$ naturally acts on $\pi_1(\Sigma_g, *)$ and this action induces an *injective* homomorphism $\mathcal{M}_{g,*} \to \mathrm{Aut}\,(\pi_1(\Sigma_g, *))$. Therefore, the action of $\mathrm{Aut}\,(\pi_1(\Sigma_g, *)) \times G$ on R_G above induces that of $\mathcal{M}_{g,*}$ on R_G. This action actually descends to the action of $\mathcal{M}_g$, the mapping class group of the closed surface, on the quotient set X_G.

2. Result

Under the notation above, the linearity condition for $\mathcal{M}_{g,*}$ is restated as follows:

Theorem 1. *Let $g > 1$ be an integer, and K an arbitrary field. The mapping class group $\mathcal{M}_{g,*}$ of the once-punctured surface is K-*linear *if and only if the action of $\mathcal{M}_g$ on $X_{\mathrm{GL}\,(n,K)} = \mathrm{Hom}(\pi_1(\Sigma_g, *), \mathrm{GL}\,(n, K))/\,\mathrm{GL}\,(n, K)$,* for some $n \geq 1$, *has a* global fixed point *which is represented by a faithful linear representation $\pi_1(\Sigma_g, *) \to \mathrm{GL}\,(n, K)$.*

Remark 2. (1) For the case of $K = \mathbb{C}$, we can combine our result with a result of Farb–Lubotzky–Minsky [1] to show that the faithful linear representation of $\pi_1(\Sigma_g, *)$ satisfying the condition in the theorem *does not exist* in the range $n \leq \sqrt{2\sqrt{g-1}}$. (2) It is known that if $\mathcal{M}_{g_0,*}$ is *not* K-linear for *some* $n_0 > 1$, then *for all* $g > g_0$, the mapping class group $\mathcal{M}_g$ of the closed surface of genus g is *not K-linear.*

The proof of Theorem 1 is given by the following three Lemmas:

Lemma 3. *For an arbitrary group G, the homomorphism of $\mathcal{M}_{g,*}$ into G is* injective *if and only if its restriction to the surface group $\pi_1(\Sigma_g, *)$ is* faithful.

Lemma 4. *Let G be an arbitrary group. If a representation $\phi \in R_G$ can be extended to a homomorphism $\mathcal{M}_{g,*} \to G$, then the element of X_G represented by ϕ is a global fixed point for the action of $\mathcal{M}_g$.*

Lemma 5. *Let K be an arbitrary field. If a linear representation $\phi \in R_{\mathrm{GL}\,(n,K)}$ represents a global fixed point in $X_{\mathrm{GL}\,(n,K)}$ for the action of $\mathcal{M}_g$, then the restriction of the* adjoint *representation of ϕ*

$$\mathrm{Ad}\,\phi : \pi_1(\Sigma_g, *) \to \mathrm{GL}\,(\mathrm{End}(n, K))$$

*to the K-*subspace $K[\phi(\pi_1(\Sigma_g, *))]$ *of* $\mathrm{End}(n, K)$ *generated by $\phi(\pi_1(\Sigma_g, *))$ extends to a finite dimensional linear representation $\Phi : \mathcal{M}_{g,*} \to \mathrm{GL}\,(K[\phi(\pi_1(\Sigma_g, *))])$.*

References

1. B. Farb, A. Lubotzky, and Y. Minsky, *Rank-1 phenomena for mapping class groups*, Duke Math. J. **106** (2001), no. 3, 581–597.
2. Y. Kasahara, *On visualization of the linearity problem of mapping class groups of surfaces*, in preparation.

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