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SYMMETRIES ON REDUCED BERS BOUNDARIES

KEN'ICHI OHSHIKA
(OSAKA UNIVERSITY)

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There are several ways to compactify Teichmüller space. In this note, we shall focus on one of them, the Bers compactification of Teichmüller space, which was first introduced by Bers in [1]. This compactification played, and still plays, an important role in the study of Kleinian groups. Indeed, first examples of (finitely generated) geometrically infinite Kleinian groups were found on the boundary of the Bers compactification.

Let us explain what is the Bers compactification first. Let S be a closed orientable surface of genus $g \geq 2$. (More in general, we can consider any hyperbolic surface of finite type.) Let $AH(S)$ denote the space of faithful discrete representations of $\pi_1(S)$ into $\mathrm{PSL}_2\mathbb{C}$ modulo conjugacy. We endow $AH(S)$ with the topology induced from the representation space. The interior of $AH(S)$ is know to coincide with the space of quasi-Fuchsian representations of $\pi_1(S)$, which we denote by $QF(S)$, by work of Ahlfors, Bers and Sullivan. Also, the simultaneous uniformisation due to Ahlfors and Bers implies that $QF(S)$ is parametrised as $qf : \mathcal{T}(S) \times \mathcal{T}(\bar{S}) \to QF(S)$, where $\mathcal{T}(S)$ denotes the Teichmüller space of S and $\mathcal{T}(\bar{S})$ the Teichmüller space of S with orientation reversing markings. Now, fix any point m_0 in $\mathcal{T}(S)$, and consider the slice $qf(\{m_0\} \times \mathcal{T}(\bar{S}))$. Bers showed that this slice is relatively compact in $AH(S)$ for any m_0. Its closure $\overline{qf(\{m_0\} \times \mathcal{T}(\bar{S}))}$ is called the Bers compactification of the Teichmüller space $\mathcal{T}(S)$, identifying $\mathcal{T}(S)$ with $\mathcal{T}(\bar{S})$ by the complex conjugation, and its boundary, which we denote by $\partial^B_{m_0}\mathcal{T}(S)$, is called the Bers boundary of the Teichmüller space.

A drawback of this compactification is the fact that it depends on the base point m_0 in the first factor of the parametrisation. Indeed, Kerckhoff and Thurston showed in [7] that there exist two points m_0, m_1 for which there is no continuous extension of the natural identification between $qf(\{m_0\} \times \mathcal{T}(\bar{S}))$ and $qf(\{m_1\} \times \mathcal{T}(\bar{S}))$ to the Bers boundaries. Moreover, they also showed that the action of the mapping class group on $\mathcal{T}(S)$ does not extend continuously to the Bers boundary.

Looking at their proof of these facts, we can see that the cause of these phenomena lies in the fact that Bers boundaries contain non-trivial quasi-conformal deformation spaces. Therefore, we can expect if we mod out Bers boundaries by collapsing every quasi-conformal deformation space into a point, these phenomena would disappear. We call thus obtained spaces the reduced Bers boundaries. Let us state the definition more formally.

Definition 1. For the Teichmüller space $\mathcal{T}(S)$ of S, let $q_{m_0} : \mathcal{T}(S) \to AH(S)$ be the Bers embedding with basepoint at m_0. Let $\partial^B_{m_0}\mathcal{T}(S)$ be the frontier of $\mathrm{Im}(q_{m_0})$. We introduce on $\partial^B_{m_0}\mathcal{T}(S)$ an equivalence relation $\sim$ such that two points $x, y \in \partial^B_{m_0}\mathcal{T}(S)$ are $\sim$-equivalent if and only if they are quasi-conformally conjugate to each other. We consider the quotient space $\partial^B_{m_0}\mathcal{T}(S)/\sim$, which we call the *reduced Bers boundary* with basepoint at m_0 and denote by $\partial^{RB}_{m_0}\mathcal{T}(S)$. We also consider the *reduced Bers compactification* with basepoint at m_0, which is $\mathcal{T}(S) \cup \partial^{RB}_{m_0}\mathcal{T}(S)$ endowed with the quotient topology induced from the Bers compactification $\mathcal{T}(S) \cup \partial^B_{m_0}\mathcal{T}(S)$. As it is clear from the context which Teichmüller space we are talking about, we omit $\mathcal{T}(S)$ and use the symbols $\partial^{RB}_{m_0}$ and $\partial^B_{m_0}$ for simplicity.

According to McMullen [6], Thurston already considered this space back in 1980's, and conjectured that this space is independent of the basepoint. We have shown that this is indeed the case.

Theorem 2. *Let m_1, m_2 be two points in $\mathcal{T}(S)$. Then there is a homeomorphism from $\partial^{RB}_{m_1}$ to $\partial^{RB}_{m_2}$ which is an extension of the natural identification between $qf(\{m_1\} \times \mathcal{T}(\bar{S}))$ and $qf(\{m_2\} \times \mathcal{T}(\bar{S}))$.*

As a corollary, we see that the mapping class group or the extended mapping class group acts on $\partial^{RB}_{m_0}$ for any $m_0 \in \mathcal{T}(S)$ as an extension of the natural action on $\mathcal{T}(S)$.

Now, let us fix a basepoint m_0 once and for all, and omitting the subcript of $\partial^{RB}_{m_0}$, denote it by ∂^{RB}. By the ending lamination theorem proved by Brock-Canary-Minsky [2], and the invariance of ending laminations under quasi-conformal deformations, we see that for any fixed m_0, there is an injection e from ∂^{RB} to the unmeasured lamination $\mathcal{UML}(S)$.

To understand its image, we introduce the following subspace.

Definition 3. We set $\mathcal{UML}_0(S)$ to be the subset of $\mathcal{UML}(S)$ consisting of unmeasured laminations λ such that for each component λ_0 of λ that is not a simple closed curve, every frontier component of the minimal supporting surface of λ_0 is contained in λ.

Then, by using the result of [8], we can easily see that the image of e coincides with $\mathcal{UML}_0(S)$. Still, we see that e is far from a homeomorphism. Indeed, we can show the following.

Proposition 4. *Neither e nor e^{-1} is continuous.*

As a topological space, ∂^{RB} is not Hausdorff, and more strongly is not T_1 either. This kind of space may look very hard to deal with. On the other hand, this non-separability is useful in showing that there are not many symmetries in ∂^{RB}.

Inspired by Papadopoulos' work [9], we have shown the following.

Theorem 1. *Suppose that $\xi(S) > 4$ (i.e. $\dim \mathcal{T}(S) > 2$). Let $f : \partial^{RB} \to \partial^{RB}$ be a homeomorphism. Then there exists a diffeomorphism $h : S \to S$ which induces f on ∂^{RB}. Furthermore, unless S is a closed surface of genus 2, two diffeomorphisms $h, h' : S \to S$ inducing the same homeomorphism on ∂^{RB} are isotopic.*

To prove this theorem, we shall introduce the notion of the adherence height.

Definition 5. A point b in ∂^{RB} is said to be *unilaterally adherent* to a in ∂^{RB} if every neighbourhood of b contains a. (We are not excluding the possibility that a is also unilaterally adherent to b although we say "unilaterally". We put this adverb to distinguish our definition from that of "adherence" by Papadopoulos [9], which is symmetric with regard to a and b.) Let $T = (a_0, \dots, a_n)$ be an ordered subset of ∂^{RB}. The set T is said to be an *adherence tower* if a_j is unilaterally adherent to $a_1, \dots, a_{j-1}$, and we call n the length of T. We define the *adherence height* of $a \in \partial^{RB}$ to be the supremum of the lengths of the adherence towers starting from a. We denote the adherence height of a by a.h.(a).

Obviously, any auto-homeomorphism of ∂^{RB} preserves the adherence height.

Furthermore, by using results in Kleinian groups, we can show the following lemma and proposition.

Lemma 6. *For a point $a \in \partial^{RB}$, we have* a.h.$(a) = \dim \pi^{-1}(a)/2$.

Let $\mathcal{C}(S)$ denote the curve complex of S. Then by considering barycentres, each simplex of $\mathcal{C}(S)$ is regarded as a point in $\mathcal{UML}_0(S)$. We denote the restriction of e^{-1} on the set of simplices of $\mathcal{C}(S)$ by ι.

Proposition 7. *Let $f : \partial^{RB} \to \partial^{RB}$ be a homeomorphism. Then, there is a simplicial automorphism $f' : \mathcal{C}(S) \to \mathcal{C}(S)$ such that $\iota(f'(c)) = f(\iota(c))$ for every simplex c of $\mathcal{C}(S)$.*

Now, applying the results of Ivanov, Luo and Korkmaz, we see that there is a diffeomorphism g inducing f' as above.

Finally, we can show that g induces the same homeomorphism as f in the entire ∂^{RB} by using the following lemma.

Lemma 8. *Let b be a point in ∂^{RB} with* $\text{a.h.}(b) = k$. *Then there is a sequence $\{a_i\}$ in ∂^{reg} which converges to b, such that for any point d other than b that is contained in the limit of $\{a_i\}$, we have* $\text{a.h.}(d) < \text{a.h.}(b)$.

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