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AN EPIMORPHISM BETWEEN KNOT GROUPS WHICH DOES NOT MAP A MERIDIAN TO A MERIDIAN (II)

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1. INTRODUCTION

Let K be a knot in S^3 and $G(K)$ the knot group. The existence of an epimorphism between knot groups defines a partial order on the set of prime knots. In [4], we consider epimorphisms which map meridians to a meridians. it is determined whether there exists such an epimorphism between the knot groups for each pair of prime knots with up to 10 crossings in [4]. The result of [4] is extended to prime knots with up to 11 crossings in [2]. On the other hand, we show an example of an epimorphism which does not map a meridian to a meridian in [9]. In this paper, we will show another expmple of an epimorphism, whose image is the knot group of the figure eight knot.

2. DEFINITION OF AN EPIMORPHISM AND MAIN THEOREM

Let K_1 be knots as depicted in Figure 1 and K_2 the figure eight knot.

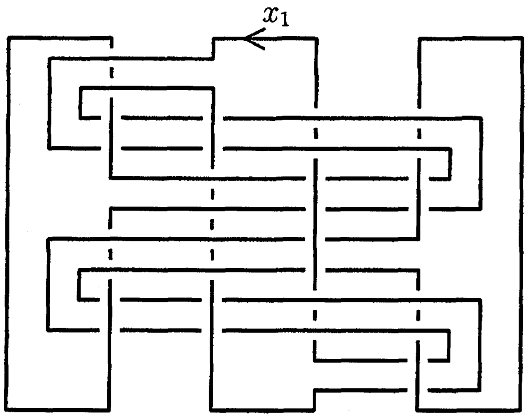

FIGURE 1. Knot K_1

The knot group $G(K_1)$ admits a Wirtinger presentation with respect to Figure 1. We denote the generators by $x_1, x_2, \dots, x_{32}$ and the defining relators are

$x_5x_1\bar{x}_5\bar{x}_2$, $x_{13}x_3\bar{x}_{13}\bar{x}_2$, $x_{25}x_3\bar{x}_{25}\bar{x}_4$, $x_{31}x_5\bar{x}_{31}\bar{x}_4$, $x_{13}x_6\bar{x}_{13}\bar{x}_5$, $x_1x_6\bar{x}_1\bar{x}_7$,
$x_{20}x_8\bar{x}_{20}\bar{x}_7$, $x_{26}x_8\bar{x}_{26}\bar{x}_9$, $x_{31}x_9\bar{x}_{31}\bar{x}_{10}$, $x_{25}x_{11}\bar{x}_{25}\bar{x}_{10}$, $x_{13}x_{11}\bar{x}_{13}\bar{x}_{12}$, $x_5x_{13}\bar{x}_5\bar{x}_{12}$,
$x_5x_{14}\bar{x}_5\bar{x}_{13}$, $x_9x_{14}\bar{x}_9\bar{x}_{15}$, $x_{26}x_{16}\bar{x}_{26}\bar{x}_{15}$, $x_{20}x_{16}\bar{x}_{20}\bar{x}_{17}$, $x_{23}x_{18}\bar{x}_{23}\bar{x}_{17}$, $x_{17}x_{18}\bar{x}_{17}\bar{x}_{19}$,
$x_7x_{20}\bar{x}_7\bar{x}_{19}$, $x_{31}x_{20}\bar{x}_{31}\bar{x}_{21}$, $x_{18}x_{22}\bar{x}_{18}\bar{x}_{21}$, $x_{28}x_{22}\bar{x}_{28}\bar{x}_{23}$, $x_{11}x_{24}\bar{x}_{11}\bar{x}_{23}$, $x_3x_{24}\bar{x}_3\bar{x}_{25}$,
$x_{31}x_{26}\bar{x}_{31}\bar{x}_{25}$, $x_7x_{26}\bar{x}_7\bar{x}_{27}$, $x_{17}x_{28}\bar{x}_{17}\bar{x}_{27}$, $x_{23}x_{28}\bar{x}_{23}\bar{x}_{29}$, $x_{28}x_{30}\bar{x}_{28}\bar{x}_{29}$, $x_{18}x_{30}\bar{x}_{18}\bar{x}_{31}$,
$x_3x_{32}\bar{x}_3\bar{x}_{31}$, $x_{11}x_{32}\bar{x}_{11}\bar{x}_1$,

where $\bar{x}_i = {x_i}^{-1}$. Note that x_1 can be regarded as a meridian of K_1 since all the generators are conjugate to one another.

We fix a presentation of the knot group $G(K_2)$:

$$G(K_2) = \langle y_1, y_2 \,|\, \bar{y}_1 y_2 y_1 \bar{y}_2 y_1 y_2 \bar{y}_1 \bar{y}_2 y_1 \bar{y}_2 \rangle.$$

where $\bar{y}_i = {y_i}^{-1}$ again.

Let $f : G(K_1) \to G(K_2)$ be a mapping defined by the image of generators of $G(K_1)$ as follows. Here we write numbers $1, 2$ for the generators y_1, y_2 respectively. For example, $11\bar{2}$ means $y_1 y_1 {y_2}^{-1}$.

$$\begin{array}{ll}
f(x_1) = 11\bar{2}, & f(x_2) = 1\bar{2}111\bar{2}\bar{1}2\bar{1}, \\
f(x_3) = 2\bar{1}\bar{2}1121\bar{2}\bar{1}, & f(x_4) = 11\bar{2}1\bar{2}12\bar{1}\bar{1}, \\
f(x_5) = 1\bar{2}1, & f(x_6) = 2\bar{1}\bar{2}111\bar{2}, \\
f(x_7) = 1\bar{2}1, & f(x_8) = 2\bar{1}2\bar{1}\bar{2}1\bar{2}121\bar{2}1\bar{2}, \\
f(x_9) = 1\bar{2}1, & f(x_{10}) = 11\bar{2}1\bar{2}12\bar{1}\bar{1}, \\
f(x_{11}) = 2\bar{1}\bar{2}1121\bar{2}\bar{1}, & f(x_{12}) = 1\bar{2}111\bar{2}\bar{1}2\bar{1}, \\
f(x_{13}) = 11\bar{2}, & f(x_{14}) = \bar{1}21\bar{2}1\bar{2}1, \\
f(x_{15}) = 11\bar{2}, & f(x_{16}) = 2\bar{1}2\bar{1}\bar{2}111\bar{2}1\bar{2}, \\
f(x_{17}) = 11\bar{2}, & f(x_{18}) = 2\bar{1}\bar{2}1121\bar{2}\bar{1}, \\
f(x_{19}) = 1\bar{2}121\bar{2}1\bar{2}\bar{1}2\bar{1}, & f(x_{20}) = 21\bar{2}1\bar{2}, \\
f(x_{21}) = 111\bar{2}\bar{1}, & f(x_{22}) = 12\bar{1}\bar{1}\bar{1}21\bar{2}1\bar{2}111\bar{2}\bar{1}, \\
f(x_{23}) = 111\bar{2}\bar{1}, & f(x_{24}) = 12\bar{1}\bar{1}\bar{1}21\bar{2}1\bar{2}111\bar{2}\bar{1}, \\
f(x_{25}) = 111\bar{2}\bar{1}, & f(x_{26}) = 21\bar{2}1\bar{2}, \\
f(x_{27}) = 1\bar{2}121\bar{2}1\bar{2}\bar{1}2\bar{1}, & f(x_{28}) = 2\bar{1}\bar{2}1121\bar{2}\bar{1}, \\
f(x_{29}) = 11\bar{2}, & f(x_{30}) = 12\bar{1}\bar{1}\bar{2}\bar{1}21\bar{2}1\bar{2}1211\bar{2}\bar{1}, \\
f(x_{31}) = 11\bar{2}, & f(x_{32}) = 12\bar{1}\bar{1}\bar{2}\bar{1}21\bar{2}1\bar{2}1211\bar{2}\bar{1}.
\end{array}$$

Theorem 2.1. *The above mapping $f : G(K_1) \to G(K_2)$ is an epimorphism which does not map a meridian of K_1 to a meridian of K_2.*

3. Proof

Theorem 2.1 is shown in this section. First, we will verify the defining relators of $G(K_1)$ vanish under the mapping f so that we prove that $f : G(K_1) \to G(K_2)$ is a group homomorphism,

$$\begin{array}{rcl}
f(x_5 x_1 \bar{x}_5 \bar{x}_2) & = & 1\bar{2}1 \cdot 11\bar{2} \cdot \bar{1}2\bar{1} \cdot 1\bar{2}12\bar{1}\bar{1}\bar{1}2\bar{1} = e, \\
f(x_{13} x_3 \bar{x}_{13} \bar{x}_2) & = & 11\bar{2} \cdot 2\bar{1}\bar{2}1121\bar{2}\bar{1} \cdot 2\bar{1}\bar{1} \cdot 1\bar{2}12\bar{1}\bar{1}\bar{1}2\bar{1} = e, \\
f(x_{25} x_3 \bar{x}_{25} \bar{x}_4) & = & 111\bar{2}\bar{1} \cdot 2\bar{1}\bar{2}1121\bar{2}\bar{1} \cdot 12\bar{1}\bar{1}\bar{1} \cdot 11\bar{2}\bar{1}2\bar{1}2\bar{1}\bar{1} \\
 & = & 111\bar{2}\bar{1}2\bar{1}\bar{2}12\bar{1}2\bar{1}\bar{1} = e, \\
f(x_{31} x_5 \bar{x}_{31} \bar{x}_4) & = & 11\bar{2} \cdot 1\bar{2}1 \cdot 2\bar{1}\bar{1} \cdot 11\bar{2}\bar{1}2\bar{1}2\bar{1}\bar{1} = e, \\
f(x_{13} x_6 \bar{x}_{13} \bar{x}_5) & = & 11\bar{2} \cdot 2\bar{1}\bar{2}111\bar{2} \cdot 2\bar{1}\bar{1} \cdot \bar{1}2\bar{1} = e, \\
f(x_1 x_6 \bar{x}_1 \bar{x}_7) & = & 11\bar{2} \cdot 2\bar{1}\bar{2}111\bar{2} \cdot 2\bar{1}\bar{1} \cdot \bar{1}2\bar{1} = e.
\end{array}$$

Then $f : G(K_1) \to G(K_2)$ is a group homomorphism by the above and similar calculations. Next, we will show that the group homomorphism $f : G(K_1) \to G(K_2)$ is surjective. We consider two elements $x_1\bar{x}_7x_1\bar{x}_{23}x_1, \bar{x}_7x_1\bar{x}_{23}x_1x_1\bar{x}_7x_1\bar{x}_{23}x_1$ of $G(K_1)$.

$$\begin{aligned} & f(x_1\bar{x}_7x_1\bar{x}_{23}x_1) = f(x_1)\,\overline{f(x_7)}\,f(x_1)\,\overline{f(x_{23})}\,f(x_1) \\ = \;& 11\bar{2}\cdot\bar{1}2\bar{1}\cdot 11\bar{2}\cdot 12\bar{1}\bar{1}\bar{1}\cdot 11\bar{2} = 11\bar{2}\bar{1}21\bar{2}12\bar{1}\bar{2} = 1, \\ & f(\bar{x}_7x_1\bar{x}_{23}x_1x_1\bar{x}_7x_1\bar{x}_{23}x_1) \\ = \;& \overline{f(x_7)}\,f(x_1)\,\overline{f(x_{23})}\,f(x_1)\,f(x_1)\,\overline{f(x_7)}\,f(x_1)\,\overline{f(x_{23})}\,f(x_1) \\ = \;& \bar{1}2\bar{1}\cdot 11\bar{2}\cdot 12\bar{1}\bar{1}\bar{1}\cdot 11\bar{2}\cdot 11\bar{2}\cdot\bar{1}2\bar{1}\cdot 11\bar{2}\cdot 12\bar{1}\bar{1}\bar{1}\cdot 11\bar{2} \\ = \;& \bar{1}21\bar{2}12\bar{1}\bar{2}11\bar{2}\bar{1}21\bar{2}12\bar{1}\bar{2} = 2. \end{aligned}$$

Since two elements 1 and 2 generate $G(K_2)$, it is shown that the group homomorphism f is surjective. Finally, we will prove that f does not map a meridian of K_1 to a meridian of K_2. We can fix meridians for K_1 and K_2 by x_1 and 1, without loss of generality. Let $\rho : G(K_2) \to SL(2;\mathbb{Z}/3\mathbb{Z})$ be a representation of $G(K_2)$ defined by

$$\rho(1) = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \qquad \rho(2) = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}.$$

We can check easily that ρ is a representation of $G(K_2)$. Besides, we get

$$\rho(f(x_1)) = \rho(11\bar{2}) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Note that the trace of $\rho(1)$ is 1. On the other hand, the trace of $\rho(f(x_1))$ is not equal to 1. Hence $f(x_1)$ is not conjugate to 1. It follows that the epimorphism f does not map a meridian of K_1 to a meridian of K_2. This completes the proof.

4. Problem

In this section, we propose some problems related to epimorphisms between knot groups. We determined whether there exist an epimorphism mapping a meridian to a meridian between the knot groups of each pair of prime knots with up to 11 crossings in [4] and [2].

Problem 4.1. *Determine whether there exist an epimorphism between the knot groups of each pair of prime knots with up to* 11 *crossings. Here we do not assume that an epimorphism maps a meridian to a meridian. In particular, does there exist such an epimorphism between* 2*-bridge knot groups?*

We note that Ohtsuki-Riley-Sakuma [7] and Lee-Sakuma [6] studied epimorphisms between 2-bridge link groups. Perhaps there exist several epimorphisms for the given knot groups. In this sense, we are interested in the following.

Problem 4.2. *Which pair of knots with up to* 11 *crossings admit an epimorphism between their knot groups which does not map a meridian to a meridian?*

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