数理解析研究所講究録
第 1778 巻 2012 年 111-116

External type of Jensen operator inequality

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In this note, we restrict ourselves to concave functions f. Usually classical Jensen's inequality is expressed by internally dividing points, or convex sum: For $\alpha_i \geqq 0$ with $\sum_i \alpha_i = 1$, a concave function f on an open interval $\mathcal{I}$ assures

$$f\left(\sum_i \alpha_i x_i\right) \geqq \sum_i \alpha_i f(x_i)$$

for all $x_i \in \mathcal{I}$, which is the same for the definition of concavity. Along this line, it is extended to various operator inequalities (cf. [4, 5]). For example, Hansen-Pedersen [6, 3] showed that if f is operator concave on $\mathcal{I}$, then

$$C^*f(X)C + D^*f(Y)D \leq f(C^*XC + D^*YD)$$

holds for selfadjoint operators X and Y with $\sigma(X)$, $\sigma(Y) \in \mathcal{I}$ and contractions C and D with $C^*C + D^*D = I$ (Recall that f is *operator concave* on $\mathcal{I}$ if

$$f\left(\frac{X+Y}{2}\right) \geqq \frac{f(X)+f(Y)}{2}$$

holds for all selfadjoint operators$X$ and Y with $\sigma(X)$, $\sigma(Y) \in \mathcal{I}$).

On the other hand, the concavity is also expressed by externally dividing points:

$$r > 0,\ v,\ w,\ (1+r)v - rw \in \mathcal{I} \implies f((1+r)v - rw) \leqq (1+r)f(v) - rf(w).$$

Then, putting $v = (1-t)x + ty$, $w = y$, $r = \frac{t}{1-t}$, we have $\frac{1}{1+r} = 1-t$, $\frac{r}{1+r} = t$ and hence

$$(1+r)v - rw = \frac{(1-t)x + ty - ty}{1-t} = x.$$

It follows that

$$f((1-t)x+ty) = f(v) \geqq \frac{1}{1+r} f((1+r)v - rw) + \frac{r}{1+r} f(w) = (1-t)f(x) + tf(y),$$

which implies the concavity of f.

Thus the purpose of this paper is to express what is an external version of the Jensen operator inequality. First we observe an external version of the classical Jensen inequality (The equivalent inequality was shown by Pečarić et. al. [1, p.83], [7, Theo. B]):

Theorem P (Pečarić-Proschan-Tong). *A function f on $\mathcal{I}$ is concave if and only if*

$$f\left(\left(1+\sum_k r_k\right)x - \sum_k r_k y_k\right) \leqq \left(1+\sum_k r_k\right) f(x) - \sum_k r_k f(y_k) \tag{1}$$

holds for all x, y_k, $z = (1+\sum_k r_k)x - \sum_k r_k y_k \in \mathcal{I}$ and nonnegative numbers r_k.

Remark 1. If $x, y_k \in \mathcal{I}$ with $x \leqq y_k$ (resp. $x \geqq y_k$), then z is indeed an external point for (x, y_k) by $z \leqq x \leqq y_k$ (resp. for (y_k, x) by $z \geqq x \geqq y_k$). Though the above result includes the inequality not only for external points, we also call such an inequality an 'external' version for the sake of convenience.

Remark 2. The original Pečarić-Proschan-Tong inequality is as follows: For $s_1 > 0$, $s_i \leqq 0$ $(i > 1)$, $S_n = \sum_{i=1}^n s_i > 0$ and z_i, $\frac{\sum_{i=1}^n s_i z_i}{S_n} \in \mathcal{I}$,

$$f\left(\frac{\sum_{i=1}^n s_i z_i}{S_n}\right) \leqq \frac{\sum_{i=1}^n s_i f(z_i)}{S_n}.$$

The equivalence between them are follows from the relations $s_1 = S_n(1+\sum_k r_k)$ and $s_{i+1} = -S_n r_i$.

So we have an external version of Jensen's operator inequality:

Theorem 1. *Let H , K and L be Hilbert spaces. A function f is operator concave on $\mathcal{I}$ if and only if*

$$f(C^*XC - D^*YD) \leq |C| f(V^*XV) |C| - D^* f(Y) D \tag{2}$$

holds for all selfadjoint operators $X \in B(K)$, $Y \in B(L)$ *with* $\sigma(X), \sigma(Y) \in \mathcal{I}$ *and operators* $C \in B(H,K)$, $D \in B(H,L)$ *with* $C^*C - D^*D = I_H$ *and* $\sigma(C^*XC - D^*YD) \in \mathcal{I}$, *where* V *is the partial isometry in the polar decomposition* $C = V|C|$.

In particular, if C *is invertible, then (2) is expressed by*

$$f(C^*XC - D^*YD) \leq C^*f(X)C - D^*f(Y)D. \tag{2'}$$

Proof. Note that $|C| = \sqrt{C^*C}$ is invertible and $\tilde{C}^*\tilde{C} + \tilde{D}^*\tilde{D} = I_H$ holds for $\tilde{C} = |C|^{-1} = \sqrt{C^*C}^{-1}$ and $\tilde{D} = D|C|^{-1} = D\tilde{C}$. Suppose f is operator concave. Then the Hansen-Pedersen-Jensen operator inequality shows

$$f(\tilde{C}^*A\tilde{C} + \tilde{D}^*B\tilde{D}) \geq \tilde{C}^*f(A)\tilde{C} + \tilde{D}^*f(B)\tilde{D}.$$

It follows that

$$\begin{aligned} |C|f(V^*XV)|C| &= |C|f\left(\tilde{C}(|C|V^*XV|C| - D^*YD)\tilde{C} + \tilde{D}^*Y\tilde{D}\right)|C| \\ &\geq |C|\tilde{C}f(C^*XC - D^*YD)\tilde{C}|C| + |C|\tilde{D}^*f(Y)\tilde{D}|C| \\ &= f(C^*XC - D^*YD) + D^*f(Y)D. \end{aligned}$$

So we have

$$f(C^*XC - D^*YD) \leq |C|f(V^*XV)|C| - D^*f(Y)D.$$

Conversely suppose (2). Let X, Y be selfadjoint operators with $\sigma(X), \sigma(Y) \in \mathcal{I}$ and $0 < t < 1$. Putting $V = (1-t)X + tY$, $W = Y$ and $r = \frac{t}{1-t} > 0$, we have $t = \frac{r}{1+r}$ and $(1+r)V - rW = \frac{(1-t)X+tY-tY}{1-t} = X$. Thus V and W satisfy $\sigma(V), \sigma(W), \sigma((1+r)V - rW) \in \mathcal{I}$. Then as a special case of the above inequality,

$$f((1+r)V - rW) \leq (1+r)f(V) - rf(W).$$

It follows that

$$f((1-t)X+tY) = f(V) \geq \frac{1}{1+r}f((1+r)V-rW) + \frac{r}{1+r}f(W) = (1-t)f(X)+tf(Y),$$

which implies the operator concavity.

If C is invertible, then V is unitary and hence $|C|f(V^*XV)|C| = |C|V^*f(X)V|C| = C^*f(X)C$, so that we have (2'). □

Considering $X = \begin{pmatrix} X_1 & & \\ & \ddots & \\ & & X_k \end{pmatrix}$, $Y = \begin{pmatrix} Y_1 & & \\ & \ddots & \\ & & Y_m \end{pmatrix}$, $C = \begin{pmatrix} C_1 \\ \vdots \\ C_k \end{pmatrix}$ and $D = \begin{pmatrix} D_1 \\ \vdots \\ D_m \end{pmatrix}$ according to the decompositions $K = K_1 \oplus \cdots \oplus K_k$ and $L = L_1 \oplus \cdots \oplus L_m$, we have

Corollary 2. *Let f be operator concave on an open interval $\mathcal{I}$. For Hilbert spaces H, $K = \oplus_i K_i$ and $L = \oplus_j L_j$, let $X_i = X_i^* \in B(K_i)$, $Y_j = Y_j^* \in B(L_j)$, $C_i \in B(H, K_i)$ and $D_j \in B(H, L_j)$ with $\sigma(X_i)$, $\sigma(Y_j) \in \mathcal{I}$. Let $C_i = V_i|C_i|$ be the polar decompositions. If $\sum_{i=1}^k C_i^*C_i - \sum_{j=1}^m D_j^*D_j = I_H$ and $\sigma(\sum_{i=1}^k C_i^*X_iC_i - \sum_{j=1}^m D_j^*Y_jD_j) \in \mathcal{I}$, then*

$$f\left(\sum_{i=1}^k C_i^*X_iC_i - \sum_{j=1}^m D_j^*Y_jD_j\right) \leq \sqrt{\sum_{i=1}^k |C_i|^2} f\left(\sum_{i=1}^k V_i^*X_iV_i\right) \sqrt{\sum_{i=1}^k |C_i|^2} - \sum_{j=1}^m D_j^* f(Y_j) D_j.$$

In particular, if $C = {}^t(C_1, \cdots, C_k)$ is invertible, then

$$f\left(\sum_{i=1}^k C_i^*X_iC_i - \sum_{j=1}^m D_j^*Y_jD_j\right) \leq \sum_{i=1}^k C_i^* f(X_i) C_i - \sum_{j=1}^m D_j^* f(Y_j) D_j.$$

Remark 3. If $C = \xi = \begin{pmatrix} \sqrt{\alpha_1} \\ \vdots \\ \sqrt{\alpha_k} \end{pmatrix}$ and $D = \eta = \begin{pmatrix} \sqrt{\beta_1} \\ \vdots \\ \sqrt{\beta_m} \end{pmatrix}$ are vectors with nonnegative entries and $\|\xi\|^2 - \|\eta\|^2 = 1$, then ξ cannot be invertible for $k > 1$. So, putting $X = \operatorname{diag}(x_1, \cdots, x_k)$ and $Y = \operatorname{diag}(y_1, \cdots, y_m)$, we only have the following numerical inequality;

$$f\left(\sum_{i=1}^k \alpha_i x_i - \sum_{j=1}^m \beta_j y_j\right) \leqq \|\xi\|^2 f\left(\sum_{i=1}^k \frac{\alpha_i x_i}{\|\xi\|^2}\right) - \sum_{j=1}^m \beta_j f(y_j)$$

holds for a operator concave function f. This also holds for a concave function f and it is equivalent to Theorem 1 by putting $\alpha = \sum_i \alpha_i$ and $x = \sum_i \alpha_i x_i$. .

Finally we show an external version for Jensen's operator inequality for projections:

Theorem 3. *Let f be a function on an open interval $\mathcal{I}$. A function f is operator concave on $\mathcal{I}$ if and only if*

$$f(\sqrt{I+P}X\sqrt{I+P}-PYP) \le \sqrt{I+P}f(X)\sqrt{I+P}-Pf(Y)P \tag{3}$$

holds for all projections P and selfadjoint operators X, Y such that

$$\sigma(X),\ \sigma(Y),\ \sigma(\sqrt{I+P}X\sqrt{I+P}-PYP) \in \mathcal{I}.$$

Proof. The operator concavity of f implies (3) by Theorem 2. Conversely let $P = I$, the identity operator. For selfadjoint operators A and B with $\sigma(A)$, $\sigma(B) \in \mathcal{I}$. Putting $X = (A+B)/2$ and $Y = B$. Then

$$\sigma(X), \sigma(Y) \in \mathcal{I} \quad \text{and} \quad \sigma(2X-Y) = \sigma(A) \in \mathcal{I}.$$

So (3) implies

$$f(A) = f(2X-Y) \le 2f(X) - f(Y) = 2f\left(\frac{A+B}{2}\right) - f(B),$$

which shows that f is operator concave. □

Remark 4. In the published paper [2], we required the condition that $\mathcal{I}$ includes 0. But we can delete this conditon.

Thus the multi-variable inequality is:

Corollary 4. *Let f be operator concave on an open interval $\mathcal{I}$. Suppose that P_i are projections with $\sum_i P_i = I$ and X_i, Y_i are selfadjoint operators. If*

$$\sigma(X_i),\ \sigma(Y_i),\ \sigma\left(\sum_i \sqrt{I+P_i}X\sqrt{I+P_i} - P_iYP_i\right) \in \mathcal{I},$$

then

$$f\left(\sum_{i=1}^k \sqrt{I+P_i}X_i\sqrt{I+P_i} - P_i^*Y_iP_i\right) \le \sum_{i=1}^k \sqrt{I+P_i}f(X_i)\sqrt{I+P_i} - P_if(Y_i)P_i.$$

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