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Collapsing behaviour of the logarithmic diffusion equation

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Abstract

I will report my result on the collapsing behaviour of the maximal solution of the equation $u_t = \Delta \log u$ in $\mathbb{R}^2 \times (0,T)$, $u(x,0) = u_0(x)$ in $\mathbb{R}^2$, near its extinction time $T = \int_{R^2} u_0 dx/4\pi$ without using the Hamilton-Yau Harnack inequality. This result extends the recent result of P. Daskalopoulos, M.A. del Pino and N. Sesum [DP2], [DS2].

In this report I will discuss my recent result [Hu5] on the collapsing behaviour of the maximal solution of the logarithmic diffusion equation:

$$\begin{cases} u_t = \Delta \log u, u > 0, & \text{in } \mathbb{R}^2 \times (0,T) \\ u(x,0) = u_0 & \text{in } \mathbb{R}^2. \end{cases} \tag{0.1}$$

(0.1) arises in many physical and geometric models. For example P.L. Lions and Toscani [LT] have shown that (0.1) arises as the diffusive limit for finite velocity Boltzmann kinetic models and T.G. Kurtz [Ku] has proved that (0.1) is the limiting density distribution of two gases moving against each other and obeying the Boltzmann equation. In [G] P.G. de Gennes showed that (0.1) also appears in the model of viscous liquid film lying on a solid surface and subjecting to long range Van der Waals interactions with fourth order term neglected.

Recently K.M. Hui [Hu3], [Hu4] (for the case $m > 0$ and $m < 0$), and J.R. Esteban, A. Rodriguez, J.L. Vazquez [ERV] (for the case $m > 0$) have shown that the solution of the porous medium equation

$$u_t = \Delta\left(\frac{u^m}{m}\right)$$

converges to the maximal solution of (0.1) as $m \to 0$. In [W1], [W2], [H], Angenent, L. Wu and R. Hamilton showed that the equation also arises in the study of the conformally equivalent metric $g_{ij} = u\delta_{ij}$ on $\mathbb{R}^2$ under the Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{0.2}$$

where R_{ij} is the Ricci curvature of the metric g_{ij}. Note that in $\mathbb{R}^2$ the scalar curvature R is given by

$$R = -\frac{\Delta \log u}{u} \tag{0.3}$$

and the Ricci curvature is given by

$$R_{ij} = \frac{1}{2} R g_{ij}. \tag{0.4}$$

By (0.3) and (0.4) the Ricci flow equation (0.2) is equivalent to the logarithmic diffusion equation:

$$u_t = \Delta \log u.$$

1 Existence and properties of solutions

The equation (0.1) has many properties different from the heat equation such as existence of infinite many solutions for any given initial L^1 data. There also does not exist any fundamental solution for (0.1) [Hu1] which suggests that conservation of mass does not hold. K.M. Hui [Hu2] by using approximation by Neumann solutions in bounded domains and P. Daskalaopoulos and M.A.del Pino [DP1] by using approximation by Dirichlet solutions in bounded domains proved independently that corresponding to each

$$0 \le u_0 \in L^p(\mathbb{R}^2) \cap L^1(\mathbb{R}^2), p > 1, 2 \le f \in L^1(0, \infty),$$

there exists a classical solution u of (0.1) in $\mathbb{R}^n \times (0, T)$ satisfying the mass loss equation,

$$\int_{\mathbb{R}^2} u(x,t)\, dx = \int_{\mathbb{R}^2} u_0\, dx - 2\pi \int_0^t f(s)\, ds \quad \forall 0 \le t < T$$

where $T = T(u_0, f) > 0$ given by

$$\int_{\mathbb{R}^2} u_0\, dx = 2\pi \int_0^T f(s)\, ds$$

is the extinction time for the solution u. Hence the solution with flux f vanishes identically to zero at time T.

Note that the maximal solution of (0.1) is the solution of (0.1) that corresponds to flux $f = 2$ which satisfies

$$\int_{\mathbb{R}^2} u(x,t)\,dx = \int_{\mathbb{R}^2} u_0\,dx - 4\pi t \quad \forall 0 \le t < T$$

with

$$T = \frac{1}{4\pi}\int_{\mathbb{R}^2} u_0\,dx.$$

For any $2 < f \in C(0,T)$ the solution u with flux f satisfies the following decay condition at infinity:

$$\lim_{|x|\to\infty} \frac{\log u}{\log |x|} = -f \quad \text{uniformly on [a,b]} \quad \forall 0 < a < b < T$$

or equivalently

$$\lim_{|x|\to\infty} \frac{r u_r}{u} = -f \quad \text{uniformly on [a,b]} \quad \forall 0 < a < b < T.$$

One would like to ask what is the asymptotic behaviour of the solution with constant flux $f = \gamma \ge 2$? When $\gamma > 2$, S.Y. Hsu [Hs1], [Hs2], by using the lap number method of Matano [M], V.A. Galaktionov and L.A. Peletier [GP], proved that if the initial value is radially symmetric and monotone decreasing and u is the solution with flux $\gamma > 2$, then there exist unique constants $\alpha > 0$, $\beta > -1/2$, $\alpha = 2\beta + 1$, depending only on γ such that the rescaled function

$$v(y,s) = \frac{u(y/(T-t)^{\beta},t)}{(T-t)^{\alpha}}, \quad s = -\log(T-t),$$

converges uniformly on every compact subset of $\mathbb{R}^2$ to $\phi_{\lambda,\beta}(y)$ for some constant $\lambda > 0$ as $s \to \infty$ where $\phi_{\lambda,\beta}(y) = \phi_{\lambda,\beta}(|y|)$ is radially symmetric and satisfies the following O.D.E.:

$$\frac{1}{r}\left(\frac{r\phi'}{\phi}\right)' + \alpha\phi + \beta r\phi' = 0 \quad \text{in } (0,\infty)$$

with

$$\phi(0) = 1/\lambda, \phi'(0) = 0.$$

In particular for $\gamma = 4$, the rescaled solution

$$v(x,s) = \frac{u(x,t)}{T-t}, \quad s = -\log(T-t),$$

converges uniformly on every compact subsets of $\mathbb{R}^2$ to the function

$$\frac{8\lambda}{(\lambda + |x|^2)^2}$$

as $s \to \infty$ for some constant $\lambda > 0$. What this said is that for solution with flux $f = 4$,

$$u(x,t) \approx \frac{8\lambda(T-t)}{(\lambda + |x|^2)^2} \quad \text{as } t \nearrow T$$

which corresponds to contracting spheres Ricci flow solution on S^2.

What about the asymptotic behaviour of the solution with flux $f = 2$? J.R. King [K] by using inner and outer asymptotic expansion and matching asymptotic method showed that if u is the solution of the logarithmic diffusion equation (0.1) with flux $f = 2$ then as t approaches the extinction time T the vanishing behaviour for solution is very different from the vanishing behaviour for the case $f \equiv \gamma > 2$. J.R. King found that for compactly supportly finite mass initial value the maximal solution behaves like

$$\frac{(T-t)^2}{\frac{T}{2}|x|^2 + e^{\frac{2T}{(T-t)}}}$$

in the inner region $(T-t)\log|x| \le T$ and behaves like

$$\frac{2t}{|x|^2(\log|x|)^2}$$

in the outer region $(T-t)\log|x| \ge T$ as $t \nearrow T$. In [DP2] P. Daskalopoulos and M.A. del Pino gave a rigorous proof of an extension of this formal result for radially symmetric initial value $u_0(r) \ge 0$ satisfying the conditions,

$$u_0(x) = u_0(|x|) \text{ is decreasing on } r = |x| \ge r_1 \text{ for some constant } r_1 > 0,$$

$$u_0(x) = \frac{2\mu}{|x|^2(\log|x|)^2}(1 + o(1)) \quad \text{as } |x| \to \infty,$$

for some constant $\mu > 0$ and

$$R_0(x) := -\frac{\Delta \log u_0}{u_0} \ge -\frac{1}{\mu} \quad \text{on } \mathbb{R}^2.$$

Later P. Daskalopoulos and N. Sesum [DS2] extended this result to the case of compactly supported $0 \le u_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. However their proof of the behaviour of the maximal solution in the outer region near the extinction time is very difficult and uses the Hamilton-Yau Harnack inequality. Recently in [Hu5] I extended their result to the case of initial value

$$0 \le u_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$$

that satisfies

$$u_0(x) = u_0(|x|) \text{ is decreasing on } r = |x| \ge r_1 \text{ for some constant } r_1 > 0,$$

$$u_0(x) = \frac{2\mu}{|x|^2(\log|x|)^2}(1 + o(1)) \quad \text{as } |x| \to \infty,$$

and

$$R_0(x) := -\frac{\Delta \log u_0}{u_0} \geq -\frac{1}{\mu} \quad \text{on } \mathbb{R}^2$$

for some constant $\mu \geq 0$ with the right hand side being replaced by $-\infty$ if $\mu = 0$ and

$$\operatorname*{ess\,inf}_{\overline{B}_{r_1}(0)} u_0 \geq \operatorname*{ess\,sup}_{\mathbb{R}^2 \setminus B_{r_2}(0)} u_0 \quad \text{for some constant } r_2 > r_1. \tag{1.1}$$

Note that (1.1) is automatically satisfied if u_0 has compact support in $\mathbb{R}^2$. In [Hu5] I proved the behaviour of the maximal solution in the outer region near the extinction time by elementary method without using the difficult Hamilton-Yau Harnack inequality for surfaces. I also obtained the behaviour of the maximal solution in the inner region as the extinction time is approached.

I will now assume that u_0 satisfies the above structural conditions and u is the maximal solution of (0.1) in $\mathbb{R}^2 \times (0, T)$ with flux $f \equiv 2$ and

$$T = \frac{1}{4\pi} \int_{\mathbb{R}^2} u_0 \, dx.$$

I will sketch of proof of [Hu5] here.

2 Inner region behaviour

By using the reflection method of D.G. Aronson and L.A. Caffarelli [AC] one has the following lemma:

Lemma 2.1. *(Lemma 1.1 of [Hu5]) The solution u satisfies*

$$u(x,t) \geq u(y,t)$$

for any $t \in (0,T)$ and $x, y \in \mathbb{R}^2$ such that $|y| \geq |x| + 2r_2$.

Then for any $0 < t < T$ there exists $x_t \in \overline{B}_{2r_2}$ such that

$$u(x_t, t) = \max_{x \in \mathbb{R}^2} u(x,t).$$

That is the maximum of $u(\cdot, t)$ is attained on the compact set $\overline{B}_{2r_2}$. We will now perform a rescaling of the solution of (0.1). Let

$$\overline{u}(x,\tau) = \tau^2 u(x,t), \qquad \tau = \frac{1}{T-t}, \tau > 1/T.$$

Then $\overline{u}$ satisfies

$$\overline{u}_\tau = \Delta \log \overline{u} + \frac{2\overline{u}}{\tau} \quad \text{in } \mathbb{R}^2 \times (1/T, \infty).$$

Let

$$R_{max}(t) = \max_{x \in \mathbb{R}^2} R(x,t)$$

and let $W(t)$ be the width function with respect to the metric $g_{ij}(t)$ as defined by P. Daskalopoulos and R. Hamilton [DH]. We now recall a result of P. Daskalopoulos and R. Hamilton [DH]:

Theorem 2.2. *([DH]) There exist positive constants $c > 0$ and $C > 0$ such that*

(i) $c(T-t) \le W(t) \le C(T-t)$

(ii) $\frac{c}{(T-t)^2} \le R_{max}(t) \le \frac{C}{(T-t)^2}$

hold for any $0 < t < T$.

By Theorem 2.2,

$$c \le \limsup_{t \nearrow T} (T-t)^2 R_{max}(t) \le C.$$

Hence the singularity is a type II singularity. Note that u satisfies the Aronson-Bénilan inequality,

$$\frac{u_t}{u} \le \frac{1}{t} \quad \text{in } \mathbb{R}^2 \times (0,T).$$

As

$$R = -\frac{\Delta \log u}{u} = -\frac{u_t}{u},$$

the Aronson-Bénilan inequality is equivalent to

$$R \ge -\frac{1}{t}.$$

So if we let

$$\overline{R}(x,\tau) = -\frac{\Delta \log \overline{u}}{\overline{u}}.$$

Then

$$\frac{2}{\tau} + \frac{2}{\tau^2 T} \ge \frac{\overline{u}_\tau}{\overline{u}} \ge -C \qquad \text{in } \mathbb{R}^2 \times (2/T, \infty). \tag{2.1}$$

Theorem 2.3. *(Theorem 1.3 of [Hu5]) For any sequence $\{\tau_k\}_{k=1}^\infty$, $\tau_k \to \infty$ as $k \to \infty$, let*

$$\overline{u}_k(y,\tau) = \alpha_k \overline{u}(\alpha_k^{\frac{1}{2}} y + x_{t_k}, \tau + \tau_k), \quad y \in \mathbb{R}^2, \tau > -\tau_k + T^{-1}$$

where

$$t_k = T - \tau_k^{-1} \quad \forall k \in \mathbb{Z}^+$$

and

$$\alpha_k = 1/\overline{u}(x_{t_k}, \tau_k).$$

Then $\{\overline{u}_k\}_{k=1}^{\infty}$ *has a subsequence* $\{\overline{u}_{k_i}\}_{i=1}^{\infty}$ *that converges uniformly on* $C^{\infty}(K)$ *for any compact set* $K \subset \mathbb{R}^2 \times (-\infty, \infty)$ *as* $i \to \infty$ *to a positive solution*

$$U(y,\tau) = \frac{1}{\lambda|y|^2 + e^{4\lambda\tau}}.$$

of equation

$$U_\tau = \Delta \log U \quad in\ \mathbb{R}^2 \times (-\infty, \infty)$$

with uniformly bounded non-negative scalar curvature and uniformly bounded width on $\mathbb{R}^2 \times (-\infty, \infty)$ *with respect to the metric* $\widetilde{g}_{ij}(t) = U(\cdot, t)\delta_{ij}$ *where* $\lambda > 0$ *is some constant.*

Proof:(Sketch) By definition,

$$\overline{u}_k(0,0) = 1 \quad \text{and} \quad \overline{u}_k(y,0) \le 1 \quad \forall y \in \mathbb{R}^2.$$

Let $a < b$. By (2.1) there exist constants $M_1 > 0$ and $k_0 \in \mathbb{Z}^+$ such that

$$\overline{u}_k(x,\tau) \le M_1 \quad \text{and } |\overline{u}_{k,\tau}(x,\tau)| \le CM_1 \quad x \in \mathbb{R}^2, a \le \tau \le b, k \ge k_0. \tag{2.2}$$

By (2.2) one can deduce the following Harnack inequality:

For any $R_1 > 0, a < b$, there exists a constant $C_2 > 0$ such that

$$\sup_{\substack{|y| \le R_1 \\ a \le \tau_1 \le b}} \overline{u}_k(y,\tau_1)^9 \le C_2 \inf_{\substack{|x| \le R_1 \\ a \le \tau_2 \le b}} \overline{u}_k(x,\tau_2) \quad \forall k \ge k_0.$$

Hence the sequence $\overline{u}_k$ is uniformly parabolic on $\overline{B}_{R_1} \times [a,b]$ and are uniformly Holder continuous in $C^{2\gamma,1\gamma}(\overline{B}_{R_1} \times [a,b])$ for any $\gamma \in \mathbb{Z}^+$. Then the sequence $\{\overline{u}_k\}_{k=1}^{\infty}$ has a subsequence which we may assume without loss of generality to be the sequence itself that converges uniformly in $C^{\infty}(K)$ as $k \to \infty$ for any compact set $K \subset \mathbb{R}^2 \times (-\infty, \infty)$ to some positive function U that satisfies the logarithmic diffusion equation. Let

$$\overline{R}_k = -\frac{\Delta \log \overline{u}_k}{\overline{u}_k}.$$

Then $\overline{R}_k$ converges uniformly on every compact subset of $\mathbb{R}^2 \times (-\infty, \infty)$ as $k \to \infty$ to the scalar curvature

$$\widetilde{R} = -\frac{\Delta \log U}{U}$$

of the metric $\widetilde{g}_{ij}(\tau) = U(\cdot, \tau)\delta_{ij}$. Moreover

$$0 \le \widetilde{R}(y,\tau) \le C \quad \forall (y,\tau) \in \mathbb{R}^2 \times (-\infty, \infty).$$

By an approximation argument the width function with respect to the metric $\widetilde{g}_{ij}(\tau) = U(\cdot,\tau)\delta_{ij}$ is uniformly bounded on $\mathbb{R}^2 \times (-\infty,\infty)$. Then by the result of P. Daskalopoulos and N. Sesum [DS1],

$$U(y,\tau) = \frac{2}{\beta(|y-y_0|^2 + \delta e^{2\beta\tau})}$$

for some $y_0 \in \mathbb{R}^2$ and constants $\beta > 0$, $\delta > 0$. Since $\overline{u}_k(y,0)$ attains its maximum at $y = 0$, $U(y,0)$ will attain its maximum at $y = 0$. Hence $y_0 = 0$.

$$U(0,0) = 1 \quad \Rightarrow \quad 1 = \frac{2}{\beta\delta}.$$

Thus

$$U(y,\tau) = \frac{1}{\lambda|y|^2 + e^{4\lambda\tau}}.$$

for some constant $\lambda > 0$. □

It can be proved that

$$\alpha_k \to \infty \qquad \text{as } k \to \infty$$

Hence we can change the origin in rescaling and have the following result:

Lemma 2.4. *(Lemma 1.10 of [Hu5]) Let*

$$q_k(y,\tau) = \alpha_k \overline{u}(\alpha_k^{\frac{1}{2}} y, \tau + \tau_k).$$

Then $q_{k_i}(y,\tau)$ converges uniformly in $C^\infty(K)$ for every compact set $K \subset \mathbb{R}^2$ to the function $U(y,\tau)$ as $\tau \to \infty$.

We will now perform a change of scaling. Let

$$\beta(\tau) = 1/\overline{u}(0,\tau),$$

$\beta_k = \beta(\tau_k)$, and

$$\overline{q}_k(y,\tau) = \beta_k \overline{u}(\beta_k^{\frac{1}{2}} y, \tau + \tau_k).$$

Then

$$\frac{\alpha_k}{\beta_k} = q_k(0,0) \to U(0,0) = 1 \qquad \text{as } k \to \infty.$$

Hence there exists $k_0 \in \mathbb{Z}^+$ and constants $c_2 > c_1 > 0$ such that

$$c_1 \le \frac{\beta_k}{\alpha_k} \le c_2 \quad \forall k \ge k_0.$$

Thus we have the following result.

Theorem 2.5. *(Theorem 1.11 of [Hu5])* $\overline{q}_k$ *has a subsequence* $\overline{q}_{k_i}$ *that converges uniformly on* $C^\infty(K)$ *for any compact set* $K \subset \mathbb{R}^2 \times (-\infty, \infty)$ *to* $U(y,\tau)$ *as* $\tau \to \infty$*. Moreover* $\beta_k \to \infty$ *as* $k \to \infty$.

Lemma 2.6. *(Proposition 1.17 and Proposition 1.18 of [Hu5])*

$$\lim_{\tau\to\infty} \frac{\beta'(\tau)}{\beta(\tau)} = \lim_{\tau\to\infty} \frac{\log\beta(\tau)}{\tau} = 2(T+\mu).$$

Let

$$\widetilde{R}_k = -\frac{\Delta \log \overline{q}_k}{\overline{q}_k}.$$

Since

$$\widetilde{R}_k(0,0) = \frac{\beta'(\tau_k)}{\beta(\tau_k)} + \frac{2}{\tau_k},$$

then

$$4\lambda = \lim_{\tau\to\infty} \widetilde{R}_k(0,0) = 2(T+\mu).$$

Let

$$\overline{q}(y,\tau) = \beta(\tau)\overline{u}(\beta(\tau)^{\frac{1}{2}}y, \tau).$$

We then have the following main theorem for inner region.

Theorem 2.7. *(Theorem 1.21 of [Hu5])* $\overline{q}(y,\tau)$ *converges uniformly on* $C^\infty(K)$ *for any compact set* $K \subset \mathbb{R}^2$ *to the function*

$$U_\mu(y) = \frac{1}{\frac{(T+\mu)}{2}|y|^2 + 1}$$

as $\tau \to \infty$.

Corollary 2.8. *(cf. [Hu5]) For any* $\varepsilon > 0$ *and* $M > 0$ *there exist* $\tau_0 > 1/T$ *and* $C > 0$ *such that*

$$\begin{cases} \left| u(x,t) - \dfrac{(T-t)^2}{\lambda|x|^2 + \beta(\tau)} \right| < u(0,t)\varepsilon & \forall |x| \le \beta(\tau)^{\frac{1}{2}}M, \tau > \tau_0 \\ u(0,t) \le C(T-t)^2 & \forall t > T - \tau_0^{-1}. \end{cases}$$

where $\tau = 1/(T-t)$.

As in [DP2], [DS2], [Hu5], we now consider the cylindrical change of variables,

$$v(\zeta,\theta,t) = r^2 u(r,\theta,t), \quad \zeta = \log r, r = |x|$$

and let

$$\widetilde{v}(\xi,\theta,\tau) = \tau^2 v(\tau\xi,\theta,t), \quad \tau = 1/(T-t), \tau \ge 1/T.$$

Then $\widetilde{v}$ satisfies

$$\tau\widetilde{v}_\tau = \frac{1}{\tau}(\log\widetilde{v})_{\xi\xi} + \tau(\log\widetilde{v})_{\theta\theta} + \xi\widetilde{v}_\xi + 2\widetilde{v} \quad \text{in } \mathbb{R} \times [0,2\pi] \times (1/T, \infty).$$

Corollary 2.9. *(Lemma 1.23 of [Hu5]) For any $\varepsilon > 0$ there exists $\tau_0 > 1/T$ such that*

$$\left|\widetilde{v}(\xi,\theta,\tau) - \frac{e^{2\tau\xi}}{\frac{T+\mu}{2}e^{2\tau\xi} + \beta(\tau)}\right| < \frac{e^{2\tau\xi}}{\beta(\tau)}\varepsilon \quad \forall \xi \le \frac{\log\beta(\tau)}{2\tau}, \theta \in [0,2\pi], \tau \ge \tau_0.$$

Corollary 2.10. *(Corollary 1.24 of [Hu5])*

$$\int_{-\infty}^{\xi^-}\int_0^{2\pi} \widetilde{v}(\xi,\theta,\tau)\,d\theta\,d\xi \to 0 \quad \text{as } \tau \to \infty$$

and

$$\lim_{\tau\to\infty} \widetilde{v}(\xi,\theta,\tau) = 0 \quad \text{uniformly on } (-\infty,\xi^-] \times [0,2\pi]$$

for any $\xi^- < T + \mu$.

3 Outer region behaviour

Let

$$\xi(\tau) = (\log\beta(\tau))/2\tau.$$

Lemma 3.1. *(Lemma 2.1 of [Hu5]) There exists constants $C_1 > 0$, $C_2 > 0$, $C_3 > 0$ and $\tau_0 > 1/T$ such that the following holds.*

(i) $\widetilde{v}(\xi,\theta,\tau) \le C_1 \quad \forall \xi \in \mathbb{R}, \theta \in [0,2\pi], \tau \ge 1/T$

(ii) $\widetilde{v}(\xi,\theta,\tau) \ge \frac{C_2}{\xi^2} \quad \forall \xi \ge \xi(\tau), \theta \in [0,2\pi], \tau \ge \tau_0$

(iii) $\widetilde{v}(\xi,\theta,\tau) \le \frac{C_3}{\xi^2} \quad \forall \xi > 0, \theta \in [0,2\pi], \tau \ge \tau_0.$

Moreover

$$\xi(\tau) = T + \mu + o(1) \quad \text{as } \tau \to \infty.$$

We now let

$$w(\xi,\theta,s) = \widetilde{v}(\xi,\theta,\tau)$$

with

$$s = \log\tau = -\log(T-t).$$

Then

$$w_s = e^{-s}(\log w)_{\xi\xi} + e^{s}(\log w)_{\theta\theta} + \xi w_\xi + 2w \quad \text{in } \mathbb{R}\times[0,2\pi]\times(-\log T,\infty).$$

The following is the main theorem for outer region.

Theorem 3.2. *(Theorem 2.3 of [Hu5]) As $\tau \to \infty$, the function $\widetilde{v}$ converges to the function*

$$V(\xi) = \begin{cases} \dfrac{2(T+\mu)}{\xi^2} & \forall \xi > T+\mu \\ 0 & \forall \xi < T+\mu. \end{cases}$$

Moreover the convergence is uniform on $(-\infty, a]$ for any $a < T+\mu$ and on $[\xi_0, \xi_0']$ for any $\xi_0' > \xi_0 > T+\mu$.

Proof:(Sketch) Let $\{s_k\}_{k=1}^{\infty}$ be a sequence such that $s_k \to \infty$ as $k \to \infty$ and

$$w_k(\xi,\theta,s) = w(\xi,\theta,s+s_k) \quad \forall \xi \in \mathbb{R}, 0 \le \theta \le 2\pi, s \ge -\log T - s_k.$$

Let

$$W_k^b(\eta,s) = \int_\eta^b \int_0^{2\pi} w_k(\xi,\theta,s)\,d\theta\,d\xi \quad \forall b \ge \eta > T+\mu, s > -\log T - s_k, k \in \mathbb{Z}^+,$$

$$W_k(\eta,s) = \int_\eta^\infty \int_0^{2\pi} w_k(\xi,\theta,s)\,d\theta\,d\xi \quad \forall \eta > T+\mu, s > -\log T - s_k, k \in \mathbb{Z}^+$$

and let $\{b_i\}_{i=1}^{\infty}$ be a monotonically increasing sequence such that $b_i > T+\mu$ for any $i \in \mathbb{Z}^+$ and $b_i \to \infty$ as $i \to \infty$.

Since

$$\int_{\mathbb{R}^2} u(x,t)\,dx = 4\pi(T-t) \quad \forall 0 < t < T,$$

$$\int_{-\infty}^\infty \int_0^{2\pi} w_k(\xi,\theta,s)\,d\theta\,d\xi = 4\pi \quad \forall s > -\log T - s_k, k \in \mathbb{Z}^+.$$

One can prove that there exists a function $\overline{w}$ and a subsequence of $\mathbb{Z}^+$ which we may assume without loss of generality to be $\mathbb{Z}^+$ itself such that

$$W_k^{b_i} \to W^{b_i} \quad \text{uniformly on } [a,b]\times[c,d] \quad b > a > T+\mu, d > c \quad \text{as } k \to \infty$$

for any $i \in \mathbb{Z}^+$ and

$$W_k \to W \quad \text{uniformly on } [a,b]\times[c,d] \quad b > a > T+\mu, d > c \quad \text{as } k \to \infty$$

where

$$W^b(\eta,s) = \int_\eta^b \int_0^{2\pi} \overline{w}(\xi,\theta,s)\,d\theta\,d\xi, \quad W(\eta,s) = \int_\eta^\infty \int_0^{2\pi} \overline{w}(\xi,\theta,s)\,d\theta\,d\xi.$$

By elementary argument one can show that

$$\eta W(\eta,s) = \overline{\eta} W(\overline{\eta},\overline{s}) \quad \forall \eta, \overline{\eta} > T+\mu, s, \overline{s} \in \mathbb{R}$$

and

$$W(T + \mu, s) = 4\pi \quad \forall s \in \mathbb{R}.$$

Letting $\overline{\eta} \to T + \mu$,

$$W(\eta, s) = \frac{4\pi(T + \mu)}{\eta} \quad \forall \eta > T + \mu, s \in \mathbb{R}$$

which will then imply the theorem after some elementary computation. □

References

[AC] D.G. Aronson and L.A. Caffarelli, *The initial trace of a solution of the porous medium equation*, Transactions A.M.S. 280 (1983), no. 1, 351–366.

[DH] P. Daskalopoulos and R. Hamilton, *Geometric estimates for the logarithmic fast diffusion equation*, Comm. Anal. Geom. 12 (2004), nos. 1–2, 143–164.

[DP1] P. Daskalopoulos and M.A. del Pino, *On a singular diffusion equation*, Comm. Anal. Geom. 3 (1995), no. 3, 523–542.

[DP2] P. Daskalopoulos and M.A. del Pino, *Type II collapsing of maximal solutions to the Ricci flow in* $\mathbb{R}^2$, Ann. Inst. H. Poincaré Anal. Non Linaire 24 (2007), 851–874.

[DS1] P. Daskalopoulos and N. Sesum, *Eternal solutions to the Ricci flow on* R^2, Int. Math. Res. Not. 2006, Art. ID 83610, 20 pp.

[DS2] P. Daskalopoulos and N. Sesum, *Type II extinction profile of maximal solutions to the Ricci flow equation*, J. Geom. Anal. 20 (2010), no. 3, 565–591.

[ERV] J.R. Esteban, A. Rodriguez, and J.L. Vazquez, *The maximal solution of the logarithmic fast diffusion equation in two space dimensions*, Advances in Diff. Eq. 2 (1997), no. 6, 867–894.

[GP] V.A. Galaktionov and L.A. Peletier, *Asymptotic behaviour near finite-time extinction for the fast diffusion equation*, Arch. Rat. Mech. Anal. 139 (1997), 83–98.

[G] P.G. de Gennes, *Wetting:statics and dynamics Reviews of Modern Physics* 57 (1985), no. 3, 827–863.

[H] R.S. Hamilton, *The Ricci flow on surfaces*, Mathematics and general relativity (Santa Cruz, CA, 1986), 237V262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988.

[Hu1] K.M. Hui, *Nonexistence of fundamental solutions of the equation* $u_t = \Delta \log u$, J. Math. Anal. Appl. 182 (1994), no. 3, 800–809.

[Hu2] K.M. Hui, *Existence of solutions of the equation* $u_t = \Delta \log u$, Nonlinear Analysis TMA 37 (1999), no. 7, 875–914.

[Hu3] K.M. Hui, *Singular limit of solutions of the equation* $u_t = \Delta(u^m/m)$ *as* $m \to 0$, Pacific J. Math. 187 (1999), no. 2, 297–316.

[Hu4] K.M. Hui, *Singular limit of solutions of the very fast diffusion equation*, Nonlinear Analysis TMA 68 (2008), no. 5, 1120–1147.

[Hu5] K.M. Hui, *Collapsing behaviour of a singular diffusion equation*, http://arxiv.org /abs/0910.5045v2.

[Hs1] S.Y. Hsu, *Asymptotic behaviour of solutions of the equation* $u_t = \Delta \log u$ *near the extinction time*, Advances in Differential Equations 8 (2003), no. 2, 161–187.

[Hs2] S.Y. Hsu, *Behaviour of solutions of a singular diffusion equation near the extinction time*, Nonlinear Analysis TMA 56 (2004), no. 1, 63–104.

[K] J.R. King, *Self-similar behaviour for the equation of fast nonlinear diffusion*, Phil. Trans. Royal Soc. London, Series A 343 (1993), 337–375.

[Ku] T.G. Kurtz, *Convergence of sequences of semigroups of nonlinear operators with an application to gas kinetics*, Trans. Amer. Math. Soc. 186 (1973), 259–272.

[LT] P.L. Lions and G. Toscani, *Diffusive limit for finite velocity Boltzmann kinetic models*, Revista Matematica Iberoamericana 13 (1997), no. 3, 473–513.

[M] H. Matano, *Nonincrease of the lap number of a solution for one dimensional semi-linear parabolic equation*, J. Fac. Sci. Univ. Tokyo, Sec. IA 29 (1982), 401–441.

[W1] L.F. Wu, *The Ricci flow on complete* R^2, Comm. in Analysis and Geometry 1 (1993), 439–472.

[W2] L.F. Wu, *A new result for the porous medium equation*, Bull. Amer. Math. Soc. 28 (1993), 90–94.