数理解析研究所講究録
第 1779 巻 2012 年 111-127

Profiles of solutions to an integral system related to the weighted Hardy-Littlewood-Sobolev inequality

東北大学大学院理学研究科　小野寺 有紹 (Michiaki Onodera)
Mathematical Institute, Tohoku University

1 Introduction

The weighted Hardy-Littlewood-Sobolev inequality of Stein and Weiss [17] states that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x|^{\alpha}|x-y|^{\lambda}|y|^{\beta}}\,dx\,dy\leq C\|f\|_r\|g\|_s \tag{1.1}$$

holds for $f\in L^r(\mathbb{R}^n)$, $g\in L^s(\mathbb{R}^n)$ with $1<r,s<\infty$, $0<\lambda<n$, $0\leq\alpha+\beta\leq n-\lambda$,

$$\frac{1}{r}+\frac{\alpha}{n}<1,\quad \frac{1}{s}+\frac{\beta}{n}<1,\quad \text{and}\quad \frac{1}{r}+\frac{1}{s}+\frac{\alpha+\beta+\lambda}{n}=2.$$

Here, $\|\cdot\|_r$ denotes the $L^r(\mathbb{R}^n)$ norm and the constant $C=C(r,s,\lambda,\alpha,\beta)$ does not depend on the choice of f and g.

To obtain the best constant for the inequality (1.1), one desires to maximize the functional

$$J(f,g):=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x|^{\alpha}|x-y|^{\lambda}|y|^{\beta}}\,dx\,dy$$

under the constraint $\|f\|_r=\|g\|_s=1$. In the case where $\alpha,\beta\geq 0$ and $\alpha+\beta+\lambda<n$, Lieb [16] proved the existence of a pair of maximizing functions f,g for this variational problem. By assuming that f and g are nonnegative functions, the corresponding system of the Euler-Lagrange equations is derived as

$$\begin{cases}\lambda_1 f(x)^{r-1}=\dfrac{1}{|x|^{\alpha}}\displaystyle\int_{\mathbb{R}^n}\frac{g(y)}{|x-y|^{\lambda}|y|^{\beta}}\,dy,\\[2mm] \lambda_2 g(x)^{s-1}=\dfrac{1}{|x|^{\beta}}\displaystyle\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{\lambda}|y|^{\alpha}}\,dy,\end{cases} \tag{1.2}$$

where λ_1 and λ_2 are the Lagrange multipliers which satisfy $\lambda_1=\lambda_2=J(f,g)$. Note that, if $(r-1)(s-1)\neq 1$, then we may assume $\lambda_1=\lambda_2=1$ by taking $c_1 f$, $c_2 g$ instead of f, g with appropriate constants c_1 and c_2. For convenience, we rewrite the system

(1.2) by $u := f^{r-1}$, $v := g^{s-1}$, $p := 1/(r-1)$ and $q := 1/(s-1)$ to obtain the following system of integral equations:

$$\begin{cases} u(x) = \dfrac{1}{|x|^\alpha} \displaystyle\int_{\mathbb{R}^n} \frac{v(y)^q}{|x-y|^\lambda |y|^\beta}\, dy, \\ v(x) = \dfrac{1}{|x|^\beta} \displaystyle\int_{\mathbb{R}^n} \frac{u(y)^p}{|x-y|^\lambda |y|^\alpha}\, dy, \end{cases} \tag{1.3}$$

where $u \in L^{p+1}(\mathbb{R}^n)$, $v \in L^{q+1}(\mathbb{R}^n)$, $0 < p, q < \infty$,

$$\frac{\alpha}{n} < \frac{1}{p+1}, \quad \frac{\beta}{n} < \frac{1}{q+1}, \quad \text{and} \quad \frac{1}{p+1} + \frac{1}{q+1} = \frac{\alpha+\beta+\lambda}{n}. \tag{1.4}$$

The determination of the functional forms of solutions to the integral system (1.3) yields the best constant for the weighted Hardy-Littlewood-Sobolev inequality (1.1). Lieb [16] classified all the maximizers of the functional $J = J(f, g)$ under the constraints $\|f\|_r = \|g\|_s = 1$ in the special case where $\alpha = \beta = 0$ and $r = s$. It was shown that any maximizer must be of the form

$$f(x) = g(x) = c\left(\frac{t}{t^2 + |x-x_0|^2}\right)^{(2n-\lambda)/2} \tag{1.5}$$

with some constants $c \in \mathbb{R}$, $t > 0$, and $x_0 \in \mathbb{R}^n$. In the paper [16], he posed the problem of the classification of all the critical points (not only maximizers) of the functional, i.e., that of all the solutions to the integral system (1.3), in the case where $\alpha = \beta = 0$, $p = q$ and $u = v$.

Letting $u = v$ reduces the system to the single equation

$$u(x) = \int_{\mathbb{R}^n} \frac{u(y)^{\frac{n+\gamma}{n-\gamma}}}{|x-y|^{n-\gamma}}\, dy, \tag{1.6}$$

where $\gamma = n - \lambda$. This integral equation corresponds to the well known differential equation

$$(-\Delta)^{\gamma/2} u = u^{(n+\gamma)/(n-\gamma)}, \tag{1.7}$$

which has been investigated by many authors. In particular, when $\gamma = 2$, Gidas, Ni and Nirenberg [7] proved the radial symmetry of positive solutions to (1.7) under the additional condition that $u(x) = O(|x|^{2-n})$ as $|x| \to \infty$, and hence the solutions must be of the form (1.5). Then, Caffarelli, Gidas and Spruck [1] obtained the same result without imposing the decay condition at infinity. Their proof was simplified by Chen and Li [3], and Li [13]. Moreover, Wei and Xu [18] studied more general equation (1.7) with γ being even numbers between 0 and n.

Later, Chen, Li and Ou [5, 6] introduced an integral form of the method of moving planes to prove the symmetry of solutions to the equation (1.6) and to the system (1.3) when $\alpha = \beta = 0$, $p, q \geq 1$ and $pq \neq 1$, and therefore they solved the open problem posed by Lieb (see [15] for a different argument by using the method of moving spheres). They also discussed about the relation between the integral equation (1.6) and the differential equation (1.7).

Now our attention turns to the integral system (1.3) for general α, β and $p, q > 0$. The symmetry of solutions was studied by Jin and Li [10]. Chen, Jin, Li and Lim [2] obtained the optimal integrability of solutions to the system when $\alpha, \beta \geq 0$, and Jin and Li [11] extended the result to the case where α or β is even negative. By using the integrability of solutions, Li and Lim [14] studied the profiles of solutions around the origin and the infinity. However, their results are restricted to the case where $p, q \geq 1$ and $pq \neq 1$, since the methods use linear operators to make a regularity lifting argument. This restriction was removed by Hang [8] when $\alpha = \beta = 0$ by developing a nonlinear technique. He proved the symmetry and regularity of solutions in this case for all $0 < p, q < \infty$. This technique was also applied to a different integral system by Hang, Wang and Yan [9].

In this paper we develop the methods of obtaining integrability, regularity and symmetry by adopting a nonlinear approach to show the profiles of solutions to the integral system (1.3) for general α, β and $0 < p, q < \infty$. This paper unifies and extends the previous results obtained by other authors and completes the study in full generality.

The following theorem shows a priori integrability of solutions in the case where $\alpha, \beta \geq 0$.

Theorem 1.1. *Suppose that a pair of nonnegative functions $u \in L^{p+1}(\mathbb{R}^n)$ and $v \in L^{q+1}(\mathbb{R}^n)$ $(0 < p, q < \infty)$ is a solution to the integral system (1.3), where $0 < \lambda < n$, $0 \leq \alpha, \beta$, $\alpha + \beta + \lambda < n$, and the condition (1.4) are satisfied. Then, $u \in L^r(\mathbb{R}^n)$ and $v \in L^s(\mathbb{R}^n)$ hold for r, s satisfying*

$$\max\left\{\frac{\alpha}{n}, \frac{q\beta + \alpha + \beta + \lambda}{n} - 1\right\} < \frac{1}{r} < \min\left\{\frac{\alpha + \lambda}{n}, \frac{q(\beta + \lambda) + \alpha + \beta + \lambda}{n} - 1\right\}, \tag{1.8}$$

$$\max\left\{\frac{\beta}{n}, \frac{p\alpha + \alpha + \beta + \lambda}{n} - 1\right\} < \frac{1}{s} < \min\left\{\frac{\beta + \lambda}{n}, \frac{p(\alpha + \lambda) + \alpha + \beta + \lambda}{n} - 1\right\}. \tag{1.9}$$

We show an analogous result in the case where α or β is strictly less than 0. Here, we may assume $\beta < 0$ without loss of generality.

Theorem 1.2. *Suppose that a pair of nonnegative functions $u \in L^{p+1}(\mathbb{R}^n)$ and $v \in L^{q+1}(\mathbb{R}^n)$ $(0 < p, q < \infty)$ is a solution to the integral system (1.3), where $0 < \lambda < n$, $\beta < 0$, $0 \leq \alpha + \beta < n - \lambda$, and the condition (1.4) are satisfied. Then, $u \in L^r(\mathbb{R}^n)$*

and $v \in L^s(\mathbb{R}^n)$ *hold for* r, s *satisfying*

$$\frac{\alpha}{n} < \frac{1}{r} < \min\left\{\frac{\alpha+\beta+\lambda}{n},\ \frac{q(\beta+\lambda)+\alpha+\beta+\lambda}{n} - 1\right\}, \tag{1.10}$$

$$\max\left\{0,\ \frac{p\alpha+\alpha+\beta+\lambda}{n} - 1\right\} < \frac{1}{s} < \min\left\{\frac{\beta+\lambda}{n},\ \frac{(p+1)(\alpha+\beta+\lambda)}{n} - 1\right\}. \tag{1.11}$$

Theorems 1.1 and 1.2 play an important role to determine the profiles of solutions to the integral system (1.3). In fact, the analysis employed by Li and Lim [14], and Lei, Li and Ma [12] can be applied to show the following result concerning the profiles of solutions. In the theorem, we use the notation $u(x) \sim A/|x|^\gamma$ as $|x| \to 0$ to mean that $\lim_{|x|\to 0} |x|^\gamma u(x) = A$. Remark that the condition $\alpha+\beta+\lambda < n$ and (1.4) imply that either $q\beta+\beta+\lambda < n$ or $p\alpha+\alpha+\lambda < n$ holds and also that either $q(\beta+\lambda)+\beta > n$ or $p(\alpha+\lambda)+\alpha > n$ holds. This fact can be easily confirmed by simple computations.

Theorem 1.3. *Suppose that a pair of nonnegative functions* $u \in L^{p+1}(\mathbb{R}^n)$ *and* $v \in L^{q+1}(\mathbb{R}^n)$ $(0 < p, q < \infty)$ *is a solution to the integral system* (1.3), *where* $0 < \lambda < n$, $0 \le \alpha+\beta < n-\lambda$, *and the condition* (1.4) *are satisfied. Then,* u *and* v *have the following profiles.*

(i) *Around the origin.*

Assume moreover that $q\beta+\beta+\lambda < n$. *Then, it holds that*

$$u(x) \sim \frac{A_0}{|x|^\alpha} \quad \text{and} \quad v(x) \sim \begin{cases} \dfrac{A_1}{|x|^\beta} & \text{if } p\alpha+\alpha+\lambda < n, \\ -\dfrac{A_2 \log|x|}{|x|^\beta} & \text{if } p\alpha+\alpha+\lambda = n, \\ \dfrac{A_3}{|x|^{p\alpha+\alpha+\beta+\lambda-n}} & \text{if } p\alpha+\alpha+\lambda > n, \end{cases}$$

as $|x| \to 0$. *Here the constants* A_0, A_1, A_2, A_3 *are given by*

$$A_0 := \int_{\mathbb{R}^n} \frac{v(y)^q}{|y|^{\lambda+\beta}}\,dy, \quad A_1 := \int_{\mathbb{R}^n} \frac{u(y)^p}{|y|^{\lambda+\alpha}}\,dy, \quad A_2 := \omega_{n-1}\left(\int_{\mathbb{R}^n} \frac{v(y)^q}{|y|^{\lambda+\beta}}\,dy\right)^p,$$
$$\text{and} \quad A_3 := \left(\int_{\mathbb{R}^n} \frac{v(y)^q}{|y|^{\lambda+\beta}}\,dy\right)^p \int_{\mathbb{R}^n} \frac{1}{|e_1 - z|^\lambda |z|^{\alpha(p+1)}}\,dz,$$

where ω_{n-1} *denotes the surface area of the unit sphere, and* $e_1 = (1, 0, \ldots, 0)$.

(ii) *Around the infinity.*

Assume moreover that $q(\beta+\lambda)+\beta > n$. *Then, it holds that*

$$u(x) \sim \frac{B_0}{|x|^{\alpha+\lambda}} \quad \text{and} \quad v(x) \sim \begin{cases} \dfrac{B_1}{|x|^{\beta+\lambda}} & \text{if } p(\alpha+\lambda)+\alpha > n, \\ \dfrac{B_2 \log|x|}{|x|^{\beta+\lambda}} & \text{if } p(\alpha+\lambda)+\alpha = n, \\ \dfrac{B_3}{|x|^{p(\alpha+\lambda)+\alpha+\beta+\lambda-n}} & \text{if } p(\alpha+\lambda)+\alpha < n, \end{cases}$$

as $|x| \to \infty$. *Here the constants* B_0, B_1, B_2, B_3 *are given by*

$$B_0 := \int_{\mathbb{R}^n} \frac{v(y)^q}{|y|^\beta}\,dy, \quad B_1 := \int_{\mathbb{R}^n} \frac{u(y)^p}{|y|^\alpha}\,dy, \quad B_2 := \omega_{n-1}\left(\int_{\mathbb{R}^n} \frac{v(y)^q}{|y|^\beta}\,dy\right)^p,$$
$$\text{and} \quad B_3 := \left(\int_{\mathbb{R}^n} \frac{v(y)^q}{|y|^\beta}\,dy\right)^p \int_{\mathbb{R}^n} \frac{1}{|e_1 - z|^\lambda |z|^{2n-(\alpha+\lambda)(p+1)}}\,dz.$$

The radial symmetry of solutions will be proved by means of an integral form of the method of moving planes introduced by Chen, Li and Ou [5, 6]. Assuming that $p, q \geq 1$, Jin and Li [10] studied the system (1.3) for general $\alpha, \beta \geq 0$. On the other hand, Hang [8] developed the method to treat the case where either $p < 1$ or $q < 1$, and proved the symmetry of solutions for $0 < p, q < \infty$ when $\alpha = \beta = 0$. We extend their results for general $0 < p, q < \infty$ and $\alpha, \beta \geq 0$.

Theorem 1.4. *Suppose the same assumption as in Theorem 1.1. Then,* u *and* v *are smooth away from the origin, radially symmetric, and strictly decreasing in the radial direction. Moreover, the center of the symmetry must be the origin unless* $\alpha = \beta = 0$.

This paper is organized as follows. In section 2, we consider integrability of solutions. By developing a nonlinear contraction mapping technique, it is shown that solutions must belong to the Lebesgue spaces with exponents in certain ranges as stated in Theorems 1.1 and 1.2. Then, Theorem 1.3 follows as a corollary. In section 3, an integral form of the method of moving planes is used to prove Theorem 1.4. In the case where $\alpha > 0$ or $\beta > 0$, the symmetric center is shown to be the origin, since solutions have singularities at the origin.

In the following sections, C denotes a generic constant and $B_R(x)$ is the ball of radius $R > 0$ with center at $x \in \mathbb{R}^n$.

2 A priori integrability of solutions

The method we use here is based on a regularity lifting argument employed in the work of Chen, Jin, Li and Lim [2] and Jin and Li [11]. They considered the operators T_1^ρ, T_2^ρ defined by

$$T_1^\rho g(x) := \frac{1}{|x|^\alpha} \int_{\mathbb{R}^n} \frac{v(y)^{q-\rho} g(y)^\rho}{|x - y|^\lambda |y|^\beta}\,dy,$$
$$T_2^\rho f(x) := \frac{1}{|x|^\beta} \int_{\mathbb{R}^n} \frac{u(y)^{p-(1/\rho)} f(y)^{1/\rho}}{|x - y|^\lambda |y|^\alpha}\,dy,$$

with $\rho = 1$. It is easy to see that any solution u, v to the system (1.3) satisfies $T_1^\rho v = u$ and $T_2^\rho u = v$. To explain the idea of their work concisely, we assume that $\|u\|_{p+1}$, $\|v\|_{q+1}$ are sufficiently small. When $\rho = 1$, the mapping T defined by $T(f, g) := (T_1^\rho g, T_2^\rho f)$ is a linear operator from $L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n)$ into itself and it can be shown that T is a contraction mapping with the unique fixed point (u, v). Here, $L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n)$ is the product space equipped with the norm $\|(f, g)\|_{p+1,q+1} := \|f\|_{p+1} + \|g\|_{q+1}$. Moreover, T also becomes a contraction mapping from $L^r(\mathbb{R}^n) \times L^s(\mathbb{R}^n)$ into itself with r, s satisfying some conditions. As shown in [2, Theorem 1], it then turns out that a unique fixed point in the space $L^r(\mathbb{R}^n) \times L^s(\mathbb{R}^n)$ must coincide with (u, v). This implies that $u \in L^r(\mathbb{R}^n)$ and $v \in L^s(\mathbb{R}^n)$.

However, the above argument is available only when $p, q > 1$, since the reason that the mapping T becomes a contraction mapping relies on the inequalities $\|T_1^\rho g\|_r \leq C\|v\|_{q+1}^{q-\rho}\|g\|_s^\rho$ and $\|T_2^\rho f\|_s \leq C\|u\|_{p+1}^{p-(1/\rho)}\|f\|_r^{1/\rho}$, i.e., $q - \rho > 0$ and $p - (1/\rho) > 0$ are required for T to be a contraction mapping. In addition, we need to take $\rho = 1$; otherwise T is no longer a contraction mapping. This prevents us from extending the above argument to the case where either p or q is smaller than 1.

In this section we consider the composite mapping $T_1^\rho T_2^\rho$ or $T_2^\rho T_1^\rho$ instead of T with general ρ, and treat all the cases $0 < p, q < \infty$. Then, as we will demonstrate later, it can be proved that the nonlinear operator $T_1^\rho T_2^\rho$ is a contraction mapping from L^r into itself when $\rho \leq 1$ and so is $T_2^\rho T_1^\rho$ when $\rho \geq 1$ with r being in a certain range. From this fact we can obtain the integrability of either u or v, and subsequently that of the other by the equations (1.3) combined with the weighted Hardy-Littlewood-Sobolev inequality. Along this way, we prove Theorem 1.1 which is the key to obtaining the profiles of solutions to the integral system (1.3) as we will see in the next section.

We should remark that this kind of nonlinear approach was employed by Hang [8], and Hang, Wang and Yan [9] to prove the regularity and symmetry of solutions to the system (1.3) and a different system of integral equations associated with a sharp inequality for harmonic functions. Here we develop the idea to show a priori integrability of solutions.

Proof of Theorem 1.1. First observe from the equality in (1.4) that the assumption $\alpha + \beta + \lambda < n$ is equivalent to the inequality $pq > 1$, and hence there exists ρ such that $1/p < \rho < q$. In what follows, we often use a variant of the weighted Hardy-Littlewood-Sobolev inequality which states that a function w defined by

$$w(x) := \frac{1}{|x|^\alpha} \int_{\mathbb{R}^n} \frac{h(y)}{|x-y|^\lambda |y|^\beta} \, dy$$

belongs to the space $L^r(\mathbb{R}^n)$ and satisfies $\|w\|_r \leq C\|h\|_\mu$, provided that $h \in L^\mu(\mathbb{R}^n)$ with

$$\frac{1}{\mu} + \frac{\beta}{n} < 1, \quad 0 < \frac{1}{\mu} + \frac{\beta + \lambda}{n} - 1, \quad \text{and} \quad \frac{1}{r} = \frac{1}{\mu} + \frac{\alpha + \beta + \lambda}{n} - 1.$$

This follows from the inequality (1.1) and a duality argument.

Step 1. Let us derive basic inequalities together with sufficient conditions for these inequalities to hold. Applying the weighted Hardy-Littlewood-Sobolev inequality and then Hölder's inequality, we have

$$\|T_1^\rho g\|_r \leq C\|v^{q-\rho} g^\rho\|_\mu \leq C\|v\|_{q+1}^{q-\rho}\|g\|_s^\rho \tag{2.1}$$

for $g \in L^s(\mathbb{R}^n)$, provided that $r, s \geq 1$ satisfy

$$\frac{1}{\mu} := \frac{q-\rho}{q+1} + \frac{\rho}{s}, \ \frac{1}{\mu} + \frac{\beta}{n} < 1, \ 0 < \frac{1}{\mu} + \frac{\beta+\lambda}{n} - 1, \ \text{and} \ \frac{1}{r} = \frac{1}{\mu} + \frac{\alpha+\beta+\lambda}{n} - 1. \tag{2.2}$$

Similarly, we see that

$$\|T_2^\rho f\|_s \leq C\|u^{p-(1/\rho)} f^{1/\rho}\|_\nu \leq C\|u\|_{p+1}^{p-(1/\rho)} \|f\|_r^{1/\rho} \tag{2.3}$$

for $f \in L^r(\mathbb{R}^n)$, provided that $r, s \geq 1$ satisfy

$$\begin{aligned} &\frac{1}{\nu} := \frac{p-(1/\rho)}{p+1} + \frac{1/\rho}{r}, \ \frac{1}{\nu} + \frac{\alpha}{n} < 1, \ 0 < \frac{1}{\nu} + \frac{\alpha+\lambda}{n} - 1, \\ &\text{and} \ \frac{1}{s} = \frac{1}{\nu} + \frac{\alpha+\beta+\lambda}{n} - 1. \end{aligned} \tag{2.4}$$

Note that, in view of (1.4), the last equalities in (2.2) and (2.4) are equivalent to each other. Moreover, we see that r, $s \geq 1$ satisfy the conditions (2.2) and (2.4) if and only if

$$\frac{\alpha}{n} < \frac{1}{r} < \frac{\alpha+\lambda}{n}, \ \ \frac{\beta}{n} < \frac{1}{s} < \frac{\beta+\lambda}{n} \ \ \text{and} \ \ \frac{1}{r} - \frac{1}{p+1} = \rho\left(\frac{1}{s} - \frac{1}{q+1}\right). \tag{2.5}$$

From (2.5) we derive the following single condition for s:

$$\begin{aligned} &\max\left\{\frac{1}{\rho}\left(\frac{\alpha}{n} - \frac{1}{p+1}\right) + \frac{1}{q+1}, \ \frac{\beta}{n}\right\} \\ &\qquad < \frac{1}{s} < \min\left\{\frac{1}{\rho}\left(\frac{\alpha+\lambda}{n} - \frac{1}{p+1}\right) + \frac{1}{q+1}, \ \frac{\beta+\lambda}{n}\right\}. \end{aligned} \tag{2.6}$$

This means that, for any given s satisfying (2.6), we can take r so that the condition (2.5) holds. Similarly, we have the following single condition for r:

$$\begin{aligned} &\max\left\{\rho\left(\frac{\beta}{n} - \frac{1}{q+1}\right) + \frac{1}{p+1}, \ \frac{\alpha}{n}\right\} \\ &\qquad < \frac{1}{r} < \min\left\{\rho\left(\frac{\beta+\lambda}{n} - \frac{1}{q+1}\right) + \frac{1}{p+1}, \ \frac{\alpha+\lambda}{n}\right\}. \end{aligned} \tag{2.7}$$

Step 2. Here we show that, depending on the value of ρ, $u \in L^r(\mathbb{R}^n)$ or $v \in L^s(\mathbb{R}^n)$ holds for r, s satisfying (2.6) and (2.7). To handle even the case where $\|u\|_{p+1}$ or $\|v\|_{q+1}$ is not small, we consider the following operators $T_1^{\rho,A}$, $T_2^{\rho,A}$ instead of T_1^ρ, T_2^ρ:

$$\begin{aligned} T_1^{\rho,A} g(x) &:= \frac{1}{|x|^\alpha} \int_{\mathbb{R}^n} \frac{v_A(y)^{q-\rho} g(y)^\rho}{|x-y|^\lambda |y|^\beta}\, dy + \frac{1}{|x|^\alpha} \int_{\mathbb{R}^n} \frac{(v(y) - v_A(y))^q}{|x-y|^\lambda |y|^\beta}\, dy, \\ T_2^{\rho,A} f(x) &:= \frac{1}{|x|^\beta} \int_{\mathbb{R}^n} \frac{u_A(y)^{p-(1/\rho)} f(y)^{1/\rho}}{|x-y|^\lambda |y|^\alpha}\, dy + \frac{1}{|x|^\beta} \int_{\mathbb{R}^n} \frac{(u(y) - u_A(y))^p}{|x-y|^\lambda |y|^\alpha}\, dy, \end{aligned}$$

where u_A and v_A are defined by

$$u_A(x) := \begin{cases} u(x) & \text{when } |x| \geq A \text{ or } |u(x)| \geq A, \\ 0 & \text{otherwise,} \end{cases}$$
$$v_A(x) := \begin{cases} v(x) & \text{when } |x| \geq A \text{ or } |v(x)| \geq A, \\ 0 & \text{otherwise.} \end{cases}$$

Then, it is easy to see that $T_2^{\rho,A}T_1^{\rho,A}v = v$ and $T_1^{\rho,A}T_2^{\rho,A}u = u$.

Let us prove that, when $\rho \geq 1$, the mapping $T_2^{\rho,A}T_1^{\rho,A}$ becomes a contraction by taking A to be sufficiently large. By the simple fact that $(a+c)^{1/\rho} - (b+c)^{1/\rho} \leq a^{1/\rho} - b^{1/\rho}$ for $a \geq b \geq 0$, $c \geq 0$ and the Minkowski inequality, we see that

$$\left| \left(T_1^{\rho,A}g_1(x)\right)^{1/\rho} - \left(T_1^{\rho,A}g_2(x)\right)^{1/\rho} \right| \leq \left(\frac{1}{|x|^\alpha} \int_{\mathbb{R}^n} \frac{v_A(y)^{q-\rho}|g_1(y) - g_2(y)|^\rho}{|x-y|^\lambda |y|^\beta} dy \right)^{1/\rho}.$$

In view of the inequalities (2.1) and (2.3), it then follows that

$$\begin{aligned} \|T_2^{\rho,A}T_1^{\rho,A}g_1 - T_2^{\rho,A}T_1^{\rho,A}g_2\|_s &\leq C\|u_A\|_{p+1}^{p-(1/\rho)}\|v_A\|_{q+1}^{(q/\rho)-1}\|g_1 - g_2\|_s \\ &\leq \frac{1}{2}\|g_1 - g_2\|_s \end{aligned} \tag{2.8}$$

for s satisfying the condition (2.6). Here the last inequality holds if A is taken to be sufficiently large. Therefore, for such a number s, $T_2^{\rho,A}T_1^{\rho,A}$ becomes a contraction mapping from $L^s(\mathbb{R}^n)$ into itself. In particular, $s = q+1$ satisfies (2.6), and hence we deduce that $v \in L^s(\mathbb{R}^n)$ (see [2, Theorem 1]). Similarly, it can be shown that, if $\rho \leq 1$, then $T_1^{\rho,A}T_2^{\rho,A}$ becomes a contraction mapping from $L^r(\mathbb{R}^n)$ into itself for large A, so that $u \in L^r(\mathbb{R}^n)$ for r satisfying (2.7).

Step 3. We are now in a position to complete the proof by taking an appropriate number ρ. We may assume that $q \geq p$ and hence $q > 1$ without loss of generality. Although the case where $p > 1$ was already treated in the paper [2], we also give the proof of this case for the sake of completeness.

Let us first consider the case where $p \leq 1$. Since $1 \leq 1/p < \rho < q$, we use the contraction mapping $T_2^{\rho,A}T_1^{\rho,A}$. In view of (2.6), ρ should be taken as small as possible to obtain the maximal integrability of v, i.e., $\rho \to 1/p$. Then, we see that $v \in L^s(\mathbb{R}^n)$ for

$$\begin{aligned} \max\left\{p\left(\frac{\alpha}{n} - \frac{1}{p+1}\right) + \frac{1}{q+1}, \frac{\beta}{n}\right\} & \\ < \frac{1}{s} < \min\left\{p\left(\frac{\alpha+\lambda}{n} - \frac{1}{p+1}\right) + \frac{1}{q+1}, \frac{\beta+\lambda}{n}\right\}&. \end{aligned} \tag{2.9}$$

This is equivalent to the condition (1.9). Moreover, with this integrability of v, it follows from the first equation in (1.3) and the weighted Hardy-Littlewood-Sobolev inequality that $u \in L^r(\mathbb{R}^n)$ for

$$\frac{1}{r} = \frac{q}{s} + \frac{\alpha+\beta+\lambda}{n} - 1, \tag{2.10}$$

where s satisfies (2.9),

$$\frac{q}{s}+\frac{\beta}{n}<1 \quad\text{and}\quad 0<\frac{q}{s}+\frac{\beta+\lambda}{n}-1.$$

Here, these three conditions for s can be represented by

$$\max\left\{\frac{1}{q}\left(1-\frac{\beta+\lambda}{n}\right),\ \frac{\beta}{n}\right\}<\frac{1}{s}<\min\left\{\frac{1}{q}\left(1-\frac{\beta}{n}\right),\ \frac{\beta+\lambda}{n}\right\}, \tag{2.11}$$

since we see from $pq>1$ that

$$p\left(\frac{\alpha}{n}-\frac{1}{p+1}\right)+\frac{1}{q+1}<\frac{1}{q}\left(1-\frac{\beta+\lambda}{n}\right)$$
$$\text{and}\quad \frac{1}{q}\left(1-\frac{\beta}{n}\right)<p\left(\frac{\alpha+\lambda}{n}-\frac{1}{p+1}\right)+\frac{1}{q+1}.$$

Therefore, by (2.10) and (2.11), we deduce that $u\in L^r(\mathbb{R}^n)$ for

$$\max\left\{\frac{\alpha}{n},\ \frac{q\beta+\alpha+\beta+\lambda}{n}-1\right\}<\frac{1}{r}<\min\left\{\frac{\alpha+\lambda}{n},\ \frac{q(\beta+\lambda)+\alpha+\beta+\lambda}{n}-1\right\}.$$

This completes the proof for the case where $p\le 1$.

Next we turn to the case where $p>1$. Then, we have two possible choices of ρ. Let us take ρ such that $1/p<1\le\rho<q$, and consider the contraction mapping $T_2^{\rho,A}T_1^{\rho,A}$. As in the previous case, taking ρ as small as possible, i.e., $\rho=1$, we see that $v\in L^s(\mathbb{R}^n)$ for

$$\max\left\{\frac{\alpha}{n}-\frac{1}{p+1}+\frac{1}{q+1},\ \frac{\beta}{n}\right\}<\frac{1}{s}<\min\left\{\frac{\alpha+\lambda}{n}-\frac{1}{p+1}+\frac{1}{q+1},\ \frac{\beta+\lambda}{n}\right\}. \tag{2.12}$$

Consequently, it follows from the first equation in (1.3) that $u\in L^r(\mathbb{R}^n)$ for

$$\frac{1}{r}=\frac{q}{s}+\frac{\alpha+\beta+\lambda}{n}-1,$$

where s satisfies the condition (2.12),

$$\frac{q}{s}+\frac{\beta}{n}<1 \quad\text{and}\quad 0<\frac{q}{s}+\frac{\beta+\lambda}{n}-1.$$

This again implies the desired integrability interval (1.8) of u. Hence, what is left to do is to prove the integrability of v as (1.9). To this end, we take ρ such that $1/p<\rho\le 1<q$, and consider the contraction mapping $T_1^{\rho,A}T_2^{\rho,A}$. In view of (2.7), we take ρ as large as possible to obtain the maximal integrability of u, i.e., $\rho=1$. Then, we see that $u\in L^r(\mathbb{R}^n)$ for

$$\max\left\{\frac{\beta}{n}-\frac{1}{q+1}+\frac{1}{p+1},\ \frac{\alpha}{n}\right\}<\frac{1}{r}<\min\left\{\frac{\beta+\lambda}{n}-\frac{1}{q+1}+\frac{1}{p+1},\ \frac{\alpha+\lambda}{n}\right\}. \tag{2.13}$$

Consequently, it follows from the second equation in (1.3) that $v \in L^s(\mathbb{R}^n)$ for

$$\frac{1}{s} = \frac{p}{r} + \frac{\alpha+\beta+\lambda}{n} - 1,$$

where r satisfies the condition (2.13),

$$\frac{p}{r} + \frac{\alpha}{n} < 1 \quad \text{and} \quad 0 < \frac{p}{r} + \frac{\alpha+\lambda}{n} - 1.$$

This implies that $v \in L^s(\mathbb{R}^n)$ holds for

$$\max\left\{\frac{\beta}{n}, \frac{p\alpha+\alpha+\beta+\lambda}{n} - 1\right\} < \frac{1}{s} < \min\left\{\frac{\beta+\lambda}{n}, \frac{p(\alpha+\lambda)+\alpha+\beta+\lambda}{n} - 1\right\},$$

as required. □

Remark 2.1. In the remaining case $\alpha+\beta+\lambda = n$, i.e., $pq = 1$, since the last inequality in (2.8) fails to hold, the regularity lifting argument does not work. As pointed out by Lieb [16, p. 369], we cannot expect the existence of maximizers for the variational problem in this case.

Theorem 1.2 can be proved by an analogous way. However, we need to be careful with each calculation since the condition $\beta < 0$ requires slight modifications.

Proof of Theorem 1.2. Let us take ρ such that $1/p < \rho < q$ as in the proof of Theorem 1.1. Since $\beta < 0$, we need an additional condition $\mu > 1$ as well as (2.2) and (2.4) so that the inequalities (2.1) and (2.3) hold. We put the conditions $\mu > 1$, (2.2) and (2.4) together to obtain

$$\frac{\alpha}{n} < \frac{1}{r} < \frac{\alpha+\beta+\lambda}{n}, \quad 0 < \frac{1}{s} < \frac{\beta+\lambda}{n} \quad \text{and} \quad \frac{1}{r} - \frac{1}{p+1} = \rho\left(\frac{1}{s} - \frac{1}{q+1}\right). \tag{2.14}$$

The condition (2.14) yields the following single condition for s:

$$\begin{aligned}\max&\left\{\frac{1}{\rho}\left(\frac{\alpha}{n} - \frac{1}{p+1}\right) + \frac{1}{q+1}, 0\right\} \\ &< \frac{1}{s} < \min\left\{\frac{1}{\rho}\left(\frac{\alpha+\beta+\lambda}{n} - \frac{1}{p+1}\right) + \frac{1}{q+1}, \frac{\beta+\lambda}{n}\right\}.\end{aligned} \tag{2.15}$$

Similarly, we have the following single condition for r:

$$\max\left\{-\frac{\rho}{q+1} + \frac{1}{p+1}, \frac{\alpha}{n}\right\} < \frac{1}{r} < \min\left\{\rho\left(\frac{\beta+\lambda}{n} - \frac{1}{q+1}\right) + \frac{1}{p+1}, \frac{\alpha+\beta+\lambda}{n}\right\}. \tag{2.16}$$

Then, as in the step 2 of the proof of Theorem 1.1, we see that $v \in L^s(\mathbb{R}^n)$ holds for s satisfying (2.15) when $\rho \geq 1$, and that $u \in L^r(\mathbb{R}^n)$ holds for r satisfying (2.16) when $\rho \leq 1$.

The next step is to choose an appropriate number ρ to obtain the desired integrability of u and v. We divide the proof into three cases. First let us consider the case where $1 \le 1/p < q$. Then, in view of the condition (2.15), by taking $\rho \to 1/p$, we see that $v \in L^s(\mathbb{R}^n)$ for

$$\begin{aligned} \max\left\{p\left(\frac{\alpha}{n} - \frac{1}{p+1}\right) + \frac{1}{q+1},\ 0\right\} \\ < \frac{1}{s} < \min\left\{p\left(\frac{\alpha+\beta+\lambda}{n} - \frac{1}{p+1}\right) + \frac{1}{q+1},\ \frac{\beta+\lambda}{n}\right\}. \end{aligned} \tag{2.17}$$

This is equivalent to the condition (1.11). Moreover, with this integrability of v, it follows from the first equation in (1.3) and the weighted Hardy-Littlewood-Sobolev inequality that $u \in L^r(\mathbb{R}^n)$ for

$$\frac{1}{r} = \frac{q}{s} + \frac{\alpha+\beta+\lambda}{n} - 1, \tag{2.18}$$

where s satisfies (2.17),

$$\frac{q}{s} < 1 \quad \text{and} \quad 0 < \frac{q}{s} + \frac{\beta+\lambda}{n} - 1.$$

Here, these three conditions for s can be represented by

$$\frac{1}{q}\left(1 - \frac{\beta+\lambda}{n}\right) < \frac{1}{s} < \min\left\{\frac{1}{q},\ \frac{\beta+\lambda}{n}\right\}. \tag{2.19}$$

Therefore, by (2.18) and (2.19), we deduce that $u \in L^r(\mathbb{R}^n)$ for

$$\frac{\alpha}{n} < \frac{1}{r} < \min\left\{\frac{\alpha+\beta+\lambda}{n},\ \frac{q(\beta+\lambda)+\alpha+\beta+\lambda}{n} - 1\right\}.$$

This completes the proof for the case where $1 \le 1/p < q$.

Next we consider the case where $1/p < 1 < q$. Let us take ρ such that $1/p < 1 \le \rho < q$. Then, by taking $\rho = 1$ in (2.15), we see that $v \in L^s(\mathbb{R}^n)$ for

$$\max\left\{\frac{\alpha}{n} - \frac{1}{p+1} + \frac{1}{q+1},\ 0\right\} < \frac{1}{s} < \min\left\{\frac{\alpha+\beta+\lambda}{n} - \frac{1}{p+1} + \frac{1}{q+1},\ \frac{\beta+\lambda}{n}\right\}. \tag{2.20}$$

Consequently, from the first equation in (1.3) and the weighted Hardy-Littlewood-Sobolev inequality it follows that $u \in L^r(\mathbb{R}^n)$ for

$$\frac{1}{r} = \frac{q}{s} + \frac{\alpha+\beta+\lambda}{n} - 1,$$

where s satisfies the condition (2.20),

$$\frac{q}{s} < 1 \quad \text{and} \quad 0 < \frac{q}{s} + \frac{\beta+\lambda}{n} - 1.$$

This implies the desired integrability interval (1.10) of u. To prove the integrability of v as (1.11), we use this integrability of u. It follows from the second equation in (1.3) that $v \in L^s(\mathbb{R}^n)$ for

$$0 < \frac{1}{s} = \frac{p}{r} + \frac{\alpha+\beta+\lambda}{n} - 1,$$

where r satisfies the condition (1.10),

$$\frac{p}{r} + \frac{\alpha}{n} < 1 \quad \text{and} \quad 0 < \frac{p}{r} + \frac{\alpha+\lambda}{n} - 1.$$

This implies that

$$\max\left\{0, \ \frac{p\alpha+\alpha+\beta+\lambda}{n} - 1\right\} < \frac{1}{s} < \min\left\{\frac{\beta+\lambda}{n}, \ \frac{(p+1)(\alpha+\beta+\lambda)}{n} - 1\right\},$$

as required.

We now deal with the last case $1/p < q \leq 1$. In view of the condition (2.16), by taking $\rho \to q$, we see that $u \in L^r(\mathbb{R}^n)$ for

$$\frac{\alpha}{n} < \frac{1}{r} < \min\left\{q\left(\frac{\beta+\lambda}{n} - \frac{1}{q+1}\right) + \frac{1}{p+1}, \ \frac{\alpha+\beta+\lambda}{n}\right\}. \tag{2.21}$$

This is equivalent to the condition (1.10). Moreover, with this integrability of u, it follows from the second equation in (1.3) that $v \in L^s(\mathbb{R}^n)$ for

$$0 < \frac{1}{s} = \frac{p}{r} + \frac{\alpha+\beta+\lambda}{n} - 1,$$

where r satisfies the condition (2.21),

$$\frac{p}{r} + \frac{\alpha}{n} < 1 \quad \text{and} \quad 0 < \frac{p}{r} + \frac{\alpha+\lambda}{n} - 1.$$

This implies that $v \in L^s(\mathbb{R}^n)$ for

$$\max\left\{0, \ \frac{p\alpha+\alpha+\beta+\lambda}{n} - 1\right\} < \frac{1}{s} < \min\left\{\frac{\beta+\lambda}{n}, \ \frac{(p+1)(\alpha+\beta+\lambda)}{n} - 1\right\}.$$

This completes the proof. □

Employing the a priori integrability of solutions obtained in Theorems 1.1 and 1.2, the profiles of solutions to the system (1.3) around the origin and the infinity as stated in Theorem 1.3 can be proved. In fact, an analysis similar to the one by Li and Lim [14], in which the case where $p, q \geq 1$, $pq \neq 1$ was treated, works also for our cases. We should remark that, if either α or β is negative, one needs more elaborate technique to obtain the result. Recently, Lei, Li and Ma [12] investigated this matter, and their argument directly applies to our case with the aid of Theorem 1.2. For this reason, we omit the proof of Theorem 1.3.

3 Radial symmetry of solutions

Here we discuss the radial symmetry of solutions to the system (1.3). Before we proceed to the proof of Theorem 1.4, we remark that the solutions are smooth away from the origin. This can be proved by the standard bootstrap argument (see Chen and Li [4], and Hang [8]). In particular, the continuity of solutions will be needed when we employ an integral form of the method of moving planes.

In the following proof, we assume $\alpha > 0$ or $\beta > 0$, since the case where $\alpha = \beta = 0$ was already studied by Hang [8].

Proof of Theorem 1.4. We may assume $q > p$ without loss of generality. Then, let us choose $\rho > 1$ so that $1/p < \rho < q$. For $\tau \in \mathbb{R}$, we define a half space $H_\tau := \{x = (x_1, x') \in \mathbb{R}^n \mid x_1 < \tau\}$ and the reflection point $x_\tau := (2\tau - x_1, x')$ of x. We also define $u_\tau(x) := u(x_\tau)$, $v_\tau(x) := v(x_\tau)$,

$$\Omega_\tau^u := \{x \in H_\tau \mid u_\tau(x) > u(x)\},$$
$$\Omega_\tau^v := \{x \in H_\tau \mid v_\tau(x) > v(x)\}.$$

Step 1. Let us take arbitrary $\tau \geq 0$ and $x \in \Omega_\tau^v$. By changing of variables, we see that

$$\begin{aligned}
v(x) &= \frac{1}{|x|^\beta}\int_{H_\tau} \frac{u(y)^p}{|x-y|^\lambda |y|^\alpha}\,dy + \frac{1}{|x|^\beta}\int_{H_\tau} \frac{u(y_\tau)^p}{|x_\tau - y|^\lambda |y_\tau|^\alpha}\,dy \\
&\geq \frac{1}{|x|^\beta}\int_{H_\tau} \frac{u(y)^p}{|x-y|^\lambda |y|^\alpha}\,dy + \frac{1}{|x_\tau|^\beta}\int_{H_\tau} \frac{u(y_\tau)^p}{|x_\tau - y|^\lambda |y_\tau|^\alpha}\,dy, \\
v_\tau(x) &= \frac{1}{|x_\tau|^\beta}\int_{H_\tau} \frac{u(y_\tau)^p}{|x-y|^\lambda |y_\tau|^\alpha}\,dy + \frac{1}{|x_\tau|^\beta}\int_{H_\tau} \frac{u(y)^p}{|x_\tau - y|^\lambda |y|^\alpha}\,dy \\
&\leq \frac{1}{|x_\tau|^\beta}\int_{H_\tau} \frac{u(y_\tau)^p}{|x-y|^\lambda |y_\tau|^\alpha}\,dy + \frac{1}{|x|^\beta}\int_{H_\tau} \frac{u(y)^p}{|x_\tau - y|^\lambda |y|^\alpha}\,dy.
\end{aligned}$$

Hence,

$$\begin{aligned}
0 &\leq v_\tau(x) - v(x) \\
&\leq \int_{H_\tau} \left(\frac{1}{|x-y|^\lambda} - \frac{1}{|x_\tau - y|^\lambda}\right)\left(\frac{u(y_\tau)^p}{|x_\tau|^\beta |y_\tau|^\alpha} - \frac{u(y)^p}{|x|^\beta |y|^\alpha}\right) dy \\
&\leq \int_{\Omega_\tau^u} \left(\frac{1}{|x-y|^\lambda} - \frac{1}{|x_\tau - y|^\lambda}\right)\left(\frac{u(y_\tau)^p}{|x|^\beta |y|^\alpha} - \frac{u(y)^p}{|x|^\beta |y|^\alpha}\right) dy \\
&\leq \int_{\Omega_\tau^u} \frac{1}{|x|^\beta |x-y|^\lambda |y|^\alpha}\left(\left(u(y_\tau)^{1/\rho}\right)^{p\rho} - \left(u(y)^{1/\rho}\right)^{p\rho}\right) dy \\
&\leq p\rho \int_{\Omega_\tau^u} \frac{u_\tau(y)^{p-(1/\rho)}}{|x|^\beta |x-y|^\lambda |y|^\alpha}\left(u_\tau(y)^{1/\rho} - u(y)^{1/\rho}\right) dy.
\end{aligned}$$

Consequently, by applying the weighted Hardy-Littlewood-Sobolev inequality and then Hölder's inequality, we see that

$$\begin{aligned}\|v_\tau - v\|_{q+1,\Omega_\tau^v} &\le C \left\|u_\tau^{p-(1/\rho)}\left(u_\tau^{1/\rho} - u^{1/\rho}\right)\right\|_{\frac{p+1}{p},\Omega_\tau^u} \\ &\le C \|u_\tau\|_{p+1,\Omega_\tau^u}^{p-(1/\rho)} \left\|u_\tau^{1/\rho} - u^{1/\rho}\right\|_{\rho(p+1),\Omega_\tau^u}.\end{aligned} \tag{3.1}$$

Now let us estimate the right hand side of (3.1). For $\tau \ge 0$ and $x \in \Omega_\tau^u$, we have

$$\begin{aligned}u(x) &\ge \frac{1}{|x|^\alpha}\int_{H_\tau}\frac{v(y)^q}{|x-y|^\lambda|y|^\beta}\,dy + \frac{1}{|x_\tau|^\alpha}\int_{H_\tau}\frac{v(y_\tau)^q}{|x_\tau-y|^\lambda|y_\tau|^\beta}\,dy \\ &\ge \frac{1}{|x|^\alpha}\int_{\Omega_\tau^v}\frac{v(y)^q}{|x-y|^\lambda|y|^\beta}\,dy + \frac{1}{|x_\tau|^\alpha}\int_{\Omega_\tau^v}\frac{v(y_\tau)^q}{|x_\tau-y|^\lambda|y_\tau|^\beta}\,dy \\ &\quad + \frac{1}{|x|^\alpha}\int_{H_\tau\setminus\Omega_\tau^v}\frac{v(y)^q}{|x_\tau-y|^\lambda|y|^\beta}\,dy + \frac{1}{|x_\tau|^\alpha}\int_{H_\tau\setminus\Omega_\tau^v}\frac{v(y_\tau)^q}{|x-y|^\lambda|y_\tau|^\beta}\,dy, \\ u_\tau(x) &\le \frac{1}{|x_\tau|^\alpha}\int_{H_\tau}\frac{v(y_\tau)^q}{|x-y|^\lambda|y_\tau|^\beta}\,dy + \frac{1}{|x|^\alpha}\int_{H_\tau}\frac{v(y)^q}{|x_\tau-y|^\lambda|y|^\beta}\,dy \\ &\le \frac{1}{|x|^\alpha}\int_{\Omega_\tau^v}\frac{v(y_\tau)^q}{|x-y|^\lambda|y|^\beta}\,dy + \frac{1}{|x_\tau|^\alpha}\int_{\Omega_\tau^v}\frac{v(y)^q}{|x_\tau-y|^\lambda|y_\tau|^\beta}\,dy \\ &\quad + \frac{1}{|x_\tau|^\alpha}\int_{H_\tau\setminus\Omega_\tau^v}\frac{v(y_\tau)^q}{|x-y|^\lambda|y_\tau|^\beta}\,dy + \frac{1}{|x|^\alpha}\int_{H_\tau\setminus\Omega_\tau^v}\frac{v(y)^q}{|x_\tau-y|^\lambda|y|^\beta}\,dy,\end{aligned}$$

and therefore from the inequality $(a+c)^{1/\rho} - (b+c)^{1/\rho} \le a^{1/\rho} - b^{1/\rho}$ for $a \ge b \ge 0$, $c \ge 0$ and the Minkowski inequality it follows that

$$\begin{aligned}0 &\le u_\tau(x)^{1/\rho} - u(x)^{1/\rho} \\ &\le \left(\frac{1}{|x|^\alpha}\int_{\Omega_\tau^v}\frac{v(y_\tau)^q}{|x-y|^\lambda|y|^\beta}\,dy + \frac{1}{|x_\tau|^\alpha}\int_{\Omega_\tau^v}\frac{v(y)^q}{|x_\tau-y|^\lambda|y_\tau|^\beta}\,dy\right)^{1/\rho} \\ &\quad - \left(\frac{1}{|x|^\alpha}\int_{\Omega_\tau^v}\frac{v(y)^q}{|x-y|^\lambda|y|^\beta}\,dy + \frac{1}{|x_\tau|^\alpha}\int_{\Omega_\tau^v}\frac{v(y_\tau)^q}{|x_\tau-y|^\lambda|y_\tau|^\beta}\,dy\right)^{1/\rho} \\ &\le \left(\int_{\Omega_\tau^v}\left(\frac{v(y_\tau)^{q/\rho} - v(y)^{q/\rho}}{|x|^{\alpha/\rho}|x-y|^{\lambda/\rho}|y|^{\beta/\rho}}\right)^\rho dy\right. \\ &\qquad\qquad \left. + \int_{\Omega_\tau^v}\left(\frac{v(y_\tau)^{q/\rho} - v(y)^{q/\rho}}{|x_\tau|^{\alpha/\rho}|x_\tau-y|^{\lambda/\rho}|y_\tau|^{\beta/\tau}}\right)^\rho dy\right)^{1/\rho} \\ &\le 2^{1/\rho}\left(\int_{\Omega_\tau^v}\frac{\left(v(y_\tau)^{q/\rho} - v(y)^{q/\rho}\right)^\rho}{|x|^\alpha|x-y|^\lambda|y|^\beta}\,dy\right)^{1/\rho} \\ &\le \frac{2^{1/\rho}q}{\rho}\left(\int_{\Omega_\tau^v}\frac{v(y_\tau)^{q-\rho}\left(v(y_\tau)-v(y)\right)^\rho}{|x|^\alpha|x-y|^\lambda|y|^\beta}\,dy\right)^{1/\rho}\end{aligned}$$

Consequently, by the weighted Hardy-Littlewood-Sobolev inequality and Hölder's inequality, we see that

$$\left\|u_\tau^{1/\rho} - u^{1/\rho}\right\|_{\rho(p+1),\,\Omega_\tau^u} \le C \left\|v_\tau\right\|_{q+1,\,\Omega_\tau^v}^{(q/\rho)-1} \left\|v_\tau - v\right\|_{q+1,\,\Omega_\tau^v}. \tag{3.2}$$

Combining the inequalities (3.1) and (3.2) yields

$$\left\|v_\tau - v\right\|_{q+1,\,\Omega_\tau^v} \le C \left\|u_\tau\right\|_{p+1,\,\Omega_\tau^u}^{p-(1/\rho)} \left\|v_\tau\right\|_{q+1,\,\Omega_\tau^v}^{(q/\rho)-1} \left\|v_\tau - v\right\|_{q+1,\,\Omega_\tau^v}. \tag{3.3}$$

Step 2. We are now in a position to move a moving plane from $x_1 = +\infty$ to the left. By the inequality (3.3), let us show that $\Omega_\tau^v = \emptyset$ for large $\tau \ge 0$. Indeed, by observing

$$\begin{aligned}\left\|u_\tau\right\|_{p+1,\,\Omega_\tau^u} &\le \left\|u\right\|_{p+1,\,\mathbb{R}^n\setminus H_\tau} \to 0 \quad \text{as } \tau \to +\infty,\\ \left\|v_\tau\right\|_{q+1,\,\Omega_\tau^v} &\le \left\|v\right\|_{q+1,\,\mathbb{R}^n\setminus H_\tau} \to 0 \quad \text{as } \tau \to +\infty,\end{aligned}$$

we can deduce that

$$\left\|v_\tau - v\right\|_{q+1,\,\Omega_\tau^v} \le \frac{1}{2}\left\|v_\tau - v\right\|_{q+1,\,\Omega_\tau^v}$$

for sufficiently large $\tau \ge 0$. This implies that $\Omega_\tau^v = \emptyset$.

Now by defining $\tau_0 := \inf\{\tau \ge 0 \mid \Omega_\sigma^v = \emptyset \text{ for } \sigma \ge \tau\}$, we will show that $\tau_0 = 0$. Let us suppose that $\tau_0 > 0$. Then, by definition, we have $v_{\tau_0}(x) \le v(x)$ for $x \in H_{\tau_0}$. we can say moreover that $v_{\tau_0} = v$. This can be confirmed by assuming $v_{\tau_0} \ne v$ and deriving a contradiction. Indeed, for $x \in H_{\tau_0}$, it follows from the inequalities

$$\begin{aligned}u(x) - u_{\tau_0}(x) &\ge \int_{H_{\tau_0}} \left(\frac{1}{|x-y|^\lambda} - \frac{1}{|x_{\tau_0}-y|^\lambda}\right)\frac{v(y)^q - v_{\tau_0}(y)^q}{|x|^\alpha |y|^\beta}\,dy > 0,\\ v(x) - v_{\tau_0}(x) &\ge \int_{H_{\tau_0}} \left(\frac{1}{|x-y|^\lambda} - \frac{1}{|x_{\tau_0}-y|^\lambda}\right)\frac{u(y)^p - u_{\tau_0}(y)^p}{|x|^\beta |y|^\alpha}\,dy > 0\end{aligned}$$

that $v_{\tau_0}(x) < v(x)$. This and the continuity of v imply that

$$\left\|v_\tau\right\|_{q+1,\,\Omega_\tau^v}^{q+1} = \int_{\mathbb{R}^n} |v(x)|^{q+1}\chi_{\Omega_\tau^v}(x_\tau)\,dx \to 0 \quad \text{as } \tau \to \tau_0,$$

since $\chi_{\Omega_\tau^v}(x_\tau) \to 0$ as $\tau \to \tau_0$ for each $x \in \mathbb{R}^n \setminus \{x_1 = \tau_0\}$. Therefore, in view of (3.3), there exists a small number $\varepsilon > 0$ such that $\Omega_\sigma^v = \emptyset$ for $\sigma \ge \tau_0 - \varepsilon$. This is a contradiction. Consequently, $v_{\tau_0} = v$, and hence $u_{\tau_0} = u$. However, this symmetry implies that u and v do not have singularities at the origin. By Theorem 1.3, this is impossible unless $\alpha = \beta = 0$. Therefore, we deduce that $\tau_0 = 0$ as required. We can repeat the above procedure in all directions, so that u and v must be radially symmetric with respect to the origin and strictly decreasing in the radial direction. $\square$

References

[1] Caffarelli, Luis A.; Gidas, Basilis; Spruck, Joel, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. *Comm. Pure Appl. Math.* **42** (1989), *no. 3*, 271–297.

[2] Chen, Wenxiong; Jin, Chao; Li, Congming; Lim, Jisun, Weighted Hardy-Littlewood-Sobolev inequalities and systems of integral equations. *Discrete Contin. Dyn. Syst.* **2005**, *suppl.*, 164–172.

[3] Chen, Wenxiong; Li, Congming, A priori estimates for prescribing scalar curvature equations. *Ann. of Math. (2)* **145** (1997), *no. 3*, 547–564.

[4] Chen, Wenxiong; Li, Congming, Regularity of solutions for a system of integral equations. *Commun. Pure Appl. Anal.* **4** (2005), *no. 1*, 1–8.

[5] Chen, Wenxiong; Li, Congming; Ou, Biao, Classification of solutions for a system of integral equations. *Comm. Partial Differential Equations* **30** (2005), *no. 1–3*, 59–65.

[6] Chen, Wenxiong; Li, Congming; Ou, Biao, Classification of solutions for an integral equation. *Comm. Pure Appl. Math.* **59** (2006), *no. 3*, 330–343.

[7] Gidas, B.; Ni, Wei Ming; Nirenberg, L., Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$. *Mathematical analysis and applications, Part A, pp. 369–402, Adv. in Math. Suppl. Stud., 7a, Academic Press, New York-London,* 1981.

[8] Hang, Fengbo, On the integral systems related to Hardy-Littlewood-Sobolev inequality. *Math. Res. Lett.* **14** (2007), *no. 3*, 373–383.

[9] Hang, Fengbo; Wang, Xiaodong; Yan, Xiaodong, Sharp integral inequalities for harmonic functions. *Comm. Pure Appl. Math.* **61** (2008), *no. 1*, 54–95.

[10] Jin, Chao; Li, Congming, Symmetry of solutions to some systems of integral equations. *Proc. Amer. Math. Soc.* **134** (2006), *no. 6*, 1661–1670.

[11] Jin, Chao; Li, Congming, Quantitative analysis of some system of integral equations. *Calc. Var. Partial Differential Equations* **26** (2006), *no. 4*, 447–457.

[12] Lei, Yutian; Li, Congming; Ma, Chao, Asymptotic radial symmetry and growth estimates of positive solutions to weighted Hardy-Littlewood-Sobolev system of integral equations. *Calc. Var. Partial Differential Equations*, in press.

[13] Li, Congming, Local asymptotic symmetry of singular solutions to nonlinear elliptic equations. *Invent. Math.* **123** (1996), *no. 2*, 221–231.

[14] Li, Congming; Lim, Jisun, The singularity analysis of solutions to some integral equations. *Commun. Pure Appl. Anal.* **6** (2007), *no. 2*, 453–464.

[15] Li, Yan Yan, Remark on some conformally invariant integral equations: the method of moving spheres. *J. Eur. Math. Soc.* **6** (2004), *no. 2*, 153–180.

[16] Lieb, Elliott H., Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. *Ann. of Math. (2)* **118** (1983), *no. 2*, 349–374.

[17] Stein, E. M.; Weiss, Guido, Fractional integrals on n-dimensional Euclidean space. *J. Math. Mech.* **7** (1958) 503–514.

[18] Wei, Juncheng; Xu, Xingwang, Classification of solutions of higher order conformally invariant equations. *Math. Ann.* **313** (1999), *no. 2*, 207–228.