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Properties of a least-energy solution to a semilinear elliptic equation with exponential nonlinearity

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1 Introduction

In this paper we shall study the following problem;

$$\begin{cases} \epsilon^2\Delta u - u + (e^{\alpha u} - \alpha u - 1) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\epsilon > 0$, $\alpha > 0$, and Ω is a bounded domain in $\mathbf{R}^2$ with the smooth boundary $\partial\Omega$. We are interested in some properties of least-energy solutions to (1.1) as $\epsilon \to 0$.

First we refer to preceding studies of problems similar to (1.1);

$$\begin{cases} \epsilon^N\Delta u - u + u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbf{R}^N$ and $N \geq 2$. Many people have studied the existence of positive solutions and the behavior of a local maximum point of solutions to (1.2) as $\epsilon \to 0$. For example Lin, Ni and Takagi [11] proved the existence of positive non-constant solutions for sufficiently small $\epsilon > 0$, and moreover the solutions are sufficiently close to a zero except for the local maximum value. Later Ni and Takagi [13], [14] showed that the solution found in [11] is a least-energy solution, and the solution has at most one local maximum point P_ϵ on the boundary $\partial\Omega$. In addition they proved that, as $\epsilon \to 0$, the maximum point P_ϵ must be located near the point where the mean curvature attains its maximum. On the other hand, Ni and Wei [15] investigated the problem (1.2) whose boundary condition is replaced by the Dirichlet boundary condition. They showed that the Dirichlet problem also has a single-peak solution. Moreover, as $\epsilon \to 0$, the maximum point P_ϵ is situated near the most distant point from the boundary.

In the cases of both the Neumann boundary condition and the Dirichlet boundary condition, the proof of the behavior of P_ϵ is due to asymptotic formulas of a critical value which corresponds to the least-energy solution. In [14] and [15], the uniqueness of a solution to

$$\begin{cases} \Delta w - w + w^p = 0 & \text{in } \mathbf{R}^N, \\ w > 0 & \text{in } \mathbf{R}^N, \\ w(z) \to 0 & \text{as } |z| \to \infty, \\ w(0) = \max\limits_{z\in\mathbf{R}^N} w(z), \end{cases} \tag{1.3}$$

is used to obtain the asymptotic formulas. In contrast, without using the uniqueness of the solution to (1.3), del Pino and Felmer [5] proved the same formulas on the behavior of the maximum point for both the Dirichlet problem and the Neumann problem. In addition Byeon [3] also showed the more general result on the behavior of P_ϵ without the uniqueness of the solution to (1.3) for the Neumann problem with $N \geq 3$. However it is difficult to know the precise profile of the solution without using the uniqueness of the solution to (1.3). Thus del Pino and Felmer's result or Byeon's result does not completely include the Ni and Takagi's result.

In the above studies, a single-peak solution is only treated. For a multi-peak solution, e.g., see Gui and Wei [8]. They constructed solutions which has many critical points in the domain by the Lyapunov-Schmidt reduction method. Similarly Gui, Wei and Winter [9] constructed solutions which have many critical points on the boundary.

Our purpose is to investigate the behavior of the local maximum point of a least-energy solution and the profile of the least-energy solution to the problem (1.1). Concerning the preceding studies of the exponential nonlinearity case, Ni and Kabeya [12] investigate the problem having the exponential nonlinearity with the Neumann boundary condition. They proved that, for the exponential nonlinearity case, the same result as the power nonlinearity case holds, that is, least-energy solutions have only one local maximum point $P_\epsilon \in \partial\Omega$.

First, to define the terminology "least-energy", we define the "energy" functional corresponding to (1.1). The functional is as follows:

$$J_\epsilon(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2\right) dx - \frac{1}{\alpha}\int_\Omega \left(e^{\alpha u} - \frac{1}{2}(\alpha u)^2 - \alpha u - 1\right) dx \qquad \text{on } H_0^1(\Omega). \quad (1.4)$$

If there exists a critical point u_ϵ of J_ϵ, then u_ϵ is a weak solution to (1.1). In addition if J_ϵ attains the smallest positive critical value at u_ϵ, then u_ϵ is said to be a least-energy solution. In our investigation, we will find a critical value of J_ϵ by using the well-known Mountain Pass Lemma. Namely we set

$$\Gamma = \{h \in C([0,1]; H_0^1(\Omega)) \mid h(0) = 0, h(1) = e_0\},$$

where $e_0 \not\equiv 0$ is a non-negative function in $H_0^1(\Omega)$ and satisfies $J_\epsilon(e_0) = 0$. Then a critical value c_ϵ characterized as

$$c_\epsilon = \inf_{h\in\Gamma} \max_{0\leq t\leq 1} J_\epsilon(h(t)) > 0$$

is found, and there exists $u_\epsilon \in H_0^1(\Omega)$ such that $J_\epsilon(u_\epsilon) = c_\epsilon$. In fact the following proposition holds;

Proposition 1.1 *Let $\epsilon_0 > 0$ be sufficiently small. Then, for any $\epsilon \in (0, \epsilon_0)$, the following two statements hold*:

(i) *There exists a positive solution $u_\epsilon \in H_0^1(\Omega)$. Moreover u_ϵ is a classical solution, and it holds that*

$$J_\epsilon(u_\epsilon) \leq C_0\epsilon^2 \quad (1.5)$$

with some constant $C_0 > 0$ which is independent of ϵ. Furthermore u_ϵ is a least-energy solution.

(ii) *The least-energy solution u_ϵ has exactly one local maximum point P_ϵ.*

Remark 1.1 *The existence of the least-energy solution u_ϵ to* (1.1) *is actually proved for any $\epsilon \in (0, \infty)$. However it is required that ϵ is restricted within a small value when we show the inequality* (1.5) *and Proposition* 1.1 (ii).

By Proposition 1.1, we see that there exists a least-energy solution, and the solution has at most one maximum point.

Next we explain the result on the profile and the behavior of the least-energy solution u_ϵ, that is, the number of maximum points and its behavior as $\epsilon \to 0$. Concerning the method to investigate those, it seems that we can apply the method used in del Pino and Felmer's study [5] for our problem by some modification. However we will investigate the more precise profile of u_ϵ by the method of Ni and Wei, and thus some preliminaries are required to make use of the method. Hereafter we state those.

First we consider the following whole space problem;

$$\begin{cases} \Delta w - w + (e^{\alpha w} - \alpha w - 1) = 0 & \text{in } \mathbf{R}^2, \\ w > 0 & \text{in } \mathbf{R}^2, \\ w(z) \to 0 & \text{as } |z| \to \infty, \\ w(0) = \max\limits_{z \in \mathbf{R}^2} w(z). \end{cases} \tag{1.6}$$

The solution w to (1.6) seems to be approximate to u_ϵ. In fact $v_\epsilon(y) := u_\epsilon(\epsilon y + P_\epsilon)$ satisfies

$$\begin{cases} \Delta v_\epsilon - v_\epsilon + (e^{\alpha v_\epsilon} - \alpha v_\epsilon - 1) = 0 & \text{in } \Omega_\epsilon^*, \\ v_\epsilon > 0 & \text{in } \Omega_\epsilon^*, \\ v_\epsilon = 0 & \text{on } \partial\Omega_\epsilon^*, \end{cases}$$

where

$$\Omega_\epsilon^* := \{ y \in \mathbf{R}^2 \mid \epsilon y + P_\epsilon \in \Omega \}. \tag{1.7}$$

As $\epsilon \to 0$, the domain Ω_ϵ^* dilates in $\mathbf{R}^2$. In fact it is proved that v_ϵ uniformly converges to w for any compact subset of $\mathbf{R}^2$. The existence of a solution w to (1.6) is guaranteed by Berestycki, Gallouët and Kavian [2], and the uniqueness of w is proved by Pucci and Serrin [16], [17]. Moreover, since we can apply the result of Gidas, Ni and Nirenberg [7] for (1.6), it is seen that w is radially symmetric and strictly decreasing in $r := |z|$ with the order of $r^{-1/2}e^{-r}$. We state these results as Lemma 2.6 again; see Section 2.

In addition, to show our result, we confirm that w is *nondegenerate*. The definition of *nondegenerate* is as follows: the solution w is said to be nondegenerate if the linearized operator

$$\Delta - 1 + \alpha(e^{\alpha w} - 1) : \ H^2(\mathbf{R}^2) \to L^2(\mathbf{R}^2)$$

has a bounded inverse in the subspace $L^2_{rad}(\mathbf{R}^2) := \{ v \in L^2(\mathbf{R}^2) \mid v(z) = v(|z|) \}$. In fact the solution w to (1.6) is nondegenerate, and it is proved in the forthcoming paper [10].

Second we introduce some notations. For the solution w to (1.6) and any bounded smooth domain $U \subset \mathbf{R}^2$, let w_U be a unique solution to the linear problem

$$\begin{cases} \Delta w_U - w_U + (e^{\alpha w} - \alpha w - 1) = 0 & \text{in } U, \\ w_U = 0 & \text{on } \partial U. \end{cases}$$

In fact, since the operator $(\Delta - 1) : H^2(U) \to L^2(U)$ is bounded and invertible, the solution w_U is unique. Moreover we apply the maximum principle for $w - w_U$, and it follows that

$$w_U(y) < w(y) \qquad \text{for any } y \in U. \tag{1.8}$$

Let Q be the most distant point from the boundary, that is, Q is the point with $d(Q, \partial\Omega) = \max_{P\in\Omega} d(P, \partial\Omega)$, where $d(P, \partial\Omega)$ denotes the distance between $P \in \Omega$ and the boundary of Ω. Adding to (1.7), we define

$$\Omega_\epsilon = \{y \in \mathbf{R}^2 \,|\, \epsilon y + Q \in \Omega\},$$

and we obtain w_{Ω_ϵ} and $w_{\Omega_\epsilon^*}$. Moreover we define

$$\begin{cases} \varphi_\epsilon(y) := w(y) - w_{\Omega_\epsilon}(y) & y \in \Omega_\epsilon, \\ \psi_\epsilon(x) := -\epsilon \log \varphi_\epsilon((x - Q)/\epsilon) & x \in \Omega, \\ V_\epsilon(y) := e^{\beta\psi_\epsilon(Q)}\varphi_\epsilon(y) & y \in \Omega_\epsilon, \end{cases} \tag{1.9}$$

and

$$\begin{cases} \tilde\varphi_\epsilon(y) := w(y) - w_{\Omega_\epsilon^*}(y) & y \in \Omega_\epsilon^*, \\ \tilde\psi_\epsilon(x) := -\epsilon \log \tilde\varphi_\epsilon((x - P_\epsilon)/\epsilon) & x \in \Omega, \\ \tilde V_\epsilon(y) := e^{\beta\tilde\psi_\epsilon(P_\epsilon)}\tilde\varphi_\epsilon(y) & y \in \Omega_\epsilon^*, \end{cases} \tag{1.10}$$

where $\beta = 1/\epsilon$. Since (1.8), the functions ψ_ϵ and $\tilde\psi_\epsilon$ are well-defined. In addition V_ϵ and $\tilde V_\epsilon$ defined above have the following properties: for any sequence $\{\epsilon_k\}$ with $\epsilon_k \to 0$ as $k \to \infty$, there exists a subsequence $\{\epsilon_{k_l}\}$ such that $V_{\epsilon_{k_l}} \to V_0$ uniformly on any compact subset of $\mathbf{R}^2$ as $l \to \infty$. Here $u = V_0$ is a solution to

$$\begin{cases} \Delta u - u = 0 & \text{in } \mathbf{R}^2, \\ u > 0 & \text{in } \mathbf{R}^2, \\ u(0) = 1. \end{cases} \tag{1.11}$$

Similarly, there exists a subsequence $\{\epsilon_{k_m}\}$ such that $\tilde V_{\epsilon_{k_m}} \to \tilde V_0$ uniformly on any compact subset of $\mathbf{R}^2$ as $m \to \infty$, where $\tilde V_0$ is also a solution to (1.11).

We define the functional associated with (1.6) as

$$I(w) := \frac{1}{2}\int_{\mathbf{R}^2} (|\nabla w|^2 + w^2) dz - \frac{1}{\alpha}\int_\Omega \left(e^{\alpha w} - \frac{1}{2}(\alpha w)^2 - \alpha w - 1\right) dz \quad \text{on } H^1(\mathbf{R}^2), \tag{1.12}$$

and, from the above preliminaries, we state Theorem 1.1;

Theorem 1.1 *Let u_ϵ, c_ϵ and P_ϵ be the least-energy solution to* (1.1)*, the critical value corresponding to u_ϵ and the maximum point of u_ϵ, respectively. Moreover the unique solution to* (1.6) *is denoted by w. Then the following statements hold:*

(i) *The critical value c_ϵ is asymptotically expanded as*

$$c_\epsilon = \epsilon^2 \left\{ I(w) + \gamma e^{-\beta\tilde\psi_\epsilon(P_\epsilon)} + o\left(e^{-\beta\tilde\psi_\epsilon(P_\epsilon)}\right)\right\} \qquad \text{as } \epsilon \to 0, \tag{1.13}$$

where $\beta = 1/\epsilon$. Here γ is a constant defined as

$$\gamma := \frac{1}{2}\int_{\mathbf{R}^2} (e^{\alpha w} - \alpha w - 1) V_* dz > 0,$$

where V_* is a unique, positive and radially symmetric solution to (1.11).

(ii) *It holds that*

$$d(P_\epsilon, \partial\Omega) \to \max_{P\in\Omega} d(P, \partial\Omega) \qquad as \;\; \epsilon \to 0.$$

Furthermore $\tilde{\psi}_\epsilon(P_\epsilon) \to 2\max_{P\in\Omega} d(P, \partial\Omega)$ *as* $\epsilon \to 0$.

(iii) *For any sequence* $\{\epsilon_k\}$ *with* $\epsilon_k \to 0$ *as* $k \to \infty$, *there exists a subsequence* $\{\epsilon_{k_l}\}$ *and* $\phi_{\epsilon_{k_l}} \in H_0^1(\Omega^*_{\epsilon_{k_l}})$ *such that*

$$u_{\epsilon_{k_l}}(x) = w_{\Omega^*_{\epsilon_{k_l}}}(y) + \exp\left(-\beta_{k_l}\tilde{\psi}_{\epsilon_{k_l}}(P_{\epsilon_{k_l}})\right)\phi_{\epsilon_{k_l}}(y) \qquad as \;\; l \to \infty,$$

where $x = \epsilon_{k_l}y + P_{\epsilon_{k_l}}$ *and* $\beta_{k_l} = 1/\epsilon_{k_l}$. *Here*

$$\|e^{-\mu|y|}(\phi_{\epsilon_{k_l}} - \phi_0)\|_{L^\infty(\Omega^*_{\epsilon_{k_l}})} \to 0 \qquad as \;\; l \to \infty, \tag{1.14}$$

where $0 < \mu < 1$. *Here* ϕ_0 *is a solution to*

$$\Delta\phi_0 - \phi_0 + \alpha(e^{\alpha w} - 1)(\phi_0 - \tilde{V}_0) = 0 \quad \text{in } \mathbf{R}^2.$$

Remark 1.2 *Although the uniqueness of a solution to* (1.11) *is not guaranteed, the following fact is known: for any solution* V *and the radially symmetric solution* V_* *(it is explicitly solved by using the modified Bessel function) to* (1.11), *it holds that*

$$\frac{1}{2}\int_{\mathbf{R}^2}(e^{\alpha w} - \alpha w - 1)V dz = \frac{1}{2}\int_{\mathbf{R}^2}(e^{\alpha w} - \alpha w - 1)V_* dz,$$

where w *is the solution to* (1.6). *This fact is used in Theorem* 1.1, *and, concerning the proof of this fact, see Caffarelli and Littman* [4].

To prove Theorem 1.1 (iii), we require the uniqueness and nondegeneracy of the solution w to (1.6). The asymptotic equality implies that u_ϵ is approximate to the solution $w_{\Omega^*_{\epsilon_{k_l}}}$ to (1.11) with $U = \Omega^*_{\epsilon_{k_l}}$, and the remainder term is exponentially small.

2 Proof of Proposition 1.1

In this section we show Proposition 1.1. Before beginning the proof, we state the next two lemmas.

Lemma 2.1 *Let* Ω *satisfy the cone condition. For* $u \in H_0^1(\Omega)$ *and* $p > 1$, *the following inequality holds:*

$$\|u\|_{L^p(\Omega)} \le K_0 p^{\frac{1}{2}}\|u\|_{H_0^1(\Omega)},$$

where the constant K_0 *is dependent on the cone property of* Ω *but is independent of* p *and* $|\Omega|$. *Moreover, for* $\xi \in [1, \infty)$, *let* $\Omega^d_\xi := \{y \mid \xi y \in \Omega\}$. *Then there also holds* $\|u\|_{L^p(\Omega^d_\xi)} \le K_0 p^{\frac{1}{2}}\|u\|_{H_0^1(\Omega^d_\xi)}$.

Concerning the cone condition, e.g. see Adams [1]. Lemma 2.1 implies that the imbedding constant K_1 is not required to be replaced when the domain Ω is dilated. This lemma is shown by the inequality

$$\|u\|_{L^p(\Omega)} \leq K_1 p^{\frac{1}{2}} |\Omega|^{\frac{1}{p}} \|u\|_{H_0^1(\Omega)}$$

with some constant $K_1 > 0$ (see Lemma 2.1 in [18]) and the strategy in which we use a family of subsets of $\mathbf{R}^2$ which have finite intersection property (see Lemma 5.14 and Corollary 5.16 in [1]).

Lemma 2.2 (Proposition 4.3 in [19]) *Let M be a bounded smooth domain, and set $\{u_j\} \subset H^1(M)$. If $u_j \to u$ weakly in $H^1(M)$ as $j \to \infty$, then $e^{lu_j} \to e^{lu}$ strongly in $L^1(M)$ as $j \to \infty$ with any $l \in \mathbb{R}$.*

By using the above lemmas, we state the proof of Proposition 1.1 (i).

Proof of Proposition 1.1 (i). If the solution $u_\epsilon \in H_0^1(\Omega)$ is found, then, from Lemma 2.2 and the Schauder regularity theorem, it is proved that u_ϵ is a classical solution. Moreover, by the maximum principle, it follows that $u_\epsilon > 0$ in Ω. Hence it suffices that we show the existence of $u_\epsilon \in H_0^1(\Omega)$.

In order to apply the Mountain Pass Lemma for (1.4), it suffices that the following three conditions are satisfied:

(a) The functional J_ϵ satisfies the Palais-Smale condition.

(b) There exists some $t_* > 0$ such that $J_\epsilon(u) \geq c > 0$ for $u \in H_0^1(\Omega)$ with $\|u\|_{H_0^1(\Omega)} = t_*$, where some constant $c > 0$.

(c) For sufficiently small $\epsilon > 0$, there exists a nonnegative function $e_0 \in H_0^1(\Omega)$ such that $J_\epsilon(e_0) = 0$ and $J_\epsilon(tv) \leq C_0\epsilon^2$ for $t \in [0,1]$, where a constant $C_0 > 0$.

The condition (a) is shown by using Lemma 2.2 which implies that the mapping $H_0^1(\Omega) \ni u \mapsto e^{lu} \in L^1(\Omega)$ is compact for any $l > 0$. Thus the compactness of J_ϵ is guaranteed, and therefore J_ϵ satisfies the Palais-Smale condition.

Next we confirm the condition (b). By using Lemma 2.1, we obtain

$$J_\epsilon(u) \geq \left\{ \frac{\epsilon^2}{2} - \sum_{k=3}^{\infty} \frac{(\alpha k^{\frac{1}{2}} K_0)^k}{k!} \|u\|_{H_0^1(\Omega)}^{k-2} \right\} \|u\|_{H_0^1(\Omega)}^2.$$

Hence it suffices that we take t_* sufficiently small.

Finally the condition (c) is confirmed. We take $\epsilon > 0$ sufficiently small, and define

$$v(x) = \begin{cases} \epsilon^2 \left(1 - \dfrac{|x|}{\epsilon}\right) & (|x| < \epsilon), \\ 0 & (|x| \geq \epsilon). \end{cases}$$

If t is sufficiently large, then $J_\epsilon(tv) < 0$. In contrast, since condition (b), $J_\epsilon(tv) > 0$ when t is sufficiently small, and hence there exists t_0 such that $J_\epsilon(t_0 v) = 0$. It suffices that we take $e_0 := t_0 v$. In addition, for the function v, we obtain

$$\begin{cases} \displaystyle\int_\Omega |\nabla v|^2 dx = D_0 \epsilon^{-2} \quad \text{with} \quad D_0 = \pi, \\ \displaystyle\int_\Omega v^s dx = D_s \epsilon^{1-s} \qquad \text{with} \quad D_s = 2\pi \int_0^1 (1-r)^s r dr \ \text{ and } \ s > 0. \end{cases} \tag{2.1}$$

The functional J_ϵ is estimated by using (2.1), and hence it follows that $J_\epsilon(tv) \leq C_0\epsilon^2$ for $t \in [0, t_0]$.

Since the conditions (a)-(c) are satisfied, we can apply the Mountain Pass Lemma and find a solution u_ϵ to (1.1). Furthermore, from

$$c_\epsilon = \inf_{h\in\Gamma} \max_{0\leq t\leq 1} J_\epsilon(h(t)) > 0,$$
$$\Gamma = \{h \in C([0,1]; H_0^1(\Omega)) \mid h(0) = 0, h(1) = e_0\},$$

and $e_0 \in \Gamma$, we see that $J_\epsilon(u_\epsilon) = c_\epsilon \leq \max_{0\leq t\leq 1} J_\epsilon(e_0) \leq C_0\epsilon^2$. Therefore the inequality (1.5) is obtained.

Finally we define

$$M[v] := \sup_{t\geq 0} J_\epsilon(tv) \qquad \text{for } v \in H_0^1(\Omega),$$

and let c_* be

$$c_* := \inf\{M[v] \mid v \in H_0^1(\Omega), v \not\equiv 0 \text{ and } v \geq 0 \text{ in } \Omega\}.$$

Then we can show that $c_\epsilon = c_*$, and therefore u_ϵ is a least-energy solution (see Lemma 3.1 in [13]). The proof is finished. ■

From Proposition 1.1 (i) and Lemma 2.1, the following results is obtained:

Lemma 2.3 *Let u_ϵ be the least-energy solution to* (1.1)*. Then, for $p \in [1, \infty)$ and sufficiently small $\epsilon > 0$, the following inequalities hold:*

$$\int_\Omega (\epsilon^2|\nabla u_\epsilon|^2 + |u_\epsilon|^2)dx = \int_\Omega u_\epsilon f(u_\epsilon)dx \leq C_1\epsilon^2,$$
$$\sup_{x\in\overline{\Omega}} u_\epsilon \leq C_1,$$
$$\int_\Omega u_\epsilon^p dx \leq M_p\epsilon^2, \tag{2.2}$$

where C and M_p are independent of ϵ.

By using the above lemma, Proposition 1.1 (ii) is proved. Hereafter u_ϵ and c_ϵ are the least-energy solution and the corresponding critical value of (1.4), respectively.

Next we show Proposition 1.1 (ii), and we refer to the following two lemmas before beginning the proof. Hereafter let P_ϵ be a local maximum point of the least-energy solution u_ϵ found in Proposition 1.1 (i), w is the unique solution to (1.6), and I is the functional defined in (1.12).

Lemma 2.4 *It holds that*

$$c_\epsilon \leq \epsilon^2(I(w) + o(1)) \qquad \text{as } \epsilon \to 0. \tag{2.3}$$

Lemma 2.5 *Let $\epsilon_1 > 0$ be sufficiently small and $R_0 > 0$ satisfy $B_{2\epsilon_1 R_0}(P_\epsilon) \subset \Omega$. Then there exists a positive constant $\eta > 0$, independent of ϵ, such that*

$$u_\epsilon(x) \geq \eta \qquad \text{for } x \in B_{\epsilon R_0}(P_\epsilon)$$

with any $\epsilon \in (0, \epsilon_1)$.

The inequality (2.3) is a rougher estimate than (1.13), and it is obtained from the result on the behavior of $w(r)$ for sufficiently large r and the fact that u_ϵ is the least-energy solution. Lemma 2.5 follows from the Harnack inequality.

Proof of Proposition 1.1 (ii). Define

$$\rho_\epsilon := \frac{d(P_\epsilon, \partial\Omega)}{\epsilon},$$

and we show that $\rho_\epsilon \to \infty$ as $\epsilon \to 0$.

First suppose that there exist some constant $C > 0$ and some sequence $\{\epsilon_j\}$ such that $\rho_{\epsilon_j} \leq C$ as $j \to \infty$. Then, since $d(P_{\epsilon_j}, \partial\Omega) \leq C\epsilon_j$, there exists some $P_0 \in \partial\Omega$ such that $P_{\epsilon_j} \to P_0$ as $j \to \infty$ by passing a subsequence if necessary (the subsequence is still denoted by $\{\epsilon_j\}$). We introduce a diffeomorphism straightening the boundary. By translation and rotation, we can set the coordinates such that P_0 is at the origin, x_1-axis is tangent to $\partial\Omega$ at P_0, and x_2-axis points to the interior of Ω. Then there exists a smooth function $\zeta_{P_0}(x_1)$, which is defined for sufficiently small $|x_1|$, such that

(i) $\zeta_{P_0}(0) = 0$, and $\zeta'_{P_0}(0) = 0$.

(ii) $\partial\Omega \bigcap \mathcal{N} = \{(x_1, x_2) | x_2 = \zeta_{P_0}(x_1)\}$ and $\Omega \bigcap \mathcal{N} = \{(x_1, x_2) | x_2 > \zeta_{P_0}(x_1)\}$, where $\mathcal{N}$ is a neighborhood of P_0.

For $y \in \mathbb{R}^2$ near 0, we define the map $x = \Phi_{P_0}(y) = (\Phi_{P_0,1}(y), \Phi_{P_0,2}(y))$ as

$$\begin{cases} \Phi_{P_0,1}(y) = y_1 - y_2\zeta'_{P_0}(y_1), \\ \Phi_{P_0,2}(y) = y_2 + \zeta_{P_0}(y_1). \end{cases}$$

Since $\zeta'_{P_0}(0) = 0$, the differential map $D\Phi_{P_0}$ satisfies $D\Phi_{P_0} = I$ (the identity map). Hence, by the inverse function theorem, Φ_{P_0} has the inverse map $y = \Psi_{P_0}(x) (= (\Phi_{P_0})^{-1}(x))$ in the small neighborhood of $x = P_0$. We assume that Ψ_{P_0} is defined in the closed ball $\overline{B_\kappa(0)} \in \mathbb{R}^2$ (in the rest of this paper, $B_R(P)$ denotes $\{x \in \mathbf{R}^2 \mid |x - P_\epsilon| < R\}$) , where $\kappa > 0$ is a sufficiently small constant.

Set

$$v_\epsilon(y) := u_\epsilon(\Phi_{P_0}(y)),$$

and hence, from (2.2), it follows that

$$\int_{B_{\kappa/\epsilon}(0)} v_\epsilon^p dx \leq M_p \qquad \text{for } p \geq 1.$$

Thus, by the Sobolev imbedding theorem and the regularity theorem for elliptic equations, there exists some function $v_0 \in C^2(\mathbf{R}^2)$ such that

$$v_\epsilon \to v_0 \quad \text{in } C^2_{loc}(\mathbf{R}^2) \qquad \text{as } \epsilon \to 0.$$

Here the function v_0 satisfies

$$\begin{cases} \Delta v_0 - v_0 + (e^{\alpha v_0} - \alpha v_0 - 1) = 0 & \text{in } \mathbf{R}^2_+, \\ v_0 > 0 & \text{in } \mathbf{R}^2_+, \\ v_0 = 0 & \text{on } \partial\mathbf{R}^2_+, \end{cases} \tag{2.4}$$

where $\mathbf{R}_+^2$ is the upper half-plane. From Theorem 1.1 in [6], the problem (2.4) only has the solution $v_0 \equiv 0$. It is inconsistent with $u_\epsilon > 0$ in Ω, and hence $\rho_\epsilon \to \infty$ as $\epsilon \to 0$.

Next we prove that u_ϵ has only one local maximum point. Suppose that u_ϵ has two local maximum points P_ϵ and P'_ϵ. Set

$$w_\epsilon(y) := u_\epsilon(\epsilon y + P_\epsilon),$$

and hence, from (2.2), it follows that

$$\int_{B_{\rho_\epsilon}(0)} w_\epsilon^p dx \leq M_p \qquad \text{for } p \geq 1$$

Hence, by the Sobolev imbedding theorem and the regularity theorem for elliptic equations, there exists some function $w_0 \in C^2(\mathbf{R}^2)$ such that

$$w_\epsilon \to w_0 \quad \text{in } C^2_{loc}(\mathbf{R}^2) \qquad \text{as } \epsilon \to 0.$$

The function w_0 is a solution to (1.6). Moreover it holds that $w_0 \equiv w$ in $\mathbf{R}^2$ from the following lemma;

Lemma 2.6 *The problem* (1.6) *has a unique solution w. Furthermore w is radially symmetric, strictly decreasing and $w(r) = O(r^{-1/2}e^{-r})$ as $r \to \infty$.*

First we assume that there exists some constant C such that $|P_\epsilon - P'_\epsilon|/\epsilon \leq C$. Then, from the above argument, u_ϵ has two maximum points in $B_R(P_\epsilon)$ (we can take R such that $R > C$). The function u_ϵ is approximate to w, this function has only one maximum point at the origin and strictly decreases for $r > 0$. It is a contradiction to the assumption that u_ϵ has two local maximum point.

Second suppose that $|P_\epsilon - P'_\epsilon|/\epsilon \to \infty$ as $\epsilon \to 0$. Here we obtain the lower estimate

$$c_\epsilon \geq \epsilon^2(I(w) + c + o(1)) \qquad \text{as } \epsilon \to 0 \tag{2.5}$$

with some constant $c > 0$.

In fact we divide $c_\epsilon = J_\epsilon(u_\epsilon)$ into two part;

$$\begin{aligned} I_1 &= \int_{B_{\epsilon R}(P_\epsilon)} \left\{ \frac{1}{2}(|\nabla u|^2 + u^2) - \left(e^{\alpha u} - \frac{1}{2}(\alpha u)^2 - \alpha u - 1 \right) \right\} dx, \\ I_2 &= J_\epsilon(u_\epsilon) - I_1, \end{aligned}$$

where $R_0 > 0$ satisfies $B_{\epsilon R_0}(P_\epsilon) \subset \Omega$. By direct calculation it follows that

$$I_1 \geq \epsilon^2(I(w) - c_1 R_0^2 + o(1)) \qquad \text{as } \epsilon \to 0 \tag{2.6}$$

where $c_1 > 0$ is independent of ϵ and R_0. In addition, from Lemma 2.5, we can obtain

$$I_2 \geq c_2 \epsilon^2, \tag{2.7}$$

where $c_2 > 0$ is independent of ϵ and R_0. In Lemma 2.5, we can take R_0 arbitrarily large if ϵ_1 is taken sufficiently small. Therefore, from (2.6) and (2.7), the inequality (2.5) holds if R_0 satisfies $c_2 - c_1 R_0^2 > 0$. Since (2.5) is inconsistent with Lemma 2.4, the number of maximum points is exactly one. Thus Proposition 1.1 is proved. ■

3 Proof of Theorem 1.1

In this section Theorem 1.1 is shown. Hereafter the least-energy solution to (1.1) is denoted by u_ϵ, and the corresponding critical value to u_ϵ is written by c_ϵ. Point P_ϵ denotes the maximum point of u_ϵ. In addition let w be the unique solution to (1.6). The functions ψ_ϵ and $\tilde{\psi}_\epsilon$ are defined in (1.9) and (1.10), respectively.

Before beginning the proof, two lemmas are stated. Recall that Q is the most distant point from the boundary $\partial\Omega$, that is, $d(Q, \partial\Omega) = \max_{P\in\Omega} d(P, \partial\Omega)$. Next define

$$\phi_\epsilon(y) = e^{\beta\tilde{\psi}_\epsilon(P_\epsilon)}(w_\epsilon(y) - w_{\Omega_\epsilon^*}(y)),$$

where $w_\epsilon(y) = u_\epsilon(\epsilon y + P_\epsilon)$, and two lemmas are as follows:

Lemma 3.1 (i) *For the critical value c_ϵ corresponding to the least-energy solution u_ϵ, the next equality and inequality hold:*

$$c_\epsilon = \epsilon^2 \left\{ I(w) + \gamma e^{-\beta\tilde{\psi}_\epsilon(P_\epsilon)} + o\left(e^{-\beta\tilde{\psi}_\epsilon(P_\epsilon)}\right)\right\}, \tag{3.1}$$
$$c_\epsilon \leq \epsilon^2 \left\{ I(w) + \gamma e^{-\beta\psi_\epsilon(Q)} + o\left(e^{-\beta\psi_\epsilon(Q)}\right)\right\},$$

as $\epsilon \to 0$. Here the constant γ is the same as that defined in Theorem 1.1 (i).

(ii) *As $\epsilon \to 0$, $\psi_\epsilon(Q) \to 2d(Q, \partial\Omega)$. Moreover, for any $\sigma_0 > 0$, there exists a sufficiently small $\epsilon_1 > 0$ such that, for any $\epsilon < \epsilon_1$,*

$$\tilde{\psi}_\epsilon(P_\epsilon) \leq (2 + \sigma_0)d(P_\epsilon, \partial\Omega).$$

Lemma 3.2 *For sufficiently small $\epsilon > 0$, the following two statements hold:*

(i) *For $s > 2$, there exists a constant C_s such that $\|e^{-\mu|y|}\phi_\epsilon\|_{W^{2,s}(\Omega_\epsilon^*)} \leq C_s$.*

(ii) *For any sequence $\{\epsilon_k\}$ with $\epsilon_k \to 0$ as $k \to \infty$, there exists a subsequence $\{\epsilon_{k_l}\}$ such that $\|e^{-\mu|y|}(\phi_{\epsilon_{k_l}} - \phi_0)\|_{L^\infty(B_{(1-\delta_2)\rho_{\epsilon_{k_l}}}(0))} \to 0$ as $l \to \infty$, where δ_2 is sufficiently small, $\rho_\epsilon = d(P_\epsilon, \partial\Omega)/\epsilon$ and ϕ_0 is a classical solution to*

$$\Delta\phi_0 - \phi_0 - \alpha(e^{\alpha w} - 1)\tilde{V}_0 = 0 \quad \text{in } \mathbf{R}^2.$$

Here $\tilde{V}_0$ is a solution to (1.11). *Furthermore $e^{-\mu|y|}\phi_0 \in W^{2,s}(\mathbb{R}^2)$ for $s > 1$.*

Lemma 3.1 is proved by a very similar argument in Sections 5 and 6 in [15], and the proof of Lemma 3.2 is almost identical to Proposition 6.1 in [15]. From the above lemmas, Theorem 1.1 is shown.

Proof of Theorem 1.1. First Theorem 1.1 (i) and (ii) follow from Lemma 3.1. In fact (1.13) is the same as (3.1). Moreover, by comparing the equality and the inequality in Lemma 3.1, we obtain

$$\tilde{\psi}_\epsilon(P_\epsilon) \geq \psi_\epsilon(Q) + o(1) \qquad \text{as } \epsilon \to 0.$$

From Lemma 3.1 (ii) it holds that, for any $\sigma_0 > 0$,

$$(2 + \sigma_0)d(P_\epsilon, \partial\Omega) \geq \tilde{\psi}_\epsilon(P_\epsilon) \geq 2d(Q, \partial\Omega) + o(1) \qquad \text{as } \epsilon \to 0. \tag{3.2}$$

On the contrary, by the definition of Q, it holds that

$$d(P_\epsilon, \partial\Omega) \leq d(Q, \partial\Omega) \qquad \text{for any } \epsilon > 0. \tag{3.3}$$

Thus, from (3.2), (3.3) and the arbitrariness of σ_0, Theorem 1.1 (ii) is shown.

Thus it suffices that Theorem 1.1 (iii) is proved. From Lemma 3.2, (1.14) holds in $B_{(1-\delta_2)\rho_{\epsilon_{k_l}}}(0)$. In addition, by using the Green function of $-\Delta + 1$ in $\mathbf{R}^2$;

$$(\phi_\epsilon - \phi_0)(y) = \int_{\mathbf{R}^2} \mathbf{G}(|z-y|) \left\{ f'(w(z))(\phi_\epsilon - \phi_0)(z) + H_\epsilon(\phi_\epsilon)(z) + f'(w(z))\tilde{V}_0(z) \right\} dz.$$

Here $\mathbf{G}$ is the Green function of $-\Delta + 1$ in $\mathbf{R}^2$, and

$$\begin{aligned} f(t) &:= e^{\alpha t} - \alpha t - 1, \\ H_\epsilon(\phi_\epsilon) &:= e^{\beta\tilde{\psi}_\epsilon(P_\epsilon)} \left\{ f(\tilde{v}_\epsilon) - f(w) - e^{-\beta\tilde{\psi}_\epsilon(P_\epsilon)} f'(w)\phi_\epsilon \right\}. \end{aligned}$$

For $\mathbf{G}$, it holds that

$$\mathbf{G}(r) \leq C\frac{e^{-r}}{(1+r)^{\frac{1}{2}}} \qquad \text{as } r \to \infty, \tag{3.4}$$

$$\mathbf{G}(r) = C(|\log r| + 1) \qquad \text{as } r \to 0, \tag{3.5}$$

where C is a positive constant. In addition, for w_ϵ and $\tilde{V}_0$, the following two inequalities are obtained:

$$w_{\epsilon_{k_l}}(z) \leq C_\delta e^{-(1-\delta)|z|}, \tag{3.6}$$

$$\tilde{V}_0(z) \leq Ce^{\left(1+\frac{\sigma_1}{2}\right)|z|}, \tag{3.7}$$

where $z \in \mathbf{R}^2$, $\delta > 0$ is taken arbitrary, and C_δ (which depends on δ) and C are positive constants. From Lemma 3.2 (i) and (3.4)-(3.7), the estimate of $e^{-\mu|y|}(\phi_{\epsilon_{k_l}} - \phi_0)$ in $\Omega^*_{\epsilon_{k_l}} \backslash B_{(1-\delta_2)\rho_{\epsilon_{k_l}}}(0)$ is directly calculated, and therefore

$$\|e^{-\mu|y|}(\phi_{\epsilon_{k_l}} - \phi_0)\|_{L^\infty(\Omega^*_{\epsilon_{k_l}} \backslash B_{(1-\delta_2)\rho_{\epsilon_{k_l}}}(0))} \to 0 \qquad \text{as } l \to \infty.$$

Therefore Theorem 1.1 (iii) follows from Lemma 3.2. The proof of Theorem 1.1 is completely finished. ■

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