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THE PHRAGMÉN-LINDELÖF THEOREM FOR L^p-VISCOSITY SOLUTIONS

KAZUSHIGE NAKAGAWA (中川 和重)
TOHOKU UNIVERSITY (東北大学)

ABSTRACT. The Phragmén-Lindelöf theorem is established for L^p-viscosity solutions of fully nonlinear second order elliptic partial differential equations with unbounded ingredients.

1. INTRODUCTION

The notion of L^p-viscosity solutions was introduced in [5] to study fully nonlinear second order elliptic partial differential equations (PDEs for short) with unbounded inhomogeneous terms. We refer to [3] (see also [4]) as a pioneering work for the regularity theory of viscosity solutions of fully nonlinear PDEs.

It turned out that the Aleksandrov-Bakelman-Pucci (ABP for short) maximum principle can be extended to L^p-viscosity solutions for fully nonlinear second order elliptic PDEs with unbounded coefficients and inhomogeneous terms in [15]. See also [18] for a generalization.

As an application of the ABP maximum principle in [15], it is known that the (boundary) weak Harnack inequality for L^p-viscosity solutions of the associated extremal PDEs in [16] holds, which implies Hölder continuity for L^p-viscosity solutions of fully nonlinear elliptic PDEs with unbounded ingredients. We also refer to [20] for Hölder continuity estimates on L^p-viscosity solutions by a different approach.

On the other hand, qualitative properties of viscosity solutions of fully nonlinear elliptic PDEs have been investigated as generalizations for classical elliptic PDE theory. For instance, the ABP maximum principle in unbounded domains in [7] and [16], the Liouville property in [11, 6], the Hadamard principle in [6], and the Phragmén-Lindelöf theorem in [8, 14]. We refer to references in [8, 11, 6] for these qualitative properties of strong/classical solutions.

Our aim here is to give a sharp estimates of the Phragmén-Lindelöf theorem in [14] when PDEs have unbounded coefficients (i.e. b in this paper). In view of the boundary weak Harnack inequality in [16], it is natural to relax the hypotheses on coefficients and inhomogeneous terms. However, for

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the weak Harnack inequality, we need to suppose that the coefficient to the first derivatives is small enough in L^n-norm. When we work in bounded domains, this is not a restriction. Since we are concerned with unbounded domains, we will need a bit more delicate analysis than those in [8].

Our paper is organized as follows: section 2 is devoted to showing the definitions and known results. In section 3, we present the ABP type estimates on L^p-viscosity subsolutions of fully nonlinear PDEs with unbounded ingredients under appropriate geometric conditions. We show the Phragmén-Lindelöf theorem in our setting in section 4.

2. Preliminaries

We consider next fully nonlinear second order PDEs in unbounded domains $\Omega \subset \mathbb{R}^n$:

$$G(x, u, Du, D^2u) = f(x) \quad \text{in } \Omega, \tag{2.1}$$

where $G : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ are given measurable functions. We also suppose that $(r, p, M) \in \mathbb{R} \times \mathbb{R}^n \times S^n \to G(x, r, p, M)$ is continuous for almost all $x \in \Omega$. Here, S^n denotes the set of $n \times n$ symmetric matrices with the standard order.

We will use the standard notation from [13]. We denote by $L^p_+(\Omega)$ the set of all nonnegative functions in $L^p(\Omega)$.

Throughout this paper, we assume that

$$p > \frac{n}{2}.$$

We note that if $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ for $p > n/2$, then we may identify u with a continuous function in Ω, and $u(x)$ is twice differentiable for almost all $x \in \Omega$.

At first, we denote the definition of L^p-viscosity solutions of (2.1).

Definition 2.1. We call $u \in C(\Omega)$ an L^p-viscosity subsolution (resp., supersolution) of (2.1) if

$$\operatorname*{ess.\,lim\,inf}_{x \to x_0} \{G(x, u(x), D\phi(x), D^2\phi(x)) - f(x)\} \le 0$$

$$\Big(\text{resp., } \operatorname*{ess.\,lim\,sup}_{x \to x_0} \{G(x, u(x), D\phi(x), D^2\phi(x)) - f(x)\} \ge 0\Big)$$

whenever $\phi \in W^{2,p}_{\mathrm{loc}}(\Omega)$ and $x_0 \in \Omega$ is a local maximum (resp., minimum) point of $u - \phi$. A function $u \in C(\Omega)$ is called an L^p-viscosity solution of (2.1) if it is both an L^p-viscosity subsolution and an L^p-viscosity supersolution of (2.1).

To make easier recalling the right inequality, we will often say that u is an L^p-viscosity solution of

$$G(x, u, Du, D^2u) \le f(x)$$

$$\big(\text{resp., } G(x, u, Du, D^2u) \ge f(x)\big),$$

if it is an L^p-viscosity subsolution (resp., supersolution) of (2.1).

In what follows, instead of (2.1), we mainly consider PDEs which do not depend on u-variable:

$$F(x, Du, D^2u) = f(x) \quad \text{in } \Omega. \tag{2.2}$$

We will assume that F is (degenerate) elliptic:

$$\begin{gathered} F(x,p,M) \leq F(x,p,N) \\ \text{for all } (x,p,M,N) \in \Omega \times \mathbb{R}^n \times S^n \times S^n \text{ provided } M \geq N. \end{gathered} \tag{2.3}$$

For fixed ellipticity constants $0 < \lambda \leq \Lambda$, we also assume that

$$\begin{gathered} \text{there exists } b \in L^q_+(\Omega) \text{ such that} \\ \mathcal{P}^-(M) - b(x)|p| \leq F(x,p,M) \quad \text{for } (x,p,M) \in \Omega \times \mathbb{R}^n \times S^n, \end{gathered} \tag{2.4}$$

where the Pucci operators $\mathcal{P}^\pm : S^n \to \mathbb{R}$ are defined by

$$\begin{gathered} \mathcal{P}^-(M) = \min\{-\text{trace}(AM) : A \in S^n : \lambda I \leq M \leq \Lambda I\}, \\ \text{and } \mathcal{P}^+(M) = \max\{-\text{trace}(AM) : A \in S^n : \lambda I \leq M \leq \Lambda I\}. \end{gathered} \tag{2.5}$$

We will use the Escauriaza's constant $p_0 = p_0(n, \lambda, \Lambda) \in [n/2, n)$, for which we refer to [12]. It is known that for $p > p_0$, and $f \in L^p(B_r(z))$, where $B_r(x) = \{y \in \mathbb{R}^n : |x-y| < r\}$, there exists a strong solution $u \in C(\overline{B}_r(z)) \cap W^{2,p}_{\text{loc}}(B_r(z))$ of

$$\mathcal{P}^-(D^2v(x)) = f(x) \quad \text{a.e. in } B_r(z)$$

under $v(x) = 0$ for $x \in \partial B_r(z)$ with estimates:

$$-C\|f^-\|_{L^p(B_r(z))} \leq v(x) \leq C\|f^+\|_{L^p(B_r(z))} \quad \text{in } B_r(z)$$

and

$$\|v\|_{W^{2,p}_{\text{loc}}(B_r(z))} \leq C'\|f\|_{L^p(B_r(z))},$$

where $C = C(n,\lambda,\Lambda,p) > 0$ and $C' = C'(n,\lambda,\Lambda,p,r) > 0$ are positive constants.

We remark that to prove the ABP maximum principle [15, Theorem 2.9], which implies the boundary weak Harnack inequality [16, Theorem 6.1], it suffices to obtain the existence of strong solutions of the above extremal equation only in balls although this fact is not clearly mentioned in [15, 16]. In fact, this existence result holds with local $W^{2,p}$-estimates for more general domains satisfying the uniform exterior cone property but the $p_0 \in [\frac{n}{2}, n)$ associated with general domains might be bigger than the above. We also notice that we may replace cubes by balls in the (boundary) weak Harnack inequality in [16] by Cabré's covering argument.

Fix $R > 0$ and $z \in \mathbb{R}^n$. Let T, $T' \subset B_R(z)$ be domains such that

$$\overline{T} \subset T', \quad \text{and} \quad \theta_0 \leq \frac{|T|}{|T'|} \leq 1 \quad \text{for some } \theta_0 > 0.$$

When we apply our weak Harnack inequality below, our choice of T and T' always satisfies the above condition.

For a given domain $A \subset \mathbb{R}^n$ and a function $v \in C(A)$, we define v_m^- on $T' \cup A$ by

$$v_m^-(x) = \begin{cases} \min\{v(x), m\} & \text{if } x \in A, \\ m & \text{if } x \in T' \setminus A, \end{cases}$$

where

$$m = \liminf_{x \to T' \cap \partial A} v(x).$$

We note that if $T' \cap \partial A \neq \emptyset$, then v_m^- is a real-valued function and that if $T' \cap \partial A \neq \emptyset$, v is a nonnegative L^p-viscosity supersolution of (2.2) and $f \leq 0$ in $T' \cap A$, then v_m^- is a nonnegative L^p-viscosity supersolution of (2.2) in T'.

Next, we recall the boundary weak Harnack inequality when PDEs have unbounded coefficients and inhomogeneous terms.

Lemma 2.2 ([16, Theorem 6.1]). *Let T, T', A be as above. Assume that $T \cap A \neq \emptyset$ and $T' \setminus A \neq \emptyset$ and that*

$$q > n, \quad q \geq p > p_0. \tag{2.6}$$

Then, there exist constants $\varepsilon_0 = \varepsilon_0(n, \lambda, \Lambda) > 0$, $r = r(n, \lambda, \Lambda, p, q) > 0$ and $C_0 = C_0(n, \lambda, \Lambda, p, q) > 0$ satisfying the following property: if $b \in L^q_+(T' \cap A)$, $f \in L^p_+(T' \cap A)$, a nonnegative L^p-viscosity solution $w \in C(T' \cap A)$ of

$$\mathcal{P}^+(D^2 w) + b(x)|Dw| \geq -f(x) \quad \text{in } T' \cap A,$$

and

$$\|b\|_{L^n(T' \cap A)} \leq \varepsilon_0, \tag{2.7}$$

then it follows that

$$\left(\frac{1}{|T|}\int_T (w^-_{T',A})^r \, dx\right)^{1/r} \leq C_0\left(\inf_T w^-_{T',A} + R\|f\|_{L^n(T' \cap A)}\right)$$

provided that $q > n$ and $q \geq p \geq n$, and

$$\left(\frac{1}{|T|}\int_T (w^-_{T',A})^r dx\right)^{1/r}$$
$$\leq C_0\left(\inf_T w^-_{T',A} + R^{2-\frac{n}{p}}\|f\|_{L^p(T' \cap A)} \sum_{k=0}^{M} R^{(1-\frac{n}{q})k}\|\mu\|^k_{L^q(T' \cap A)}\right)$$

provided that $q > n > p > p_0$, where $M = M(n, p, q) \geq 1$ is an integer.

In the next section, we will establish some local and global ABP type estimates on L^p-viscosity subsolutions for (2.2). Finally, we recall the notations concerning the shape of domains from [8].

Definition 2.3 (Local geometric condition). Let $\sigma, \tau \in (0, 1)$. We call $y \in \Omega$ a $G_{\sigma,\tau}$ point of Ω if there exist $R = R_y > 0$ and $z = z_y \in \mathbb{R}^n$ such that

$$y \in B_R(z), \quad \text{and} \quad |B_R(z) \backslash \Omega_{y, B_R(z), \tau}| \geq \sigma |B_R(z)|, \tag{2.8}$$

where $\Omega_{y,B_R(z),\tau}$ is the connected component of $B_{\frac{R}{\tau}}(z) \cap \Omega$ containing y. For $\sigma, \tau \in (0,1)$, and $R_0 > 0$, $\eta \geq 0$, we call $y \in \Omega$ a $G_{\sigma,\tau}^{R_0,\eta}$ point in Ω if y is a $G_{\sigma,\tau}$ point in Ω with $R = R_y > 0$ and $z = z_y$ satisfying

$$R \leq R_0 + \eta|y|. \tag{2.9}$$

Definition 2.4 (Global geometric condition). We call Ω a weak-G domain if any $y \in \Omega$ is a $G_{\sigma,\tau}^{R_0,\eta}$ point in Ω.

Remark 2.5. For the sake of simplicity of notations, for a $G_{\sigma,\tau}$ point $y \in \Omega$, we will write B_y for $B_{\frac{R_y}{\tau}}(z_y)$, where $R_y > 0$ and $z_y \in \mathbb{R}^n$ are from Definition 2.3.

We refer the reader to [21] and [8] for examples of weak-G domains Ω. We also refer to [1] for a generalization.

3. ABP TYPE ESTIMATES

In this section, we first present pointwise estimates on L^p-viscosity subsolutions of (2.2), which is often referred as the Krylov-Safonov growth lemma. For simplisity, throughout this paper, we assume that $p \geq n$. In what follows, we fix $\sigma, \tau \in (0,1)$ and $R_0 > 0$. Let $y \in \Omega$ be a $G_{\sigma,\tau}^{R_0,\eta}$ point with $\eta \geq 0$. It is possible to apply our weak Harnack inequality in B_y, which is from Definition 2.3, if $\|b\|_{L^n(B_y\cap\Omega)} \leq \varepsilon_0$. Here and later, $\varepsilon_0 > 0$ is the constant from Lemma 2.2.

Even if $\|b\|_{L^n(B_y\cap\Omega)} > \varepsilon_0$, we may use Cabré's covering argument; we divide B_y into small pieces so that we may apply the weak Harnack inequality in each piece. We then obtain the weak Harnack inequality in B_y by combining all the inequalities for small pieces.

However, since we need the estimates uniform in $y \in \Omega$, this argument does not work immediately because of unboundedness of $\{R_y\}_{y\in\Omega}$ when $\eta > 0$.

To avoid this difficulty, we will suppose a decay rate of b:
for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{R>1} \int_E R^n b(Rx)^n \, dx < \varepsilon \quad \text{for } E \subset A, |E| < \delta, \tag{3.1}$$

where $A = \Omega \cap \{x \in \mathbb{R}^n | \frac{1}{4}\min\{1/(1+\eta), (\sigma/4)^{1/n}\} < |x| < 2 + 1/\tau\}$.

Lemma 3.1 (pointwise estimate). *Assume that* (2.3), (2.6) *and* (2.4) *hold with $b \in L^q_+(\Omega)$. Let $\eta > 0$ and $y \in \Omega$ be a $G_{\sigma,\tau}^{R_0,\eta}$ point in Ω with $R = R_y > 0$ and $z = z_y \in \mathbb{R}^n$. Then, there exist $\kappa = \kappa(n, \lambda, \Lambda, \sigma, \tau, R_0, \eta) \in (0,1)$ and $\varepsilon = \varepsilon(n, \sigma, \eta) > 0$ satisfying the following property: if $w \in C(\Omega)$ is an L^p-viscosity subsolution of* (2.2) *with $f \in L^p_+(\Omega)$, then we have the following properties: (i) If $|y| \leq R_0$ and $p \geq n$, then*

$$w(y) \leq \kappa \sup_{B_y\cap\Omega} w^+ + (1-\kappa) \limsup_{x\to B_y\cap\partial\Omega} w^+ + R_0\|f\|_{L^n(B_y\cap\Omega)}.$$

(ii) Assume that (3.1) *is satisfied and that* $|y| > R_0$. *If* $p \geq n$, *then*

$$w(y) \leq \kappa \sup_{B_y \cap \Omega} w^+ + (1-\kappa) \limsup_{x \to B_y \cap \partial\Omega} w^+ + R\|f\|_{L^n(B_y \cap \Omega \setminus B_{\varepsilon R}(0))}.$$

Remark 3.2. To get the weak maximum principle (Lemma 4.1 below), it is important to have the term $\|f\|_{L^p(B_y \cap \Omega \setminus B_{\varepsilon R}(0))}$ instead of $\|f\|_{L^p(B_y \cap \Omega)}$ in the estimates of the assertion (ii) above.

Proof. First of all, we shall omit giving the proof in the case of $\|b\|_{L^q(\Omega)} = 0$ because it is easy to do it, and we suppose that $\|b\|_{L^q(\Omega)} > 0$.

It is enough to show the assertion when $\hat{C} := \limsup_{x \to B_y \cap \partial\Omega} w^+(x) = 0$. In fact, after having established the assertion when $\hat{C} = 0$, we may apply the result to $w - \hat{C}$ to prove the assertion in the general case.

Due to (2.4), w is an L^p-viscosity solution of

$$\mathcal{P}^-(D^2 w) - b(x)|Dw| \leq f(x) \quad \text{in } \Omega.$$

Setting $C_w = \sup_{B_y \cap \Omega} w^+$, we immediately see that $v(x) := C_w - w(x)$ is an L^p-viscosity solution of

$$\mathcal{P}^+(D^2 v) + b(x)|Dv| \geq -f(x) \quad \text{in } \Omega.$$

We shall first prove (ii).

Case (ii) $|y| > R_0$:

Taking $\varepsilon = \frac{1}{4}\min\{\frac{1}{1+\eta}, (\frac{\sigma}{4})^{\frac{1}{n}}\} \in (0, \frac{1}{2}\min\{\frac{1}{1+\eta}, (\frac{\sigma}{4})^{\frac{1}{n}}\})$. Note that $2\varepsilon < 1/(1+\eta)$ and $(2\varepsilon)^n < \sigma/4$. We set $T = B_R(z) \backslash \overline{B}_{2\varepsilon R}(0)$ and $T' = B_y \backslash \overline{B}_{\varepsilon R}(0)$. Observe that

$$2\varepsilon R < \frac{R}{1+\eta} \leq \frac{R_0 + \eta|y|}{1+\eta} < |y|$$

and consequently $y \in T = B_R(z) \setminus \overline{B}_{2\varepsilon R}(0)$. Let A be the connected component of $T' \cap \Omega$ which contains y. We have

$$\begin{aligned} |T \backslash A| &\geq |T \backslash \Omega_{y, B_R(z), \tau}| \\ &\geq |B_R(z) \backslash \Omega_{y, B_R(z), \tau}| - |B_{2\varepsilon R}(0)| \\ &\geq \sigma |B_R(0)| - (2\varepsilon)^n |B_R(0)| \\ &\geq \frac{\sigma}{2} |B_R(0)| \\ &\geq \frac{\sigma}{2} |T|. \end{aligned}$$

Since

$$T' \cap \partial A \subset T' \cap \partial(T' \cap \Omega) \subset T' \cap (\partial T' \cup \partial\Omega) = T' \cap \partial\Omega, \tag{3.2}$$

in view of $\hat{C} \leq 0$, we have

$$\liminf_{x \to T' \cap \partial A} v(x) = C_w - \limsup_{x \to T' \cap \partial A} w(x) \geq C_w. \tag{3.3}$$

Now, we verify (2.7). By (3.1), if $\|Rb(R\,\cdot)\|_{L^n(A)} \leq \varepsilon_0$, we see that

$$\|b\|_{L^n(T' \cap A)} \leq \|Rb(R\,\cdot)\|_{L^n(A)} \leq \varepsilon_0.$$

Setting $m = \liminf_{x \to T' \cap \partial A} v(x)$, we use (3.3) to show for any $r > 0$,

$$\left(\frac{\sigma}{2}\right)^{1/r} C_w \le \left(\frac{|T \backslash A|}{|T|}\right)^{1/r} C_w \le \left(\frac{1}{|T|}\int_{T \backslash A} m^r dx\right)^{1/r} \le \left(\frac{1}{|T|}\int_T (v_m^-)^r dx\right)^{1/r}.$$

Since $y \in A$, we have

$$\inf_T v_m^- \le v(y) = C_w - w(y). \tag{3.4}$$

Thus, letting $r > 0$ be the constant from Lemma 2.2, we have

$$\left(\frac{\sigma}{2}\right)^{1/r} C_w \le C_0 \left(\inf_T v_m^- + R\|f\|_{L^n(T' \cap A)}\right) \le C_0 \left(C_w - w(y) + R\|f\|_{L^n(T' \cap \Omega)}\right).$$

Therefore, we conclude that the assertion (ii) holds with $\kappa = 1 - (\frac{\sigma}{2})^{1/r} \min\{C_0^{-1}, 1\} > 0$ in the case where $\|Rb(R\,\cdot)\|_{L^n(A)} \le \varepsilon_0$.

Next assume that $\|Rb(R\,\cdot)\|_{L^n(A)} > \varepsilon_0$. In this case, we can show that using a Cabré's covering argument.

Case (i) $|y| \le R_0$:
Since we have $R \le (1+\eta)R_0$ in this case, we may regard Ω as a bounded domain. Thus, we can use the standard covering argument by Cabré without using (3.1). Setting $T = B_R(z)$, $T' = B_{\frac{R}{\tau}}(z)$ and $A = \Omega_{y, B_R(z), \tau}$, we have

$$|T \backslash A| = |B_R(z) \setminus \Omega_{y, B_R(z), \tau}| \ge \sigma |B_R(z)| \ge \frac{\sigma}{2}|T|.$$

We shall only give a proof when $\|b\|_{L^n(T' \cap A)} \le \varepsilon_0$.

Following the same argument as in case (ii) with the above inequality, and new A, T, T', we have

$$\begin{aligned}\left(\frac{\sigma}{2}\right)^{1/r} C_w &\le C_0 \left(\inf_T v_m^- + R_0\|f\|_{L^n(B_y \cap \Omega)}\right) \\ &\le C_0 \left(C_w - w(y) + R_0\|f\|_{L^n(B_y \cap \Omega)}\right).\end{aligned}$$

Therefore, we conclude that the assertion holds with the same $\kappa \in (0,1)$ as in case (ii). □

When $\Omega \subset \mathbb{R}^n$ is a weeak-G domain, we derive the ABP maximum principle for L^p-viscosity subsolutions bounded from above of (2.2).

Theorem 3.3 (ABP maximum principle in unbounded domains). *Assume* (2.6), (2.3) *and* (2.4) *with* $b \in L^q_+(\Omega)$ *satisfying* (3.1). *Let* $\eta > 0$ *and* $\Omega \subset \mathbb{R}^n$ *be a weak-G domain. Assume also*

$$\sup_{y \in \Omega, |y| > R_0} R_y \|f\|_{L^n(A_y \cap \Omega)} < \infty \tag{3.5}$$

Let $\frac{1}{4}\min\{\frac{1}{1+\eta}, (\frac{\sigma}{4})^{1/n}\} \le \varepsilon < \min\{\frac{1}{1+\eta}, (\frac{\sigma}{4})^{1/n}\}$. *Then, there exists*

$$C = C(n, \lambda, \Lambda, p, q, \varepsilon, \sigma, \tau, R_0, \eta) > 0$$

satisfying the following properties: if $w \in C(\Omega)$ is an L^p-viscosity subsolution bounded from above of (2.2) *with $f \in L^p_+(\Omega)$, then it follows that*

$$\begin{aligned}\sup_{\Omega} w \le \limsup_{x\to\partial\Omega} w^+(x) + C \sup_{y\in\Omega,|y|>R_0} R_y\|f\|_{L^n(A_y\cap\Omega)} \\ + CR_0 \sup_{y\in\Omega,|y|\le R_0} \|f\|_{L^n(B_y\cap\Omega)}.\end{aligned} \tag{3.6}$$

Here, $A_y = B_{\frac{R_y}{\tau}}(z_y)\setminus B_{\varepsilon R_y}(0)$ and $B_y = B_{\frac{R_y}{\tau}}(z_y)$.

Proof. We take the supremum over $y \in \Omega$ with the estimates in Lemma 3.1 to conclude the inequalities (3.6). □

4. Phragmén-Lindelöf theorem

In this section, we show that the weak maximum principle holds for PDEs with zero-order terms. As before, assuming that Ω is a weak-G domain, for each $y \in \Omega$, we use the notations $R_y > 0$ and $z_y \in \mathbb{R}^n$. Also, B_y and A_y, respectively, denote $B_{\frac{R_y}{\tau}}(z_y)$ and $B_{\frac{R_y}{\tau}}(z_y)\setminus B_{\varepsilon R_y}(0)$ for $\varepsilon \in [\frac{1}{4}\min\{\frac{1}{1+\eta},(\frac{\sigma}{4})^{1/n}\}, \frac{1}{2}\min\{\frac{1}{1+\eta},(\frac{\sigma}{4})^{1/n}\})$.

Lemma 4.1. *Assume* (2.3), (2.6) *and* (2.4) *with $b \in L^q_+(\Omega)$ satisfying* (3.1). *Let $\eta > 0$ and Ω be a weak-G domain. Then, there exists $c_0 = c_0(n,\lambda,\Lambda,p,q,\sigma,\tau,R_0,\eta) > 0$ satisfying the following property:*
if $c \in L^n_+(\Omega)$, $w \in C(\Omega)$ is an L^p-viscosity solution bounded from above of

$$F(x,Dw,D^2w) - c(x)w^+ \le 0 \quad \text{in } \Omega \tag{4.1}$$

such that

$$\limsup_{x\to\partial\Omega} w(x) \le 0, \tag{4.2}$$

and

$$K_0 := \max\left\{\sup_{y\in\Omega,|y|>R_0} \|\langle\cdot\rangle c(\cdot)\|_{L^n(A_y\cap\Omega)}, \sup_{y\in\Omega,|y|\le R_0} \|c\|_{L^n(B_y\cap\Omega)}\right\} \le c_0, \tag{4.3}$$

then $w \le 0$ in Ω.

Proof. Note that by (2.4), w is an L^n-viscosity solution of

$$\mathcal{P}^-(D^2w) - b(x)|Dw| - c(x)w^+ \le 0.$$

We apply Theorem 3.3 with $f = cw^+$ to obtain that when $|y| \le R_0$,

$$R_0\|cw^+\|_{L^n(B_y\cap\Omega)} \le R_0 \sup_{\Omega} w^+\|c\|_{L^n(B_y\cap\Omega)} \le R_0K_0 \sup_{\Omega} w^+.$$

On the other hand, when $|y| > R_0$, we have

$$R_y\|cw^+\|_{L^n(A_y\cap\Omega)} \le \frac{R_y}{\sqrt{1+(\varepsilon R_y)^2}} \sup_{\Omega} w^+\|\langle\cdot\rangle c(\cdot)\|_{L^n(A_y\cap\Omega)} \le \frac{K_0}{\varepsilon} \sup_{\Omega} w^+. \tag{4.4}$$

Choosing $\varepsilon_1 = \frac{1}{4}\min\{\frac{1}{1+\eta}, (\frac{\sigma}{4})^{1/n}\}$ for instance, we have

$$\sup_{\Omega} w \le C_3 \max\left\{R_0, \frac{1}{\varepsilon_1}\right\} c_0 \sup_{\Omega} w^+$$

for some constant $C_3 > 0$. Taking $c_0 < 1/(C_3 \max\{R_0, 1/\varepsilon_1\})$, this end the proof. □

Theorem 4.2 (Phragmén-Lindelöf theorem). *Assume* (2.3), (2.6) *and* (2.4) *with* $b \in L^q_+(\Omega)$ *satisfying* (3.1). *Let* $\eta > 0$ *and* Ω *be a weak-G domain. There exists a positive constant* $\alpha > 0$ *such that if* $w \in C(\Omega)$ *is an* L^p*-viscosity solution of*

$$F(x, Dw, D^2w) \le 0 \quad \text{in } \Omega \tag{4.5}$$

with (4.2) *holds and*

$$w^+(x) = O(|x|^\alpha) \quad \text{as } |x| \to \infty, \tag{4.6}$$

then $w \le 0$ *in* Ω.

Proof of Theorem 4.2. Define a positive smooth function

$$\xi(x) = \langle x\rangle^\alpha,$$

where $\alpha > 0$ will be fixed later. Setting $u = w/\xi$, which is bounded from above. A straightforward calculation shows that

$$\frac{|D\xi|}{\xi}(x) \le \frac{\alpha}{\langle x\rangle}, \quad \frac{|D^2\xi|}{\xi}(x) \le \frac{C_4\alpha}{\langle x\rangle^2}$$

for some $C_4 > 0$. Thus, we see that u is an L^n-viscosity solution of

$$\mathcal{P}^-(D^2u) - \gamma_1(x)|Du| - \alpha\gamma_2(x)u^+ \le 0 \quad \text{in } \Omega,$$

where

$$\gamma_1(x) = \frac{C_5\alpha}{\langle x\rangle} + b(x), \quad \gamma_2(x) = \frac{C_6}{\langle x\rangle}\left(\frac{1}{\langle x\rangle} + b(x)\right)$$

for some positive constants $C_5, C_6 > 0$. We easily see that γ_1 satisfies (3.1).

We next show that (4.3) holds for γ_2. Direct calculation implies

$$\tilde{K}_0 := \max\{\sup_{y\in\Omega, |y|>R_0} \|\langle\cdot\rangle\gamma_2(\cdot)\|_{L^n(A_y\cap\Omega)}, \sup_{y\in\Omega,|y|\le R_0} \|\gamma_2\|_{L^n(B_y\cap\Omega)}\} < +\infty \tag{4.7}$$

is boounded. Thus, $K_0 = \alpha\tilde{K}_0$ is small when $\alpha > 0$ is small enough.

Therefore, using Lemma 4.1 with $b = \gamma_1$ and $c = \gamma_2$, we get $u \le 0$. This imlies $w \le 0$. □

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Kazushige Nakagawa
Mathematical Institute, Tohoku University
6-3, Aoba, Aramaki, Aoba-ku, SENDAI 980-8578 ,JAPAN
E-mail address: knakagawa@math.tohoku.ac.jp