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Colorings and eventual colorings

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Abstract. 本稿では、可分距離空間上における不動点を持たない同相写像の力学的な性質の幾つかを扱う。与えられた周期点の集合が零次元である不動点を持たない同相写像 $f : X \to X$ と、任意の自然数 p に対して、 coloring number を一般化した eventual coloring number $C(f,p)$ を定義しその性質を調べる。特に、空間 X が有限次元である場合、X が 2 つの閉集合 C_1 と C_2 に分割でき、X のすべての元が f を p 回施す間に C_1 と C_2 の間を行き来するような X の次元に依存した自然数 p が存在することを紹介する。

1 Introduction

All spaces are assumed nonempty separable metric spaces and all maps are continuous functions. Let $\mathbb{N}$ be the set of all natural numbers, i.e., $\mathbb{N} = \{1, 2, 3, \cdots\}$. For a (separable metric) space X, $\dim X$ denotes the topological dimension of X. Let $P(f)$ be the set of all periodic points of map $f : X \to X$, i.e.,

$$P(f) = \{x \in X \mid f^j(x) = x \text{ for some } j \in \mathbb{N}\}.$$

Let $f : X \to X$ be a fixed-point free closed map of a separable metric space X. In this paper, we assume that all maps are closed maps. A subset C of X is called a *color* (see [10]) of f provided that $f(C) \cap C = \emptyset$. Note that $f(C) \cap C = \emptyset$ if and only if $C \cap f^{-1}(C) = \emptyset$. We say a cover $\mathcal{C}$ of X is a *coloring* of f if each element C of $\mathcal{C}$ is a color of f. The minimal cardinality $C(f)$ of closed (or open) colorings of f is called the *coloring number* of f. Many mathematicians have investigated the coloring number $C(f)$ (see [1-4], [6] and [8-10]).

Theorem 1.1. (Aarts, Fokkink and Vermeer [1]) *Let $f : X \to X$ be a fixed-point free involution of a (separable) metric space X with $\dim X = n < \infty$. Then $C(f) \leq n + 2$.*

Theorem 1.2. (Aarts, Fokkink and Vermeer [1]) *Let $f : X \to X$ be a fixed-point free homeomorphism of a (separable) metric space X with $\dim X = n < \infty$. Then $C(f) \leq n + 3$.*

Now, we will define more general notion of color. Let $f : X \to X$ be a fixed-point free map of a space X and $p \in \mathbb{N}$. A subset C of X is *eventually colored within* p of f if $\bigcap_{i=0}^{p} f^{-i}(C) = \emptyset$. It is easy to see that C is a color of f if and only if C is eventually colored within 1. Then we have the following proposition.

Proposition 1.3. *Let $f : X \to X$ be a fixed-point free map of a separable metric space X and $p \in \mathbb{N}$. Then the followings hold.*
(1) A subset C of X is eventually colored within p of f if and only if each point $x \in C$ wanders off C within p, i.e., for each $x \in C$, $f^i(x) \notin C$ with some $i \leq p$.
(2) If a subset C of X satisfies the condition $\bigcap_{i=0}^{p} f^i(C) = \emptyset$, then C is eventually colored within p of f.
(3) If f is an injective map, then a subset C of X is eventually colored within p of f if and only if C satisfies the condition $\bigcap_{i=0}^{p} f^i(C) = \emptyset$.

Remark. In general, the converse assertion of (2) in the proposition above is not true. Let $X = \{a, b, c\}$ be a set consisting three points and let $f : X \to X$ be the map defined by $f(a) = b, f(b) = c, f(c) = b$. Then $C = \{a, b\}$ is eventually colored within 2 of f, but $\bigcap_{i=0}^{p} f^i(C) \neq \emptyset$ $(p \in \mathbb{N})$.

Next, we define the eventual coloring number $C(f,p)$ as follows. A cover $\mathcal{C}$ of X is called an *eventual coloring within* p if each element C of $\mathcal{C}$ is eventually colored within p. The minimal cardinality $C(f,p)$ of all closed (or open) eventual colorings within p is called the *eventual coloring number* of f within p. Note that $C(f,1) = C(f)$. If there is some $p \in \mathbb{N}$ with $C(f,p) < \infty$, we say that f is eventually colored. Similarly, we can consider the index $C^+(f,p)$ defined by

$$\min\{|\mathcal{C}|;\ \mathcal{C} \text{ is a closed (open) cover of } X \text{ such that for each } C \in \mathcal{C}, \bigcap_{i=0}^{p} f^i(C) = \emptyset\}.$$

By the definitions, we see that $C(f,p) \leq C^+(f,p)$. In section 3, we show that $C(f,p) = C^+(f,p)$ if X is compact.

In this paper, we need the following notions. A finite cover $\mathcal{C}$ of X is a *closed partition* of X provided that each element C of $\mathcal{C}$ is closed, $int(C) \neq \emptyset$ and $C \cap C' = bd(C) \cap bd(C')$ for any $C, C' \in \mathcal{C}$. Let $\mathcal{B}$ be a collection of subsets of a space X with $\dim X = n < \infty$. Then we say that $\mathcal{B}$ is *in general position in* X (see [7]) provided that if $\mathcal{S} \subset \mathcal{B}$ with $|\mathcal{S}| \leq n+1$, then $\dim(\bigcap\{S |\ S \in \mathcal{S}\}) \leq n - |\mathcal{S}|$. By a *swelling* of a family $\{A_s\}_{s \in S}$ of subsets of a space X, we mean any family $\{B_s\}_{s \in S}$ of subsets of X such that $A_s \subset B_s$ $(s \in S)$ and for every finite set of indices $s_1, s_2, ..., s_m \in S$,

$$\bigcap_{i=1}^{m} A_{s_i} \neq \emptyset \text{ if and only if } \bigcap_{i=1}^{m} B_{s_i} \neq \emptyset.$$

Conversely, for any cover $\{B_s\}_{s \in S}$ of X, a cover $\{A_s\}_{s \in S}$ of X is a *shrinking* of $\{B_s\}_{s \in S}$ if $A_s \subset B_s$ $(s \in S)$. The following facts are well-known;

(1) for any locally finite collection $\mathcal{F}$ of closed subsets of a space X, $\mathcal{F}$ has a swelling consisting of open subsets of X (e.g., see [10, Proposition 3.2.1]) and

(2) for any open cover $\mathcal{U}$ of X, $\mathcal{U}$ has a closed shrinking cover of X (e.g., see [10, Proposition A.7.1]).

Hence we see that if $f : X \to X$ is a closed map and a closed finite cover $\mathcal{B}$ of X is an eventual coloring of f, then we can find an open swelling $\mathcal{C}$ of $\mathcal{B}$ which is an eventual coloring of f.

2 Two indices which evaluate eventual coloring numbers

In this section, we will define indices $\varphi_n(k)$ and $\tau_n(k)$. First, for each $n = 0, 1, 2, ...,$ and each $k = 0, 1, 2, ..., n+1$, we define the index $\varphi_n(k)$ as follows: Put $\varphi_n(0) = 1$. For each $k = 1, 2, ..., n+1$, by induction on k we define the index $\varphi_n(k)$ by

$$\varphi_n(k) = 2\varphi_n(k-1) + [n/(n+2-k)] \cdot (\varphi_n(k-1) + 1),$$

where $[x] = \max\{m \in \mathbb{N} \cup \{0\} | \ m \leq x\}$ for $x \in [0, \infty)$. Next, we will consider another index $\tau_n(k)$ defined by

$$\tau_n(k) = k(2n+1) + 1$$

for each $n = 0, 1, 2, ...,$ and each $k = 0, 1, 2, ..., n+1$. Thus, we obtain following main theorem in this paper.

Theorem 2.1. ([5, Theorem 2.3, Theorem 2.6]) *Let $f : X \to X$ be a fixed-point free homeomorphism of a separable metric space X with $\dim X = n < \infty$. If $\dim P(f) \leq 0$, then*

$$C(f, \min\{\varphi_n(k), \tau_n(k)\}) \leq n + 3 - k$$

for each $k = 0, 1, 2, ..., n+1$.

Remark. If we do not assume $\dim P(f) \leq 0$, the above theorem is not true. Let $f : S^n \to S^n$ be the antipodal map of the n-dimensional sphere S^n. Note that $P(f) = S^n$ and $C(f, p) = C(f, 1) = n + 2$ for any $p \in \mathbb{N}$.

In fact we have the following tables of values of two indices.

$\varphi_n(k)$

n \ k	0	1	2	3	4	5	6
0	1	2	-	-	-	-	-
1	1	2	7	-	-	-	-
2	1	2	7	30	-	-	-
3	1	2	7	22	113	-	-
4	1	2	7	22	90	544	-
5	1	2	7	22	69	278	1951

$\tau_n(k)$

n \ k	0	1	2	3	4	5	6
0	1	2	-	-	-	-	-
1	1	4	7	-	-	-	-
2	1	6	11	16	-	-	-
3	1	8	15	22	29	-	-
4	1	10	19	28	37	46	-
5	1	12	23	34	45	56	67

The way of the construction of $\tau_n(k)$ is similar to way of $\varphi_n(k)$. This way is repainting a color one by one. Thus, we expect that repainting many colors all at once will bring a better index than $\tau_n(k)$ and $\varphi_n(k)$ to us. But this way is very complicated.

Now we have the following general problem for eventual coloring numbers.

Problem 2.2. *For each $n \geq 0$ and each $1 \leq k \leq n+1$, determine the minimal number $\mu_n(k)$ of natural numbers p satisfying the condition; if $f : X \to X$ is any fixed-point free homeomorphism of a separable metric space X such that $\dim X = n$ and $\dim P(f) \leq 0$, then $C(f,p) \leq n+3-k$.*

By comparing two indices $\varphi_n(k)$ and $\tau_n(k)$, we have a partial answer to the above problem.

Corollary 2.3. *Suppose that $f : X \to X$ is a fixed-point free homeomorphism of a separable metric space X and $\dim P(f) \leq 0$.*
(1) If $\dim X = 0$, then $C(f,2) = 2$.
(2) If $\dim X = 1$, then $C(f,7) = 2$.
(3) If $\dim X = 2$, then $C(f,16) = 2$.
(4) If $\dim X = 3$, then $C(f,29) = 2$.
(5) If $\dim X = 4$, then $C(f,46) = 2$.
In other words, $\mu_0(1) = 2, \mu_1(2) \leq 7, \mu_2(3) \leq 16, \mu_3(4) \leq 29$ and $\mu_4(5) \leq 46$.

In addition, we have the following result which is the case $C(f,p) = 2$.

Corollary 2.4. *Let $f : X \to X$ be a fixed-point free homeomorphism of a separable metric space X with $\dim X = n < \infty$. If $\dim P(f) \leq 0$, then there is some $p \in \mathbb{N}$ with $p \leq \min\{\varphi_n(n+1), \tau_n(n+1)\}$ such that*

$$C(f,p) = 2.$$

In other words, X can be divided into two closed subsets C_1, C_2 (i.e., $X = C_1 \cup C_2$) and there is some $p \in \mathbb{N}$ such that if $x \in C_i$ ($i \in \{1,2\}$), there is a strictly increasing sequence $\{n_x(k)\}_{k=1}^{\infty}$ of natural numbers such that $1 \leq n_x(1) \leq p$, $n_x(k+1) - n_x(k) \leq p$ and if $j \in \{1,2\}$ with $j \neq i$, then

$$f^{n_x(k)}(x) \in C_j - C_i \text{ (k:odd)}, \quad f^{n_x(k)}(x) \in C_i - C_j \text{ (k:even)}.$$

3 Eventual coloring numbers on compact metric spaces

In this section, we consider eventual coloring numbers of fixed-point free maps of compact metric spaces. Let X be a compact metric space and let $f : X \to X$ be a map. Consider the inverse limit (X,f) of f, i.e.

$$(X,f) = \{(x_i)_{i=0}^{\infty} |\ x_i \in X, f(x_i) = x_{i-1} \text{ for } i \in \mathbb{N}\} \subset X^{\infty} = \prod_{j=0}^{\infty} X_j.$$

Then we have the shift homeomorphism $\widetilde{f} : (X, f) \to (X, f)$ of f and the natural projection $p_j : (X, f) \to X_j = X$ $(j \geq 0)$ defined by

$$\widetilde{f}((x_i)_{i=0}^{\infty}) = (f(x_i))_{i=0}^{\infty}, \quad p_j((x_i)_{i=0}^{\infty}) = x_j.$$

Note that $p_j \cdot \widetilde{f} = f \cdot p_j$. We see that if $f : X \to X$ is a fixed-point free map of a compact metric space X, then $\widetilde{f} : (X, f) \to (X, f)$ is a fixed-point free homeomorphism. By a modification of the proof of [1, Theorem 6], we have the following theorem which is a more precise result than [1, Theorem 6].

Theorem 3.1. *Let $f : X \to X$ be a fixed-point free map of a compact metric space X and let $\widetilde{f} : (X, f) \to (X, f)$ be the shift homeomorphism of f. Then for $p \in \mathbb{N}$,*

$$C(f, p) = C^+(f, p) = C(\widetilde{f}, p).$$

Corollary 3.2. (cf. [1, Theorem 6]) *Let $f : X \to X$ be a fixed-point free map of a compact metric space X with $\dim X = n < \infty$. If $\dim P(f) \leq 0$, then there is $p \in \mathbb{N}$ with $p \leq \min\{\varphi_n(k), \tau_n(k)\}$ such that*

$$C(f, p) \leq n + 3 - k$$

for each $k = 0, 1, 2, ..., n + 1$.

Example. There are a (zero-dimensional) separable metric space X and a fixed-point free map $f : X \to X$ such that $\dim P(f) \leq 0$ and
(1) f is closed,
(2) f is finite-to-one, and
(3) f cannot be eventually colored within any $p \in \mathbb{N}$.

Remark. In the statement of Theorem 1.2, "a separable metric space X" can be replaced with "a paracompact space X" (see [M. A. van Hartskamp and J. Vermeer, On colorings of maps, Topology and its Applications 73 (1996), 181-190]). Hence Theorem 2.1 is also true for the case that X is a paracompact space.

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