数理解析研究所講究録
第 1783 巻 2012 年 155-168

ON THE CURVATURE OF HOLOMORPHIC LINE BUNDLES WITH PARTIALLY VANISHING COHOMOLOGY.

SHIN-ICHI MATSUMURA

ABSTRACT. The Andreotti-Grauert vanishing theorem asserts, a partial curvature positivity of a holomorphic line bundle implies asymptotic vanishing of certain higher cohomology groups for tensor powers of the line bundle. This report introduces recent results on Demailly-Peternell-Schneider problem, which asks whether the converse implication of the Andreotti-Grauert vanishing theorem holds.

1. INTRODUCTION

On a complex projective manifold, positivity concepts of a (holomorphic) line bundle are important. In particular, in the theory of several complex variables and algebraic geometry, a positive line bundle plays a central role. A positive line bundle is characterized in various ways. For instance, some positive multiple gives an embedding to the projective space (geometric characterization), all higher cohomology groups of some positive multiple are zero (cohomological characterization), and the intersection number with any subvariety is positive (numerical characterization). In this report, we study the characterizations of a q-positive line bundle which is the generalization of a usual positive line bundle. The main purpose of this report is to introduce recent developments on relations between q-positivity and q-ampleness of a line bundle.

Throughout this report, let X be a compact complex manifold of dimension n. Sometimes we may suppose that X is Kähler or projective. First we define q-positivity and q-ampleness of a line bundle. Let L be a holomorphic line bundle on X and q an integer with $0 \leq q \leq n-1$.

Definition 1.1. (1) A holomorphic line bundle L on X is called *q-positive*, if there exists a (smooth) hermitian metric h whose Chern curvature

$\sqrt{-1}\Theta_h(L)$ has at least $(n-q)$ positive eigenvalues at any point on X as a $(1,1)$-form.

(2) A holomorphic line bundle L on X is called (*cohomologically*) *q-ample*, if for any coherent sheaf $\mathcal{F}$ on X there exists a positive integer $m_0 = m_0(\mathcal{F}) > 0$ such that

$$H^i\big(X, \mathcal{F} \otimes \mathcal{O}_X(L^{\otimes m})\big) = 0 \quad \text{for } i > q,\ m \geq m_0.$$

The above definition of a q-ample line bundle may seem to be different from the definition in [DPS96]. However, we have that they are actually same. It is proved in [Tot10, Theorem 7.1]. Andreotti and Grauert gave a relation between q-positive line bundles and q-ample line bundles. They proved that q-positivity of a line bundle leads to q-ampleness. It is so-called the Andreotti-Grauert vanishing theorem.

Theorem 1.2. ([AG62, Théorèm 14], [DPS96, Proposition 2.1]). *A q-positive line bundle is always a q-ample line bundle.*

Note a 0-positive line bundle is equal to a positive line bundle in the usual sense by the definition. It is well-known that a positive line bundle corresponds with an ample line bundle. A line bundle is called *ample*, if the complete linear system of some positive multiple of the line bundle gives an embedding to the projective space. Recall that the Serre vanishing theorem says, an ample line bundle is always 0-ample. Thus the Andreotti-Grauert vanishing theorem can be seen as the generalization of the Serre vanishing theorem. Remark that the converse implication of the Serre vanishing theorem holds, which gives the characterization of ample (positive) line bundles in terms of cohomological properties.

It is of interest to know whether the converse implication of the Andreotti-Grauert theorem holds. It is a natural question, however, it has been an open problem for a long time except the case when $q = 0$. This problem was first posed by Demailly, Peternell and Schneider in [DPS96].

Problem 1.3. ([DPS96]). *If a line bundle is q-ample, is the line bundle q-positive* ?

In this report, we mainly discuss the recent results on Problem 1.3. This report is organized in the following way: In section 2, we introduce the results of [Mat11], which claims Problem 1.3 is affirmatively solved in some situations. For example, the results asserts that the problem is true for any line bundle on a smooth projective surface or, under the assumption that a line bundle is semi-ample. In his paper [Ott11], Ottem gave a counterexample to Problem 1.3 on a higher dimensional manifold by investigating the properties of ample subvarieties. Section 3 is devoted to the study of the counterexample and some observations of ample (positive) subvarieties.

Acknowledgment. The author would like to express his deep gratitude to his supervisor Professor Shigeharu Takayama for useful comments. He also would like to thanks Professor Makoto Abe, Hideaki Kazama and Kazuko Matsumoto for fruitful discussions on Problem 3.2. He is supported by the Grant-in-Aid for Scientific Research (KAKENHI No. 23-7228) and the Grant-in-Aid for JSPS fellows.

2. On partial answers under various situations

2.1. **On line bundles on smooth projective surfaces.** In this subsection, we give the sketch of the proof of the following theorem, which claims Problem 1.3 is affirmative on a smooth projective surface without any assumptions on a line bundle. See [Mat11, Section 2] for the precise argument.

Theorem 2.1. *On a smooth projective surface, the converse of the Andreotti-Grauert vanishing theorem holds. That is, the following conditions are equivalent.*
(A) L *is* 1*-ample.*
(B) L *is* 1*-positive.*

We need to give a numerical characterization of a $(n-1)$-ample line bundle on a smooth projective variety for the proof of the theorem above. For this purpose, we shall establish Proposition 2.2. The equivalence between (1) and (2) in Proposition 2.2 is proved by using the Serre duality.

Thanks to the deep result of [BDPP], the dual cone of (the numerical classes of) pseudo-effective line bundles is equal to the closure of the cone of strongly movable curves. This fact implies the equivalence between (2) and (3) .

Proposition 2.2. *Let L be a line bundle on a smooth projective variety X of dimension n. Then the following properties are equivalent.*
(1) *L is $(n-1)$-ample.*
(2) *The dual line bundle $L^{\otimes -1}$ is not pseudo-effective.*
(3) *There exists a strongly movable curve C on X such that the degree of L on C is positive.*

Here a curve C is called a strongly movable curve if

$$C = \mu_*\big(A_1 \cap \cdots \cap A_{n-1}\big)$$

for suitable very ample divisors A_i on $\tilde{X}$, where $\mu : \tilde{X} \to X$ is a birational morphism. See [BDPP, Definition 1.3] for more details.

On a smooth projective surface, the closure of the cone of strongly movable curves agree with the closure of the cone of ample line bundles (that is, the nef cone). Therefore a 1-ample line bundle on a smooth projective surface is characterized by the intersection number with an ample line bundle as follows:

Corollary 2.3. *Let L be a line bundle on a smooth projective surface X. Then the following properties are equivalent.*
(1) *L is 1-ample.*
(2) *There exists an ample line bundle H on X such that the intersection number $(H \cdot L)$ is positive.*

The difficulty of the proof of Theorem 2.1 is to construct a metric whose curvature is q-positive from numerical properties (such as property (3) in Proposition 2.2 or property (2) in Corollary 2.3). In order to overcome the difficulty, we prove the following theorem by using solutions of Monge-Ampère equations.

Theorem 2.4. *Let L be a line bundle on a compact Kähler manifold X of dimension n and ω a Kähler form on X. Assume that the intersection* *number* $\left(L\cdot\{\omega\}^{n-1}\right)$ *is positive. Then* L *is* $(n-1)$*-positive. That is, there exists a smooth hermitian metric* h *whose Chern curvature* $\sqrt{-1}\Theta_h(L)$ *has at least* 1 *positive eigenvalue at any point on* X.

Here $\{\omega\} \in H^{1,1}(X,\mathbb{R})$ means the cohomology class of ω. It follows Theorem 2.1 from Theorem 2.4 and Corollary 2.3. Further Theorem 2.4 gives the following corollary which can be seen as the generalization of [FO09, Theorem 1] to a pseudo-effective line bundle. In [FO09], Fuse and Ohsawa showed $(n-1)$-positivity of a $\mathbb{Q}$-effective line bundle. They use $(n-1)$-completeness of a non-compact complex manifold in the proof. We make use of Monge-Ampère equations instead of $(n-1)$-completeness of a non-compact complex manifold.

Corollary 2.5. *Let* L *be a pseudo-effective line bundle on a compact Kähler manifold* X. *Assume that the first Chern class* $c_1(L)$ *of* L *is not zero. Then* L *is* $(n-1)$*-positive.*

A pseudo-effective line bundle (which is not numerically trivial) is $(n-1)$-ample (see Proposition 2.2). Therefore it can be expected to be $(n-1)$-positive if the converse of the Andreotti-Grauert theorem holds. Corollary 2.5 asserts that is true at least on a compact Kähler manifold.

2.2. **The case when a line bundle is semi-ample.** In this subsection, the various characterizations of q-positivity of a semi-ample line bundle are given on an arbitrary compact complex manifold. A line bundle is called semi-ample, if the holomorphic global sections of some positive multiple of the line bundle has no common zero set. Thus a semi-ample line bundle gives a holomorphic map to the projective space. See [Laz04] for more details on a semi-ample line bundle. Theorem 2.6 provides the characterization of fibre dimensions of a holomorphic map in terms of q-positivity.

Theorem 2.6. *Let* $\Phi : X \longrightarrow Y$ *be a holomorphic map (possibly not surjective) from* X *to a compact complex manifold* Y. *Then the condition* (B) *implies* (A). *Moreover, if* X *is projective, the converse holds.*

(A) *Fix a Hermitian form* ω (*that is, a positive definite* $(1,1)$*-form*) *on* Y. *Then there exists a function* $\varphi \in C^\infty(X,\mathbb{R})$ *such that the* $(1,1)$*-form* $\Phi^*\omega + dd^c\varphi$ *is* q*-positive* (*that is, the form has at least* $(n-q)$ *positive eigenvalues at any point on* X *as a* $(1,1)$*-form*).

(B) *The map* Φ *has fibre dimensions at most* q.

When the map is the holomorphic map to the projective space associated to a sufficiently large multiple of a semi-ample line bundle, the condition (B) in Theorem 2.6 is equivalent to q-ampleness of L (see [So78, Proposition 1.7]). It leads to the following corollary:

Theorem 2.7. *Assume a line bundle* L *on a compact complex manifold* X *is semi-ample.*

Then the following conditions (A), (B) *and* (C) *are equivalent.*

(A) L *is* q*-positive.*

(B) *The semi-ample fibration of* L *has fibre dimensions at most* q.

(C) L *is* q*-ample.*

Moreover if X *is projective, the conditions above are equivalent to the condition* (D).

(D) *For every subvariety* Z *with* $\dim Z > q$, *there exists a curve* C *on* Z *such that the degree of* L *on* C *is positive.*

The condition (B) (resp.(C), (D)) gives the geometric (resp. cohomological, numerical) characterization of a q-positive line bundle. In particular, the converse of the Andreotti-Grauert theorem holds for a semi-ample line bundle on any compact complex manifold. In the condition (D), the projectivity of X is effectively worked when we construct a curve where the degree of L is positive. Note that the equivalence between the condition (B) and (C) due to [So78]. The original part of [Mat11] is to construct a hermitian metric whose curvature is q-positivity from the condition (B).

2.3. **The case when a line bundle is big.** In this subsection, we consider Zariski-Fujita type theorems (Theorem 2.8) in order to investigate q-positivity of a big line bundle. It reduces q-positivity of a big line bundle to that of the restriction to the non-ample locus. That is, the converse of the Andreotti-Grauert theorem for a big line bundle is reduced to the case of smaller dimensional varieties. See [ELMNP] or [Bou04, Section 3.5] for the definition and properties of a non-ample locus. (Sometimes a non-ample locus is called an augmented base locus or a non-Kähler locus.)

Theorem 2.8. *Assume that L is a big line bundle on a smooth projective variety and the following condition $(*)$ holds.*

$(*)$ *The restriction of L to the non-ample locus $\mathbb{B}_+(L)$ is q-positive.*
Then L is q-positive on X.

Recall that a 0-positive line bundle is a positive line bundle in the usual sense (that is, an ample line bundle). Hence Theorem 2.8 implies that L is ample on X if the restriction of L to the non-ample locus is ample. It can be seen as the parallel to the Zariski-Fujita theorem (see [Zar89] and [Fuj83] for the Zariski-Fujita theorem).

If L is q-positive on X, the restriction to any subvariety on X is always q-positive. However the converse does not hold in general. Theorem 2.8 says the converse holds when a subvariety is equal to the non-ample locus. In his paper [Bro11], Brown showed the similar statement holds for a q-ample line bundle. See [Bro11, Theorem 1.1] for the precise statement. Remark that q-positivity can be defined even if a subvariety has singularities. Therefore a q-positive line bundle can be defined even if the non-ample locus has singularities. See Definition 2.9 for the precise definition.

Definition 2.9. Let V be a subvariety on X. The restriction $L|_V$ of L to V is called q-positive if there exists a real-valued continuous function φ on V with the following condition:

For every point on V, there exist a neighborhood U on X and a C^2-function $\tilde{\varphi}$ on U such that, $\tilde{\varphi}|_{V\cap U} = \varphi$ and the $(1,1)$-form $\sqrt{-1}\Theta_h(L) + dd^c\tilde{\varphi}$ has at least $(n-q)$-positive eigenvalues on U.

When the dimension of the non-ample locus is less than or equal to q, the condition $(*)$ in Theorem 2.8 is automatically satisfied. Thus it follows the corollary from Theorem 2.8.

Corollary 2.10. *Assume the dimension of the non-ample locus of L is less than or equals to q. Then L is q-positive.*

It is known that, L is q-ample under the assumption in Corollary 2.10. (cf.[Kür10], [Mat10, Theorem 1.6]). Corollary 2.10 asserts that q-positivity has the same property.

2.4. **On the relation with Holomorphic Morse inequalities.** In this subsection, we study the asymptotic cohomology of a line bundle, which is defined as follows. It is closely related with the Andreotti-Grauert vanishing theorem.

Definition 2.11. Let L be a line bundle on a compact complex manifold X of dimension n. Then the asymptotic q-cohomology of L is defined to be

$$\hat{h}^q(L) := \limsup_{m\to\infty} \frac{n!}{m^n} h^q(X, \mathcal{O}_X(L^{\otimes m}))$$

In his paper [Dem85], Demailly gave a relation between the dimension of the asymptotic cohomology of a line bundle and certain Monge-Ampère integrals of the curvature. It is so-called Demailly's holomorphic Morse inequality. For simplicity, we assume that X is projective.

Theorem 2.12. ([Dem85]). *For every holomorphic line bund L on a projective manifold X of dimension n, one has the (weak) Morse inequality*

$$\hat{h}^q(L) \leq \inf_{h:\text{hermitian metric on } L} \int_{X(h,q).} (\sqrt{-1}\Theta_h(L))^n(-1)^q,$$

where h runs through smooth hermitian metrics on L, and $X(h,q)$ is the set defined by

$$X(h,q) := \left\{x \in X \mid \sqrt{-1}\Theta_h(L) \text{ has a signature } (n-q,q) \text{ at } x.\right\}.$$

The holomorphic Morse inequality would be seen as an asymptotic version of the Andreotti-Grauert vanishing theorem. In his paper [Dem10-A], Demailly conjectured that the inequality would actually be an equality. The conjecture has the similarity to Problem 1.3. He proved the converse of holomorphic Morse inequalities on surfaces in [Dem10-B]. Its result can be seen as a "partial" converse of the Andreotti-Grauert theorem.

2.5. **Example.** Thanks to Theorem 2.1, a 1-ample line bundle is always 1-positive on a smooth projective surface. However, in general, it is difficult to construct a concrete metric whose curvature is 1-positive. Thus it seems to be worth collecting examples which can be explicitly computed. This subsection is devoted to give such examples. For simplicity, we use an additional notation for line bundles in this subsection.

Example 2.13. Let X be the product of two 1-dimensional projective spaces. Denote by $p_i : X \to \mathbb{P}^1$, the i-th projection ($i = 1, 2$). Then a line bundle L on X can be written as

$$L_{(a,b)} = a\, p_1^*\mathcal{O}_{\mathbb{P}^1}(1) + b\, p_2^*\mathcal{O}_{\mathbb{P}^1}(1)$$

with integers a, b. Here $\mathcal{O}_{\mathbb{P}^1}(1)$ is the hyperplane bundle on $\mathbb{P}^1$. From a simple computation (or Corollary 2.3), L (which parametrized by integers a, b) is 1-ample if and only if $a > 0$ or $b > 0$. Then a metric on L which is induced by the pullback of suitable multiple of the Fubini-study metric has a 1-positive curvature.

Example 2.14. Let E be an elliptic curve. We set $X := E \times E$ with projections $p_i : X \to E$ ($i = 1, 2$). We consider line bundles

$$F_1 := p_1^*(\mathcal{O}_E(p)), \qquad F_2 := p_2^*(\mathcal{O}_E(p)), \qquad \Gamma := \mathcal{O}_X(\Delta),$$

where p is a point on E and $\Delta \subset X = E \times E$ is the diagonal divisor. It is known that an arbitrary line bundle on X can be written as a linear combination of F_1, F_2 and Γ. Thanks to Proposition 2.2, L is 1-ample if and only if $-L$ is not pseudo-effective. Since the automorphism group of X (which is a connected algebraic group) acts transitively on X, the pseudo-effective cone corresponds with the nef cones. Thus, the line bundle L is 1-ample if and only if $(L^2) < 0$ or $(L \cdot A) > 0$, where A is an ample line bundle such as $A := F_1 + F_2 + \Gamma$. The intersection numbers among F_1, F_2 and Γ can be computed as follows:

$$(\Gamma \cdot F_1) = (\Gamma \cdot F_2) = (F_1 \cdot F_2) = 1,$$
$$(\Gamma^2) = (F_1^2) = (F_2^2) = 0.$$

By the argument above, we have the following proposition.

Proposition 2.15. *A line bundle $L = aF_1 + bF_2 + c\Gamma$ is 1-ample if and only if*

$$a + b + c > 0 \quad \textit{or}$$
$$ab + bc + ca < 0.$$

Now we construct a metric on L whose curvature is 1-positive under the condition above on a, b and c. Denote by h, a hermitian metric on $\mathcal{O}_E(p)$ such that the pull-back of the Chern curvature by the universal covering $\mathbb{C} \to E$ is equal to $du \wedge d\bar{u}$. Here u is a (standard) coordinate on $\mathbb{C}$. On the other hand, we can construct a metric k on Γ whose Chern curvature can be written as:

$$\begin{aligned}\Theta_k(\Gamma) &= dd^c|z - w|^2 \\ &= dz \wedge d\bar{z} + dw \wedge d\bar{w} - dz \wedge d\bar{w} - dw \wedge d\bar{z}.\end{aligned}$$

Here (z, w) is a local coordinate on X which is induced by the universal covering $\mathbb{C}^2 \to X$. Then the Chern curvature of

$$L = aF_1 + bF_2 + c\Gamma$$

associated to a metric $p_1^*(h^{\otimes a}) \otimes p_2^*(h^{\otimes b}) \otimes k^{\otimes c}$ is

$$(a + c)dz \wedge d\bar{z} + (b + c)dw \wedge d\bar{w} - cdz \wedge d\bar{w} - cdw \wedge d\bar{z}.$$

Eigenvalues of the curvature are solutions of the equation

$$\det \begin{pmatrix} (a + c) - x & -c \\ -c & (b + c) - x \end{pmatrix} = 0.$$

The a necessary and sufficient condition that the equation has at least 1-positive solution is

$$a + b + 2c > 0 \quad \text{or}$$
$$ab + bc + ca < 0.$$

It is easy to see that this condition is equivalent to the condition in Proposition 2.15.

3. Counter examples to Problem 1.3

In this subsection, we study Ottem's counterexample to Problem 1.3. Further we investigate the converse implication of Andreotti-Grauert vanishing theorem on a non-compact manifold.

By the (classical) Andreotti-Grauert vanishing theorem, a q-complete complex space is always cohomologically q-complete. Let us confirm the definitions. Let M be a non-compact, irreducible and reduced analytic space of dimension n and q an integer with $0 \leq q \leq (n-1)$.

Definition 3.1. (1) M is called *q-complete*, if there exists a (smooth) exhaustive function $\varphi \in C^\infty(M, \mathbb{R})$ whose Levi-form $\sqrt{-1}\partial\overline{\partial}\varphi$ has at least $(n-q)$ positive eigenvalues at any point on M as a $(1,1)$-form.
(2) M is called *cohomologically q-complete*, if for any coherent sheaf $\mathcal{F}$ on M,

$$H^i(M, \mathcal{F}) = 0 \quad \text{for } i > q.$$

It is natural to ask whether the converse implication holds. It is a non-compact version of Problem 1.3.

Problem 3.2. *If M is cohomologically q-complete, is M q-complete ?*

In their paper [ES80], Eastwood and Suria proved that the problem above is affirmatively solved, if M is a domain with a smooth boundary in a Stein manifold. Another proof is given for a domain with a smooth boundary in $\mathbb{C}^n$ in [Wat94].

It is well-known that any non-compact complex space of dimension n is cohomologically $(n-1)$-compete. If Problem 3.2 is true, any non-compact complex space of dimension n should be $(n-1)$-compete. In the case when complex space is non-singular, Greene and Wu proved $(n-1)$-completeness of non-compact analytic space in [GW75]. In the case when complex space has singularities, that is proved by Ohsawa (see [Oh84]).

In this section, we show that the observation for Ottem's example gives a counterexample to Problem 1.3. See [Ott11, Section 10] for the example. The originality of the counterexample is due to Ottem.

Proposition 3.3. *For a pair (n, q) of positive integers with $n/2 - 1 < q < n-2$, there exists a complex manifold M of dimension n such that M is cohomologically q-complete, but not q-complete. In particular, Problem 3.2 is negative in general.*

Proof. We give the proof only in the case when $(n, q) = (4, 1)$. (A slight change in the proof gives the proof of other cases.)

We consider a smooth Enriques surface S in the projective space $\mathbb{P}^4$. Then we shall show that the complement $\mathbb{P}^4 \setminus S$ is cohomologically 1-complete, but not 1-complete. We denote by M, the complement $\mathbb{P}^4 \setminus S$. Since S is an Enriques surface, the fundamental group $\pi_1(S)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Therefore, we have $H^1(S, \mathbb{Q}) = 0$, (which is isomorphic to $H^1(\mathbb{P}^4, \mathbb{Q})$). Thus we can conclude that M is cohomologically 1-complete from [Og73, Theorem 4.4].

It remains to show that M is not 1-complete. We assume that M is 1-complete for a contradiction. By the definition, there exists an exhaustive function $\varphi \in C^\infty(M, \mathbb{R})$ such that $\sqrt{-1}\partial\bar{\partial}\varphi$ has at least 3 positive eigenvalues at any point on M. We can assume that $\varphi \geq 0$ since φ is exhaustive. Now we consider a function f on $\mathbb{P}^4$ which is defined to be

$$f := \begin{cases} 1/\varphi & \text{if } x \notin S, \\ 0 & \text{others.} \end{cases}$$

A simple computation implies that the critical points of f on M are equal to that of φ. Therefore we have

$$\sqrt{-1}\partial\bar{\partial}f = \frac{-\sqrt{-1}\partial\bar{\partial}\varphi}{\varphi^2}$$

at the critical points of φ on M. It implies that the index (the number of negative eigenvalues of the Hessian) at the critical points is greater than or equal to 3. Note that the index of the Hessian of a smooth function is equal to the number of the negative eigenvalues of the Levi-form. Therefore by applying the standard Morse theory, for an arbitrary number $\delta > 0$ we have that, X is obtained from W_δ by successively attaching cells of dimension ≥ 3. Here W_δ is $f^{-1}([0,\delta])$. In particular, we have the isomorphism $\pi_i(W_\delta, S) \cong \pi_i(\mathbb{P}^4, S)$ for $i = 0, 1, 2$ for any $\delta > 0$. Since we triangulate $\mathbb{P}^4$ with S as a subcomplex, we can take a neighborhood U of S which deformation retracts onto S. Since φ is an exhaustive function, f is continuous. Thus, W_δ is contained in U for a sufficiently small $\delta > 0$, since f has a positive valued on M. Then we have the following commutative diagram

$$\begin{tikzcd} & \pi_i(\mathbb{P}^4, S) & \\ \pi_i(W_\delta, S) \arrow[ur, "\cong"] \arrow[rr] & & \pi_i(U, S). \arrow[ul] \end{tikzcd}$$

The diagonal map on the left is an isomorphism for $i = 0, 1, 2$. Since U can retracts onto S, we have $\pi_i(U, S) = 0$ for any i. Therefore we obtain $\pi_i(\mathbb{P}^4, S) = 0$ for $i = 0, 1, 2$.

By the argument above,we have that, if M is 1-complete, then the map $\pi_1(S) \to \pi_1(\mathbb{P}^4)$ (which is induced by the inclusion map) should be an isomorphism. However, since $\mathbb{P}^4$ is simply connected, it is a contradiction. $\square$

References

[AG62] Andreotti A, Grauert H. *Théorème de finitude pour la cohomologie des espaces complexes.* Bull. Soc. Math. France **90** (1962), 193–259.

[Ba80] Barlet, D. *Convexité au voisinage d'un cycle.* (French) Functions of several complex variables, IV (Sem. Francois Norguet, 1977–1979) (French), pp. 102–121, Lecture Notes in Math., **807** , Springer, Berlin, 1980.

[BDPP] Boucksom S, Demailly J P, Paun M, Peternell T. *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension.* preprint, arXiv:math/0405285v1.

[Bro11] Brown M V. *Big q-ample Line Bundles.* preprint, arXiv:1105.3449v1, to appear in Compositio Mathematica.

[Bou04] Boucksom S. *Divisorial Zariski decompositions on compact complex manifolds.* Ann. Sci. Ecole Norm. Sup. (4) **37** (2004), no. 1, 45–76.

[Dem85] Demailly J P. *Champs magnétiques et inégalités de Morse pour la d''-cohomologie.* Ann. Inst. Fourier (Grenoble) **35** (1985), 189-229

[Dem10-A] Demailly J P. *Holomorphic Morse inequalities and asymptotic cohomology groups: a tribute to Bernhard Riemann.* Milan J. Math. **78** (2010), no. 1, 265–277.

[Dem10-B] Demailly J P. *A converse to the Andreotti-Grauert theorem.* preprint, arXiv:1011.3635v2.

[DPS96] Demailly J P, Peternell T, Schneider M. *Holomorphic line bundles with partially vanishing cohomology.* Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), 165–198, Israel Math. Conf. Proc., **9**, Bar-Ilan Univ 1996.

[ELMNP] Ein L, Lazarsfeld R, Muştă M, Nakamaye M, Popa M. *Asymptotic invariants of base loci.* Ann. Inst. Fourier (Grenoble) **56** (2006), no. 6, 1701–1734.

[ES80] Eastwood M G, Suria G V. *Cohomologically complete and pseudoconvex domains.* Comment. Math. Helv. **55** (1980), no. 3, 413–426.

[FO09] Fuse H, Ohsawa T. *On a curvature property of effective divisors and its application to sheaf cohomology.* Publ. Res. Inst. Math. Sci. **45** (2009), no. 4, 1033–1039.

[Fuj83] Fujita T. *Semipositive line bundles.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. **30** (1983), no. 2, 353–378.

[GW75] Greene R E, Wu H. *Embedding of open Riemannian manifolds by harmonic functions.* Ann. Inst. Fourier (Grenoble) **25** (1975), no. 1, vii, 215–235.

[Kür10] Küronya A. *Positivity on subvarieties and vanishing of higher cohomology.* preprint, arXiv:1012.1102v1.

[Laz04] Lazarsfeld R. *Positivity in Algebraic Geometry I.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Vols. 48-49., Springer Verlag, Berlin, 2004.

[Mat10] Matsumura S. *Restricted volumes and divisorial Zariski decompositions.* preprint, arXiv:1005.1503v1.

[Mat11] Matsumura S. *Asymptotic cohomology vanishing and a converse to the Andreotti-Grauert theorem on a surface.* preprint, arXiv:1104.5313v1.

[Og73] Ogus A. *Local cohomological dimension of algebraic varieties.* Ann. of Math. (2) **98** (1973), 327–365.

[Oh84] Ohsawa T. *Completeness of noncompact analytic spaces.* Publ. Res. Inst. Math. Sci. **20** (1984), no. 3, 683–692.

[Ott11] Ottem J C. *Ample subvarieties and q-ample divisors.* preprint, arXiv:1105.2500v1.

[So78] Sommese A J. *Submanifolds of Abelian varieties.* Math. Ann. **233** (1978), no. 3, 229–256.

[Tot10] Totaro B. *Line bundles with partially vanishing cohomology.* preprint, arXiv:1007.3955v1, to appear in Journal of the European Mathematical Society.

[Wat94] Watanabe K. *Vanishing cohomology and q-pseudoconvexity* Proceedings of the Second Korean-Japanese Colloquium on Finite or Infinite Dimensional Complex Analysis, Fukuoka, Japan, 27–29 July (1994), 251–264.

[Yau78] Yau S T. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation.* I. Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411.

[Zar89] Zariski O. *The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface.* Ann. of Math. (2) **76** 1962 560–615.

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO, 3-8-1 KOMABA, TOKYO, 153-8914, JAPAN.

E-mail address: `shinichi@ms.u-tokyo.ac.jp`