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# ASYMPTOTIC ANALYSIS OF OSCILLATORY INTEGRALS AND LOCAL ZETA FUNCTIONS

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## 1. Introduction

In this article, we announce some results in the paper [4].

We investigate oscillatory integrals, that is, integrals of the form

$$I(\tau)=\int_{\mathbb{R}^n} e^{i\tau f(x)}\varphi(x)\chi(x)dx, \tag{1.1}$$

for large values of the real parameter $\tau$, where $f$, $\varphi$, $\chi$ are real-valued smooth functions defined on $\mathbb{R}^n$ and $\chi$ is a cut-off function with small support which identically equals one in a neighborhood of the origin in $\mathbb{R}^n$. Here $f$ and $\varphi\chi$ are called the *phase* and the *amplitude*, respectively.

By the principle of stationary phase, the main contribution in the behavior of the integral (1.1) as $\tau\to+\infty$ is given by the local properties of the phase around its critical points. We assume that the phase has a critical point at the origin, i.e., $\nabla f(0)=0$. The following deep result has been obtained by using Hironaka's resolution of singularities [17] (cf. [21]). If $f$ is real analytic on a neighborhood of the origin and the support of $\chi$ is contained in a sufficiently small neighborhood of the origin, then the integral $I(\tau)$ has an asymptotic expansion of the form

$$I(\tau)\sim e^{i\tau f(0)}\sum_{\alpha}\sum_{k=1}^{n}C_{\alpha k}\tau^{\alpha}(\log\tau)^{k-1}\quad \text{as } \tau\to+\infty, \tag{1.2}$$

where $\alpha$ runs through a finite number of arithmetic progressions, not depending on $\varphi$ and $\chi$, which consist of negative rational numbers. Our interest focuses the largest $\alpha$ occurring in (1.2). Let $S(f,\varphi)$ be the set of pairs $(\alpha,k)$ such that for each neighborhood of the origin in $\mathbb{R}^n$, there exists a cut-off function $\chi$ with support in this neighborhood for which $C_{\alpha k}\neq 0$ in the asymptotic expansion (1.2). We denote by $(\beta(f,\varphi),\eta(f,\varphi))$ the maximum of the set $S(f,\varphi)$ under the lexicographic ordering, i.e. $\beta(f,\varphi)$ is the maximum of values $\alpha$ for which we can find $k$ so that $(\alpha,k)$ belongs to $S(f,\varphi)$; $\eta(f,\varphi)$ is the maximum of integers $k$ satisfying that $(\beta(f,\varphi),k)$ belongs to $S(f,\varphi)$. We call $\beta(f,\varphi)$ *oscillation index* of $(f,\varphi)$ and $\eta(f,\varphi)$ its *multiplicity*. (This multiplicity, less one, is equal to the corresponding multiplicity in [1],p.183.)

From various points of view, the following is an interesting problem: What kind of information of the phase and the amplitude determines (or estimates) the oscillation index $\beta(f,\varphi)$ and its multiplicity $\eta(f,\varphi)$? There have been many interesting studies concerning this problems ([26],[7],[23],[8],[6],[14],[15],[16], etc.). In particular, the significant work of Varchenko [26] shows the following by using the theory of toric varieties: By the geometry of the Newton polyhedron of $f$, the oscillation index can be estimated and, moreover, this index and its multiplicity can be exactly determined when $\varphi(0) \neq 0$, under a certain nondegenerate condition of the phase (see Theorem 2.1 in Section 2). Since his study, the investigation of the behavior of oscillatory integrals has been more closely linked with the theory of singularities. Refer to the excellent expositions [1],[20] for studies in this direction. Besides [26], recent works of Greenblatt [12],[13],[14],[15],[16] are also interesting. He explores a certain resolution of singularities, which is obtained from an elementary method, and investigates the asymptotic behavior of $I(\tau)$. His analysis is also available for a wide class of phases without the above nondegenerate condition.

We generalize and improve the above results of Varchenko [26]. To be more precise, we are especially interested in the behavior of the integral (1.1) as $\tau \to +\infty$ when $\varphi$ has a zero at a critical point of the phase. Indeed, under some assumptions, we obtain more accurate results by using the Newton polyhedra of not only the phase but also the amplitude. Closely related issues have been investigated by Arnold, Gusein-Zade and Varchenko [1] and Pramanik and Yang [22], and they obtained similar results to ours. From the point of view of our investigations, their results will be reviewed in Remark 2.8 in Section 2 and Section 6.4. In our results, delicate geometrical conditions of the Newton polyhedra of the phase and the amplitude affect the behavior of oscillatory integrals. There exist some faces of the Newton polyhedron of the amplitude, which play a crucial role in determining the oscillation index and its multiplicity. Furthermore, in order to determine the oscillation index in general, we need not only geometrical properties of their Newton polyhedra but also information about the coefficients of the terms, corresponding to the above faces, in the Taylor series of the amplitude.

It is known (see, for instance, [18],[20],[1], and Section 5 in this paper) that the asymptotic analysis of oscillatory integral (1.1) can be reduced to an investigation of the poles of the functions $Z_+(s)$ and $Z_-(s)$ (see (4.1) below), which are similar to the local zeta function

$$Z(s) = \int_{\mathbb{R}^n} |f(x)|^s \varphi(x)\chi(x)dx, \tag{1.3}$$

where $f$, $\varphi$, $\chi$ are the same as in (1.1) with $f(0) = 0$. The substantial analysis in this paper is to investigate the properties of poles of the local zeta function $Z(s)$

and the functions $Z_{\pm}(s)$ by using the Newton polyhedra of the functions $f$ and $\varphi$. See Section 4 for more details.

Many problems in analysis, including partial differential equations, mathematical physics, harmonic analysis and probability theory, lead to the need to study the behavior of oscillatory integrals of the form (1.1) as $\tau \to +\infty$. We explain the original motivation for our investigation. In the function theory of several complex variables, it is an important problem to understand boundary behavior of the Bergman kernel for pseudoconvex domains. In [19], the special case of domains of finite type is considered and the behavior as $\tau \to +\infty$ of the Laplace integral

$$\tilde{I}(\tau) = \int_{\mathbb{R}^n} e^{-\tau f(x)} \varphi(x) dx$$

plays an important role in boundary behavior of the above kernel. Here $f$,$\varphi$ are $C^\infty$ functions satisfying certain conditions. The computation of asymptotic expansion of the above kernel in [19] requires precise analysis of $\tilde{I}(\tau)$ when $\varphi$ has a zero at the critical point of $f$. Our analysis in this paper can be applied to the case of the above Laplace integrals. See also [2],[3].

*Notation and Symbols.*

- We denote by $\mathbb{Z}_+, \mathbb{Q}_+, \mathbb{R}_+$ the subsets consisting of all nonnegative numbers in $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, respectively.
- We use the multi-index as follows. For $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}_+^n$, define
$$|x| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}, \quad \langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n,$$
$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \langle \alpha \rangle = \alpha_1 + \cdots + \alpha_n.$$
- For $A, B \subset \mathbb{R}^n$ and $c \in \mathbb{R}$, we set
$$A + B = \{a + b \in \mathbb{R}^n; a \in A \text{ and } b \in B\}, \quad c \cdot A = \{ca \in \mathbb{R}^n; a \in A\}.$$
- We express by $\mathbf{1}$ the vector $(1, \dots, 1)$ or the set $\{(1, \dots, 1)\}$.
- For a $C^\infty$ function $f$, we denote by Supp($f$) the support of $f$, i.e., Supp($f$)= $\overline{\{x \in \mathbb{R}^n; f(x) \neq 0\}}$.

## 2. Definitions and main results

2.1. **Newton polyhedra.** Let us explain some necessary notions to state our main theorems.

Let $f$ be a real-valued $C^\infty$ function defined on a neighborhood of the origin in $\mathbb{R}^n$, which has the Taylor series $\sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha x^\alpha$ at the origin. Then, the *Taylor support* of $f$ is the set $S_f = \{\alpha \in \mathbb{Z}_+^n; c_\alpha \neq 0\}$ and the *Newton polyhedron* of $f$ is the integral polyhedron:

$$\Gamma_+(f) = \text{the convex hull of the set } \bigcup \{\alpha + \mathbb{R}_+^n; \alpha \in S_f\} \text{ in } \mathbb{R}_+^n$$

(i.e., the intersection of all convex sets which contain $\bigcup\{\alpha+\mathbb{R}^n_+;\alpha\in S_f\}$). The union of the compact faces of the Newton polyhedron $\Gamma_+(f)$ is called the *Newton diagram* $\Gamma(f)$ of $f$, while the topological boundary of $\Gamma_+(f)$ is denoted by $\partial\Gamma_+(f)$. The *principal part* of $f$ is defined by $f_0(x)=\sum_{\alpha\in\Gamma(f)\cap\mathbb{Z}^n_+}c_\alpha x^\alpha$. For a compact subset $\gamma\subset\partial\Gamma_+(f)$, let $f_\gamma(x)=\sum_{\alpha\in\gamma\cap\mathbb{Z}^n_+}c_\alpha x^\alpha$. $f$ is said to be *nondegenerate* over $\mathbb{R}$ with respect to the Newton polyhedron $\Gamma_+(f)$ if for every compact face $\gamma\subset\Gamma(f)$, the polynomial $f_\gamma$ satisfies

$$\nabla f_\gamma=\left(\frac{\partial f_\gamma}{\partial x_1},\dots,\frac{\partial f_\gamma}{\partial x_n}\right)\neq(0,\dots,0)\quad\text{on the set }\{x\in\mathbb{R}^n;x_1\cdots x_n\neq 0\}.$$

$f$ is said to be *convenient* if the Newton diagram $\Gamma(f)$ intersects all the coordinate axes.

Let $f,\varphi$ be real-valued $C^\infty$ functions defined on a neighborhood of the origin in $\mathbb{R}^n$ and assume that $\Gamma(f)$ and $\Gamma(\varphi)$ are nonempty. We define the *Newton distance* of $(f,\varphi)$ by

$$d(f,\varphi)=\min\{d>0;d\cdot(\Gamma_+(\varphi)+\mathbf{1})\subset\Gamma_+(f)\}. \tag{2.1}$$

It is easy to see $d(f,\varphi)=\max\{d>0;\partial\Gamma_+(f)\cap d\cdot(\Gamma_+(\varphi)+\mathbf{1})\neq\emptyset\}$. The number $d(f,\varphi)$ corresponds to what is called the *coefficient of inscription* of $\Gamma_+(\varphi)$ in $\Gamma_+(f)$ in [1], p 254. (Their definition in [1] must be slightly modified.) Let $\Gamma(\varphi,f)$ be the subset in $\mathbb{R}^n$ defined by

$$\Gamma(\varphi,f)+\mathbf{1}=\left(\frac{1}{d(f,\varphi)}\cdot\partial\Gamma_+(f)\right)\cap(\Gamma_+(\varphi)+\mathbf{1}).$$

In the above definition, $\partial\Gamma_+(\varphi)$ can be used instead of $\Gamma_+(\varphi)$. Note that $\Gamma(\varphi,f)$ is some union of faces of $\Gamma_+(\varphi)$.

Let $\Gamma^{(k)}$ be the union of $k$-dimensional faces of $\Gamma_+(f)$. Then $\Gamma_+(f)$ is stratified as $\Gamma^{(0)}\subset\Gamma^{(1)}\subset\cdots\subset\Gamma^{(n-1)}(=\partial\Gamma_+(f))\subset\Gamma^{(n)}(=\Gamma_+(f))$. Let $\tilde\Gamma^{(k)}=\Gamma^{(k)}\setminus\Gamma^{(k-1)}$ for $k=1,\dots,n$ and $\tilde\Gamma^{(0)}=\Gamma^{(0)}$. A map $\rho_f:\Gamma_+(f)\to\{0,1,\dots,n\}$ is defined as $\rho_f(\alpha)=k$ if $\alpha\in\tilde\Gamma^{(n-k)}$. In other words, $\rho_f(\alpha)$ is the codimension of the face of $\Gamma_+(f)$, whose relative interior contains the point $\alpha$. We define the *Newton multiplicity* of $(f,\varphi)$ by

$$m(f,\varphi)=\max\{\rho_f(d(f,\varphi)(\alpha+\mathbf{1}));\ \alpha\in\Gamma(\varphi,f)\}.$$

Let $\Gamma_0$ be the subset of $\Gamma(\varphi,f)$ defined by

$$\Gamma_0=\{\alpha\in\Gamma(\varphi,f);\rho_f(d(f,\varphi)(\alpha+\mathbf{1}))=m(f,\varphi)\},$$

which is called the *essential set* on $\Gamma(\varphi,f)$. Note that $\Gamma_0$ is a disjoint union of faces of $\Gamma_+(\varphi)$.

Consider the case $\varphi(0)\neq0$. Then $\Gamma_+(\varphi)=\mathbb{R}^n_+$. In this case, $d(f,\varphi)$ and $m(f,\varphi)$ are denoted by $d_f$ and $m_f$, respectively. (Note that $d(f,\varphi)\le d_f$ for general $\varphi$.) It is easy to see that the point $q=(d_f,\dots,d_f)$ is the intersection of the line $\alpha_1=\cdots=\alpha_n$ in $\mathbb{R}^n$ and $\partial\Gamma_+(f)$, and that $m_f=\rho_f(q)$. $\Gamma(\varphi,f)=\Gamma_0=\{0\}$. More generally, in the case that $\Gamma_+(\varphi) = \{p\} + \mathbb{R}_+^n$ with $p \in \mathbb{Z}_+^n$, the geometrical meanings of the quantities $d(f,\varphi)$ and $m(f,\varphi)$ will be considered in Proposition 4.2 below.

2.2. **Results on oscillatory integrals.** Let us explain our results relating to the behavior of the oscillatory integral $I(\tau)$ in (1.1) as $\tau \to +\infty$.

Throughout this subsection, $f$, $\varphi$, $\chi$ satisfy the following conditions: Let $U$ be an open neighborhood of the origin in $\mathbb{R}^n$.

(A) $f : U \to \mathbb{R}$ is a real analytic function satisfying that $f(0) = 0$, $|\nabla f(0)| = 0$ and $\Gamma(f) \neq \emptyset$;
(B) $\varphi : U \to \mathbb{R}$ is a $C^\infty$ function satisfying $\Gamma(\varphi) \neq \emptyset$;
(C) $\chi : \mathbb{R}^n \to \mathbb{R}_+$ is a $C^\infty$ function which identically equals one in some neighborhood of the origin and has a small support which is contained in $U$.

As mentioned in the Introduction, it is known that the oscillatory integral (1.1) has an asymptotic expansion of the form (1.2). Before stating our results, we recall a part of famous results due to Varchenko in [26]. In our language, they are stated as follows.

**Theorem 2.1** (Varchenko [26]). *Suppose that $f$ is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron. Then*

(i) *$\beta(f,\varphi) \leq -1/d_f$ for any $\varphi$;*
(ii) *If $\varphi(0) \neq 0$ and $d_f > 1$, then $\beta(f,\varphi) = -1/d_f$ and $\eta(f,\varphi) = m_f$;*
(iii) *The progression $\{\alpha\}$ in (1.2) belongs to finitely many arithmetic progressions, which are obtained by using the theory of toric varieties based on the geometry of the Newton polyhedron $\Gamma_+(f)$. (See Remark 2.6, below.)*

Now, let us explain our results. First, we investigate more precise situation in the estimate in the part (i) of Theorem 2.1. Indeed, when $\varphi$ has a zero at the origin, the oscillation index $\beta(f,\varphi)$ can be more accurately estimated by using the Newton distance $d(f,\varphi)$, which is called "the coefficient of inscription of $\Gamma_+(\varphi)$ in $\Gamma_+(f)$" in [1].

**Theorem 2.2.** *Suppose that* (i) *$f$ is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron and* (ii) *at least one of the following conditions is satisfied:*

(a) *$f$ is convenient;*
(b) *$\varphi$ is convenient;*
(c) *$\varphi$ is real analytic on $U$;*
(d) *$\varphi$ is expressed as $\varphi(x) = x^p \tilde{\varphi}(x)$ on $U$, where $p \in \mathbb{Z}_+^n$ and $\tilde{\varphi}$ is a $C^\infty$ function defined on $U$ with $\tilde{\varphi}(0) \neq 0$.*

*Then, we have $\beta(f,\varphi) \leq -1/d(f,\varphi)$.*

*Remark* 2.3. A more precise estimate for $I(\tau)$ is obtained as follows: If the support of $\chi$ is contained in a sufficiently small neighborhood of the origin, then there exists a positive constant $C$ independent of $\tau$ such that

$$|I(\tau)| \leq C\tau^{-1/d(f,\varphi)}(\log \tau)^{A-1} \quad \text{for } \tau \geq 1,$$

where

$$A := \begin{cases} m(f,\varphi) & \text{if } 1/d(f,\varphi) \text{ is not an integer,} \\ \min\{m(f,\varphi)+1, n\} & \text{otherwise.} \end{cases}$$

*Remark* 2.4. Let us consider the above theorem under the assumptions (i), (ii)-(d) without the condition: $\tilde{\varphi}(0) \neq 0$. Then the estimate $\beta(f,\varphi) \leq -1/d(f,\varphi)$ does not always hold. In fact, consider the two-dimensional example: $f(x_1,x_2) = x_1^2$, $\varphi(x_1,x_2) = x_1^2(x_1^2 + e^{-1/x_2^2})$. The proof of Theorem 2.2, however, implies that the estimate $\beta(f,\varphi) \leq -1/d(f,x^p)$ holds under the above assumptions. This assertion with $p = (0,\dots,0)$ shows the assertion (i) in Theorem 2.1.

Vassiliev [25] obtained a similar result to that in the case of (d).

*Remark* 2.5. The condition (d) implies $\Gamma_+(\varphi) = \{p\} + \mathbb{R}_+^n$. When $\varphi$ is a $C^\infty$ function, however, the converse is not true in general. We give an example in Section 6.2, which shows that the assumption (d) cannot be replaced by the condition: $\Gamma(\varphi) = \{p\} + \mathbb{R}_+^n$ in Theorem 2.2.

*Remark* 2.6. From the proof of the above theorem, we can see that under the same condition, the progression $\{\alpha\}$ in (1.2) is contained in the set

$$\left\{ -\frac{\tilde{l}(a) + \langle a \rangle + \nu}{l(a)}; a \in \tilde{\Sigma}^{(1)}, \nu \in \mathbb{Z}_+ \right\} \cup (-\mathbb{N}),$$

where the symbols $l(a)$, $\tilde{l}(a)$ and $\tilde{\Sigma}^{(1)}$ are as in Theorem 4.7, below. This explicitly shows the assertion (iii) in Theorem 2.1.

Next, let us give an analogous result to the part (ii) in Theorem 2.1, due to Varchenko. Indeed, the following theorem deals with the case that the equation $\beta(f,\varphi) = -1/d(f,\varphi)$ holds.

**Theorem 2.7.** *Suppose that* (i) *$f$ is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron,* (ii) *at least one of the following two conditions is satisfied:*

(a) *$d(f,\varphi) > 1$;*

(b) *$f$ is nonnegative or nonpositive on $U$,*

*and* (iii) *at least one of the following two conditions is satisfied:*

(c) *$\varphi$ is expressed as $\varphi(x) = x^p\tilde{\varphi}(x)$ on $U$, where every component of $p \in \mathbb{Z}_+^n$ is even and $\tilde{\varphi}$ is a $C^\infty$ function defined on $U$ with $\tilde{\varphi}(0) \neq 0$;*

(d) *$f$ is convenient and $\varphi_{\Gamma_0}$ is nonnegative or nonpositive on $U$.*

*Then the equations $\beta(f,\varphi) = -1/d(f,\varphi)$ and $\eta(f,\varphi) = m(f,\varphi)$ hold.*

*Remark* 2.8. Considering the assumptions: (i), (ii)-(a), (iii)-(c) with $p = (0,\dots,0)$ in the above theorem, we see the assertion (ii) in Theorem 2.1.

Pramanik and Yang [22] obtained a similar result in the case that the dimension is two and $\varphi(x) = |g(x)|^\epsilon$ where $g$ is real analytic and $\epsilon$ is positive. Their result in Theorem 3.1 (a) does not need any additional assumptions. We explain this reason roughly. They use the *weighted Newton distance*, whose definition is different from our Newton distance. The definition of their distance is more intrinsic and is based on a good choice of coordinate system, which induces a clear resolution of singularity. Moreover, the nonnegativity of $\varphi$ implies the positivity of the coefficient of the expected leading term of the asymptotic expansion (1.2). On the other hand, in our case, the corresponding coefficient possibly vanishes without the assumption (c) or (d). See Sections 6.1 and 6.3.

*Remark* 2.9. If $\varphi(0) = 0$ and $\varphi$ takes the local minimal (resp. the local maximal) at the origin, then $\varphi_{\Gamma_0}$ is nonnegative (resp. nonpositive) on some neighborhood of the origin.

*Remark* 2.10. It is easy to show that Theorem 2.7 can be rewritten in a slightly stronger form by replacing the condition (c) by the following (c$'$):

(c$'$) $\varphi$ is expressed as $\varphi(x) = \sum_{j=1}^{l} x^{p_j}\tilde{\varphi}_j(x)$ on $U$, where $p_j \in \mathbb{Z}_+^n$ and $\tilde{\varphi}_j \in C^\infty(U)$ for all $j$ satisfies that if $p_j \in \Gamma_0$, then every component of $p_j$ is even and $\tilde{\varphi}_j(0) > 0$ (or $\tilde{\varphi}_j(0) < 0$) for all $j$.

We will give an example in Section 6.3, which satisfies the conditions (a), (d) but does not satisfy the condition (c$'$). (Consider the case that the parameter $t$ satisfies $0 < |t| < 2$ in the example.)

*Remark* 2.11. In the one-dimensional case, the conditions (c) and (d) are equivalent.

Lastly, let us discuss a "symmetrical" property with respect to the phase and the amplitude. Observe the one-dimensional case. Let $f, \varphi$ satisfy that $f(0) = f'(0) = \cdots = f^{(q-1)}(0) = \varphi(0) = \varphi'(0) = \cdots = \varphi^{(p-1)}(0) = 0$ and $f^{(q)}(0)\varphi^{(p)}(0) \neq 0$, where $p, q \in \mathbb{N}$ are even. Applying the computation in Chapter 8 in [24] (see also Section 6.1 in this paper), we can see that if the support of $\chi$ is sufficiently small, then

$$\int_{-\infty}^{\infty} e^{i\tau x f(x)}\varphi(x)\chi(x)dx \sim \tau^{-\frac{p+1}{q+1}} \sum_{j=0}^{\infty} C_j \tau^{-j/(q+1)} \qquad \text{as } \tau \to \infty,$$

where $C_0$ is a nonzero constant. Note that the above expansion can be obtained for $C^\infty$ functions $f$ and $\varphi$. In particular, $\beta(xf, \varphi) = -\frac{p+1}{q+1}$ holds. Similarly, we can get $\beta(x\varphi, f) = -\frac{q+1}{p+1}$. From this observation, the following question seems interesting: When does the equality $\beta(x^1 f, \varphi)\beta(x^1\varphi, f) = 1$ hold in higher dimensional case? The following theorem is concerned with this question.

**Theorem 2.12.** *Let $f$, $\varphi$ be nonnegative or nonpositive real analytic functions defined on $U$. Suppose that both $f$ and $\varphi$ are convenient and nondegenerate over $\mathbb{R}$ with respect to their Newton polyhedra. Then we have $\beta(x^1 f, \varphi)\beta(x^1\varphi, f) \geq 1$. Moreover, the following two conditions are equivalent:*

(i) $\beta(x^1 f, \varphi)\beta(x^1\varphi, f) = 1$;

(ii) *There exists a positive rational number* $d$ *such that* $\Gamma_+(x^1 f) = d \cdot \Gamma_+(x^1\varphi)$. *If the condition* (i) *or* (ii) *is satisfied, then we have* $\eta(x^1 f, \varphi) = \eta(x^1\varphi, f) = n$.

## 3. Toric resolution

The purpose of this section is to give the resolution of the singularities of the critical points of some functions from the theory of toric varieties.

3.1. **Cones and fans.** In order to construct a toric resolution obtained from the Newton polyhedron, we recall the definitions of important terminology: *cone* and *fan*.

A *rational polyhedral cone* $\sigma \subset \mathbb{R}^n$ is a cone generated by finitely many elements of $\mathbb{Z}^n$. In other words, there are $u_1, \ldots, u_k \in \mathbb{Z}^n$ such that

$$\sigma = \{\lambda_1 u_1 + \cdots + \lambda_k u_k \in \mathbb{R}^n; \lambda_1, \ldots, \lambda_k \geq 0\}.$$

We say that $\sigma$ is *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$.

The *fan* is defined to be a finite collection $\Sigma$ of cones in $\mathbb{R}^n$ with the following properties:

- Each $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone;
- If $\sigma \in \Sigma$ and $\tau$ is a face of $\sigma$, then $\tau \in \Sigma$;
- If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of each.

For a fan $\Sigma$, the union $|\Sigma| := \bigcup_{\sigma\in\Sigma} \sigma$ is called the *support* of $\Sigma$. For $k = 0, 1, \ldots, n$, we denote by $\Sigma^{(k)}$ the set of $k$-dimensional cones in $\Sigma$. The *skeleton* of a cone $\sigma \in \Sigma$ is the set of all of its primitive integer vectors (i.e., with components relatively prime in $\mathbb{Z}_+$) in the edges of $\sigma$. It is clear that the skeleton of $\sigma$ generates $\sigma$ itself. Thus, the set of skeletons of the cones belonging to $\Sigma^{(k)}$ is also expressed by the same symbol $\Sigma^{(k)}$.

3.2. **Simplicial subdivision.** We denote by $(\mathbb{R}^n)^*$ the dual space of $\mathbb{R}^n$ with respect to the standard inner product. For $a = (a_1, \ldots, a_n) \in (\mathbb{R}^n)^*$, define

$$l(a) = \min\{\langle a, \alpha\rangle; \alpha \in \Gamma_+(f)\} \tag{3.1}$$

and $\gamma(a) = \{\alpha \in \Gamma_+(f); \langle a, \alpha\rangle = l(a)\}(= \Gamma_+(f) \cap H(a, l(a)))$. We introduce an equivalence relation $\sim$ in $(\mathbb{R}^n)^*$ by $a \sim a'$ if and only if $\gamma(a) = \gamma(a')$. For any $k$-dimensional face $\gamma$ of $\Gamma_+(f)$, there is an equivalence class $\gamma^*$ which is defined by

$$\gamma^* = \{a \in (\mathbb{R}^n)^*; \gamma(a) = \gamma, \text{ and } a_j \geq 0 \text{ for } j = 1, \ldots, n\}.$$

It is easy to see that the closure of $\gamma^*$ is an $(n-k)$-dimensional strongly convex rational polyhedral cone in $(\mathbb{R}^n)^*$. Moreover, the collection of the closures of $\gamma^*$ gives a fan $\Sigma_0$. Note that $|\Sigma_0| = \mathbb{R}^n_+$.

It is known that there exists a *simplicial subdivision* $\Sigma$ of $\Sigma_0$, that is, $\Sigma$ is a fan satisfying the following properties:

- The fans $\Sigma_0$ and $\Sigma$ have the same support;

- Each cone of $\Sigma$ lies in some cone of $\Sigma_0$;
- The skeleton of any cone belonging to $\Sigma$ can be completed to a base of the lattice dual to $\mathbb{Z}^n$.

3.3. **Construction of toric varieties.** Fix a simplicial subdivision $\Sigma$ of $\Sigma_0$. For $n$-dimensional cone $\sigma \in \Sigma$, let $a^1(\sigma), \dots, a^n(\sigma)$ be the skeleton of $\sigma$, ordered once and for all. Here, we set the coordinates of the vector $a^j(\sigma)$ as

$$a^j(\sigma) = (a_1^j(\sigma), \dots, a_n^j(\sigma)).$$

With every such cone $\sigma$, we associate a copy of $\mathbb{C}^n$ which is denoted by $\mathbb{C}^n(\sigma)$. We denote by $\pi(\sigma) : \mathbb{C}^n(\sigma) \to \mathbb{C}^n$ the map defined by $\pi(\sigma)(y_1, \dots, y_n) = (x_1, \dots, x_n)$ with

$$x_j = y_1^{a_j^1(\sigma)} \cdots y_n^{a_j^n(\sigma)}, \qquad j = 1, \dots, n. \tag{3.2}$$

Let $X_\Sigma$ be the union of $\mathbb{C}^n(\sigma)$ for $\sigma$ which are glued along the images of $\pi(\sigma)$. Indeed, for any $n$-dimensional cones $\sigma, \sigma' \in \Sigma$, two copies $\mathbb{C}^n(\sigma)$ and $\mathbb{C}^n(\sigma')$ can be identified with respect to a rational mapping: $\pi^{-1}(\sigma') \circ \pi(\sigma) : \mathbb{C}^n(\sigma) \to \mathbb{C}^n(\sigma')$ (i.e. $x \in \mathbb{C}^n(\sigma)$ and $x' \in \mathbb{C}^n(\sigma')$ will coalesce if $\pi^{-1}(\sigma') \circ \pi(\sigma) : x \mapsto x'$). Then it is known that

- $X_\Sigma$ is an $n$-dimensional complex algebraic manifold;
- The map $\pi : X_\Sigma \to \mathbb{C}^n$ defined on each $\mathbb{C}^n(\sigma)$ as $\pi(\sigma) : \mathbb{C}^n(\sigma) \to \mathbb{C}^n$ is proper.

The manifold $X_\Sigma$ is called the *toric variety* associated with $\Sigma$. The transition functions between local maps of the manifold $X_\Sigma$ are real on the real part of the manifold $X_\Sigma$ which will be denoted by $Y_\Sigma$. The restriction of the projection $\pi$ to $Y_\Sigma$ is also denoted by $\pi$. Then we have

- $Y_\Sigma$ is an $n$-dimensional real algebraic manifold;
- The map $\pi : Y_\Sigma \to \mathbb{R}^n$ defined on each $\mathbb{R}^n(\sigma)$ as $\pi(\sigma) : \mathbb{R}^n(\sigma) \to \mathbb{R}^n$ is proper.

3.4. **Resolution of singularities.** For $I \subset \{1, \dots, n\}$, define the set $T_I$ in $\mathbb{R}^n$ by

$$T_I = \{y \in \mathbb{R}^n; y_j = 0 \text{ for } j \in I, \ y_j \neq 0 \text{ for } j \notin I\}. \tag{3.3}$$

The following proposition shows that $\pi : Y_\Sigma \to \mathbb{R}^n$ is a real resolution of the singularity of the critical point of a real analytic function satisfying the nondegenerate property.

**Proposition 3.1** ([26], Lemma 2.13, Lemma 2.15). *Suppose that $f$ is a real analytic function in a neighborhood $U$ of the origin. Then we have the following.*

(i) *There exists a real analytic function $f_\sigma$ defined on the set $\pi(\sigma)^{-1}(U)$ such that $f_\sigma(0) \neq 0$ and*

$$(f \circ \pi(\sigma))(y_1, \dots, y_n) = y_1^{l(a^1(\sigma))} \cdots y_n^{l(a^n(\sigma))} f_\sigma(y_1, \dots, y_n). \tag{3.4}$$

(ii) *The Jacobian of the mapping* $\pi(\sigma)$ *is equal to*

$$J_{\pi(\sigma)}(y) = \pm y_1^{\langle a^1(\sigma)\rangle - 1} \cdots y_n^{\langle a^n(\sigma)\rangle - 1}. \tag{3.5}$$

(iii) *The set of the points in* $\mathbb{R}^n$ *in which* $\pi(\sigma)$ *is not an isomorphism is a union of coordinate planes.*

*Moreover, if* $f$ *is nondegenerate over* $\mathbb{R}$ *with respect to* $\Gamma_+(f)$ *and* $\pi(\sigma)(T_I) = 0$, *then the set* $\{y \in T_I; f_\sigma(y) = 0\}$ *is nonsingular, that is, the gradient of the restriction of the function* $f_\sigma$ *to* $T_I$ *does not vanish at the points of the set* $\{y \in T_I; f_\sigma(y) = 0\}$.

Next, we consider the case of $C^\infty$ functions.

**Proposition 3.2.** *Let* $\varphi$ *be a* $C^\infty$ *function defined on a neighborhood of the origin. When* $\varphi$ *is convenient or real analytic near the origin, define* $\tilde{l}(a) = \min\{\langle a, \alpha\rangle; \alpha \in \Gamma_+(\varphi)\}$ *for* $a \in \mathbb{Z}_+^n$. *Otherwise, define* $\tilde{l}(a) = \min\{\langle a, \alpha\rangle; \alpha \in \Gamma_+(\varphi)\}$ *for* $a \in \mathbb{N}^n$ *and* $\tilde{l}(a) = 0$ *for* $a \in \mathbb{Z}_+^n \setminus \mathbb{N}^n$. *Then, for* $\sigma \in \Sigma^{(n)}$, $\varphi \circ \pi(\sigma)$ *can be expressed as*

$$\varphi(\pi(\sigma)(y)) = \left(\prod_{j=1}^{n} y_j^{\tilde{l}(a^j(\sigma))}\right) \varphi_\sigma(y), \tag{3.6}$$

*where* $\varphi_\sigma$ *is a* $C^\infty$ *function defined on a neighborhood of the origin. (Needless to say, if* $\varphi$ *is real analytic, so is* $\varphi_\sigma$.)

## 4. Poles of local zeta functions

Throughout this section, the functions $f$, $\varphi$, $\chi$ always satisfy the conditions (A), (B), (C) in the beginning of Section 2.2.

The purpose of this section is to investigate the properties of poles of the functions:

$$Z_+(s) = \int_{\mathbb{R}^n} f(x)_+^s \varphi(x)\chi(x)dx, \quad Z_-(s) = \int_{\mathbb{R}^n} f(x)_-^s \varphi(x)\chi(x)dx, \tag{4.1}$$

where $f(x)_+ = \max\{f(x), 0\}$ and $f(x)_- = \max\{-f(x), 0\}$ and the *local zeta function*:

$$Z(s) = \int_{\mathbb{R}^n} |f(x)|^s \varphi(x)\chi(x)dx. \tag{4.2}$$

From the properties of $Z_+(s)$ and $Z_-(s)$, we can easily obtain analogous properties of $Z(s)$ by using the relationship: $Z(s) = Z_+(s) + Z_-(s)$.

It is easy to see that the above functions are holomorphic functions in the region $\mathrm{Re}(s) > 0$. Moreover, it is known (see [20],[1], etc.) that if the support of $\chi$ is sufficiently small, then these functions can be analytically continued to the complex plane as meromorphic functions and their poles belong to finitely many arithmetic progressions constructed from negative rational numbers. More precisely, Varchenko [26] describes the positions of the candidate poles and their orders by using the toric resolution constructed in Section 3. In this section, we give more accurate results in the case that $\varphi$ has a zero at the origin.

4.1. **The monomial case.** First, let us consider the case that the function $\varphi$ is a monomial, i.e., $\varphi(x) = x^p = x_1^{p_1} \cdots x_n^{p_n}$ with $p = (p_1, \ldots, p_n) \in \mathbb{Z}_+^n$. Fedorjuk [9] was the first to consider this kind of issue in two-dimensional case. Moreover, there have been closely related studies to ours in [7],[8],[5],[6], which contain other interesting results.

**Theorem 4.1.** *Suppose that* (i) $f$ *is nondegenerate over* $\mathbb{R}$ *with respect to its Newton polyhedron and* (ii) $\varphi(x) = x^p$ *with* $p \in \mathbb{Z}_+^n$. *If the support of* $\chi$ *is contained in a sufficiently small neighborhood of the origin, then the poles of the functions* $Z_+(s)$, $Z_-(s)$ *and* $Z(s)$ *are contained in the set*

$$\left\{ -\frac{\langle a, p+\mathbf{1}\rangle + \nu}{l(a)};\ \nu \in \mathbb{Z}_+,\ a \in \tilde{\Sigma}^{(1)} \right\} \cup (-\mathbb{N}),$$

*where* $l(a)$ *is as in* (3.1) *and* $\tilde{\Sigma}^{(1)} = \{a \in \Sigma^{(1)}; l(a) > 0\}$.

For $p \in \mathbb{Z}_+^n$, we define

$$\beta(p) = \max \left\{ -\frac{\langle a, p+\mathbf{1}\rangle}{l(a)}; a \in \tilde{\Sigma}^{(1)} \right\}. \tag{4.3}$$

If $s = \beta(p)$ is a pole of $Z_\pm(s)$, $Z(s)$, then we denote by $\eta_\pm(p)$, $\hat{\eta}(p)$ the order of its pole, respectively. For $\sigma \in \Sigma^{(n)}$, let

$$A_p(\sigma) = \left\{ j \in B(\sigma); \beta(p) = -\frac{\langle a^j(\sigma), p+\mathbf{1}\rangle}{l(a^j(\sigma))} \right\} \subset \{1, \ldots, n\}.$$

The following proposition shows the relationship between "the values of $\beta(p)$, $\eta_\pm(p)$, $\hat{\eta}(p)$" and "the geometrical conditions of $\Gamma_+(f)$ and the point $p$".

**Proposition 4.2.** *Let* $q = (q_1, \ldots, q_n)$ *be the point of the intersection of* $\partial\Gamma_+(f)$ *with the line joining the origin and the point* $p + \mathbf{1} = (p_1 + 1, \ldots, p_n + 1)$. *Then*

$$-\beta(p) = \frac{p_1+1}{q_1} = \cdots = \frac{p_n+1}{q_n} = \frac{\langle p\rangle + n}{\langle q\rangle} = \frac{1}{d(f, x^p)},$$

$$\eta_\pm(p), \hat{\eta}(p) \le \begin{cases} \rho_f(q) & \text{if } 1/d(f, x^p) \text{ is not an integer}, \\ \min\{\rho_f(q)+1, n\} & \text{otherwise}, \end{cases}$$

*where* $\rho_f$ *and* $d(\cdot,\cdot)$ *are as in Section* 2.1. *Note that* $m(f, x^p) = \rho_f(q) = \rho_f(d(f, x^p)(p+\mathbf{1}))$.

*Remark* 4.3. In the case when $n = 2$ or 3, $\rho_f(q)$ is equal to $\min\{\hat{m}_p, n\}$, where $\hat{m}_p$ is the number of the $(n-1)$-dimensional faces of $\Gamma_+(f)$ containing the point $q$. This, however, does not generally hold for $n \ge 4$.

Next, let us consider the coefficients of the Laurent expansions of $Z_+(s)$ and $Z_-(s)$ at the poles. When $d(f, x^p) > 1$, we compute the coefficients of $(s - \beta(p))^{-m(f,x^p)}$ in the Laurent expansions of $Z_\pm(s)$, $Z(s)$. Let

$$C_\pm = \lim_{s\to\beta(p)} (s-\beta(p))^{m(f,x^p)} Z_\pm(s), \quad C = \lim_{s\to\beta(p)} (s-\beta(p))^{m(f,x^p)} Z(s),$$

respectively.

**Theorem 4.4.** *Suppose that* (i) *$f$ is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron,* (ii) *$\varphi(x) = x^p$, where every component of $p \in \mathbb{Z}_+^n$ is even, and* (iii) *$d(f, x^p) > 1$. If the support of $\chi$ is contained in a sufficiently small neighborhood of the origin, then $C_+$ and $C_-$ are nonnegative and $C = C_+ + C_-$ is positive.*

The following proposition is concerned with the poles of $Z_+(s)$ and $Z_-(s)$, which are induced by the set of zeros of $f_\sigma$.

**Proposition 4.5.** *Suppose that the conditions* (i), (ii) *in Theorem* 4.1 *are satisfied and* (iii) *$d(f, x^p) < 1$. Let $1, \dots, k_*$ be all the natural numbers strictly smaller than $-\beta(p) = 1/d(f, x^p)$. If the support of $\chi$ is contained in a sufficiently small neighborhood of the origin, then $Z_+(s)$ and $Z_-(s)$ have at $s = -1, \dots, -k_*$ poles of order not higher than* 1 *and do not have other poles in the region* $\mathrm{Re}(s) > \beta(p)$. *Moreover, let $a_k^+$, $a_k^-$ be the residues of $Z_+(s)$, $Z_-(s)$ at $s = -k$, respectively, then we have $a_k^+ = (-1)^{k-1} a_k^-$ for $k = 1, \dots, k_*$.*

*Remark* 4.6. We can easily generalize the results in this subsection as follows. The same assertions in Theorem 4.1, Theorem 4.4 and Proposition 4.5 can be obtained, even if $x^p$ is replaced by $x^p\tilde{\varphi}(x)$ where $\tilde{\varphi} \in C^\infty(U)$ with $\tilde{\varphi}(0) \neq 0$. Here, in the case of Theorem 4.4, when $\tilde{\varphi}(0) < 0$, "positive" and "nonnegative" must be changed to "negative" and "nonpositive", respectively.

4.2. **The convenient case.** Next, let us consider the poles of $Z_\pm(s)$ in (4.1) and $Z(s)$ in (4.2) in the case that $f$ or $\varphi$ is convenient, i.e., the associated Newton polyhedron intersects all the coordinate axes.

**Theorem 4.7.** *Suppose that* (i) *$f$ is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron and* (ii) *at least one of the following conditions is satisfied:*

(a) *$f$ is convenient;*
(b) *$\varphi$ is convenient;*
(c) *$\varphi$ is real analytic on a neighborhood of the origin.*

*If the support of $\chi$ is contained in a sufficiently small neighborhood of the origin, then the poles of the functions $Z_+(s)$, $Z_-(s)$ and $Z(s)$ are contained in the set*

$$\left\{ -\frac{\tilde{l}(a) + \langle a \rangle + \nu}{l(a)};\ \nu \in \mathbb{Z}_+,\ a \in \tilde{\Sigma}^{(1)} \right\} \cup (-\mathbb{N}),$$

*where* $l(a)$ *is as in* (3.1), $\tilde{l}(a)$ *is as in Proposition* 3.2, *below, and* $\tilde{\Sigma}^{(1)}$ *is as in Theorem* 4.1, *and*

$$\max\left\{-\frac{\tilde{l}(a)+\langle a\rangle}{l(a)};\ a\in\tilde{\Sigma}^{(1)}\right\}=-\frac{1}{d(f,\varphi)}.$$

*Moreover, for each* $Z_+(s)$,$Z_-(s)$ *and* $Z(s)$, *if* $s=-1/d(f,\varphi)$ *is a pole, then its order is not larger than*

$$\begin{cases} m(f,\varphi) & \text{if } 1/d(f,\varphi) \text{ is not an integer,}\\ \min\{m(f,\varphi)+1,n\} & \text{otherwise.}\end{cases}$$

Next, when $d(f,\varphi)>1$, we consider the coefficients of $(s+1/d(f,\varphi))^{-m(f,\varphi)}$ in the Laurent expansions of $Z_\pm(s)$ and $Z(s)$. Let

$$\begin{aligned} C_\pm &= \lim_{s\to -1/d(f,\varphi)}(s+1/d(f,\varphi))^{m(f,\varphi)}Z_\pm(s),\\ C &= \lim_{s\to -1/d(f,\varphi)}(s+1/d(f,\varphi))^{m(f,\varphi)}Z(s), \end{aligned}\tag{4.4}$$

respectively.

**Theorem 4.8.** *Suppose that* (i) $f$ *is convenient and nondegenerate over* $\mathbb{R}$ *with respect to its Newton polyhedron,* (ii) $\varphi_{\Gamma_0}$ *is nonnegative* (*resp. nonpositive*) *on a neighborhood of the origin and* (iii) $d(f,\varphi)>1$. *If the support of* $\chi$ *is contained in a sufficiently small neighborhood of the origin, then* $C_+$ *and* $C_-$ *are nonnegative* (*resp. nonpositive*) *and* $C=C_++C_-$ *is positive* (*resp. negative*).

The following proposition is concerned with the poles of $Z_+(s)$ and $Z_-(s)$, which are induced by the set of zeros of $f_\sigma$.

**Proposition 4.9.** *Suppose that the conditions* (i), (ii) *in Theorem* 4.8 *are satisfied and* (iii) $d(f,\varphi)<1$. *Let* $1,\dots,k_*$ *be all the natural numbers strictly smaller than* $1/d(f,\varphi)$. *If the support of* $\chi$ *is contained in a sufficiently small neighborhood of the origin, then* $Z_+(s)$ *and* $Z_-(s)$ *have at* $s=-1,\dots,-k_*$ *poles of order not higher than 1 and they do not have other poles in the region* $\mathrm{Re}(s)>-1/d(f,\varphi)$. *Moreover, let* $a_k^+$, $a_k^-$ *be the residues of* $Z_+(s)$, $Z_-(s)$ *at* $s=-k$, *respectively, then we have* $a_k^+=(-1)^{k-1}a_k^-$ *for* $k=1,\dots,k_*$.

4.3. **Remarks.** In this subsection, let us consider Theorem 4.4 (with Remark 4.6) and Theorem 4.8 under the additional assumption: $f$ is nonnegative or nonpositive near the origin. The following theorem shows that the same assertions can be obtained without the assumption: $d(f,\varphi)>1$.

**Theorem 4.10.** *Suppose that* (i) $f$ *is nondegenerate over* $\mathbb{R}$ *with respect to its Newton polyhedron,* (ii) $f$ *is nonnegative or nonpositive on a neighborhood of the origin and* (iii) *at least one of the following condition is satisfied:*

(a) $\varphi$ *is expressed as* $\varphi(x) = x^p\tilde{\varphi}(x)$ *on a neighborhood of the origin, where every component of* $p \in \mathbb{Z}_+^n$ *is even and* $\tilde{\varphi}(0) > 0$ *(resp.* $\tilde{\varphi}(0) < 0$*);*

(b) $f$ *is convenient and* $\varphi_{\Gamma_0}$ *is nonnegative (resp. nonpositive) on a neighborhood of the origin.*

*If the support of* $\chi$ *is contained in a sufficiently small neighborhood of the origin, then* $C_+$ *and* $C_-$ *are nonnegative (resp. nonpositive) and* $C = C_+ + C_-$ *is positive (resp. negative), where* $C_\pm$, $C$ *are as in* (4.4).

4.4. **Certain symmetrical properties.** We denote by $\beta_\pm(f,\varphi)$, $\hat{\beta}(f,\varphi)$ the largest poles of $Z_\pm(s)$, $Z(s)$ and by $\eta_\pm(f,\varphi)$, $\hat{\eta}(f,\varphi)$ their orders, respectively.

**Theorem 4.11.** *Let* $f,\varphi$ *be nonnegative or nonpositive real analytic functions defined on a neighborhood of the origin. Suppose that* $f$ *and* $\varphi$ *are convenient and nondegenerate over* $\mathbb{R}$ *with respect to their Newton polyhedra. If the support of* $\chi$ *is contained in a sufficiently small neighborhood of the origin, then we have*

$$\beta_\pm(x^1 f,\varphi)\beta_\pm(x^1\varphi,f) \le 1 \text{ and } \hat{\beta}(x^1 f,\varphi)\hat{\beta}(x^1\varphi,f) \le 1 \tag{4.5}$$

*Moreover, the following two conditions are equivalent:*

(i) *The equality holds in each estimate in* (4.5)*;*

(ii) *There exists a positive rational number* $d$ *such that* $\Gamma_+(x^1 f) = d \cdot \Gamma_+(x^1\varphi)$.

*If the condition* (i) *or* (ii) *is satisfied, then we have* $\eta_\pm(x^1 f,\varphi) = \eta_\pm(x^1\varphi,f) = \hat{\eta}(x^1\varphi,f) = n$.

## 5. Relationship between $I(\tau)$ and $Z_\pm(s)$

It is known (see [18], [20], [1], etc.) that the study of the asymptotic behavior of the oscillatory integral $I(\tau)$ in (1.1) can be reduced to an investigation of the poles of the functions $Z_\pm(s)$ in (4.1). Indeed, every result in Section 2 can been easily obtained from the results in Section 4. Here, we roughly explain a relationship between $I(\tau)$ and $Z_\pm(s)$. Let $f$,$\varphi$,$\chi$ satisfy the conditions (A),(B),(C) in Section 2.2. Suppose that the support of $\chi$ is sufficiently small.

Define the Gelfand-Leray function: $K : \mathbb{R} \to \mathbb{R}$ as

$$K(t) = \int_{W_t} \varphi(x)\chi(x)\omega, \tag{5.1}$$

where $W_t = \{x \in \mathbb{R}^n; f(x) = t\}$ and $\omega$ is the surface element on $W_t$ which is determined by $df \wedge \omega = dx_1 \wedge \cdots \wedge dx_n$. $I(\tau)$ and $Z_\pm(s)$ can be expressed by using $K(t)$: Changing the integral variables in (1.1),(4.1), we have

$$I(\tau) = \int_{-\infty}^{\infty} e^{i\tau t}K(t)dt = \int_0^{\infty} e^{i\tau t}K(t)dt + \int_0^{\infty} e^{-i\tau t}K(-t)dt, \tag{5.2}$$

$$Z_\pm(s) = \int_0^{\infty} t^s K(\pm t)dt, \tag{5.3}$$

respectively. Applying the inverse formula of the Mellin transform to (5.3), we have

$$K(\pm t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Z_{\pm}(s)t^{-s-1}ds, \tag{5.4}$$

where $c > 0$ and the integral contour follows the line $\mathrm{Re}(s) = c$ upwards. Recall that $Z_+(s)$ and $Z_-(s)$ are meromorphic functions and their poles exist on the negative part of the real axis. By deforming the integral contour as $c$ tends to $-\infty$ in (5.4), the residue formula gives the asymptotic expansions of $K(t)$ as $t \to \pm 0$. Substituting these expansions of $K(t)$ into (5.2), we can get an asymptotic expansion of $I(\tau)$ as $\tau \to +\infty$.

Through the above calculation, we see more precise relationship for the coefficients. If $Z_+(s)$ and $Z_-(s)$ have the Laurent expansions at $s = -\lambda$:

$$Z_{\pm}(s) = \frac{B_{\pm}}{(s+\lambda)^{\rho}} + O\left(\frac{1}{(s+\lambda)^{\rho-1}}\right),$$

respectively, then the corresponding part in the asymptotic expansion of $I(\tau)$ has the form

$$B\tau^{-\lambda}(\log\tau)^{\rho-1} + O(\tau^{-\lambda}(\log\tau)^{\rho-2}).$$

Here a simple computation gives the following relationship:

$$B = \frac{\Gamma(\lambda)}{(\rho-1)!}\left[e^{i\pi\lambda/2}B_+ + e^{-i\pi\lambda/2}B_-\right], \tag{5.5}$$

where $\Gamma$ is the Gamma function.

## 6. Examples

In this section, we give some examples of the phase and the amplitude in the integral (1.1), which clarifies the subtlety of our results in Sections 2 and 4. Throughout this section, we always assume that $f$, $\varphi$, $\chi$ satisfy the conditions (A), (B), (C) in Section 2. (In Examples 1, 2, each $f$, $\varphi$, $\varphi_t$ satisfies the respective condition.)

6.1. **The one-dimensional case.** Let us compute the asymptotic expansion of $I(\tau)$ in (1.1) as $\tau \to +\infty$ in the one-dimensional case by using our analysis in this paper. As mentioned in Section 2, the results below can also be obtained by using the analysis in [24]. Note that the computation below is valid for $C^\infty$ phases. From the assumptions $\Gamma_+(f), \Gamma_+(\varphi) \neq \emptyset$, $f$, $\varphi$ can be expressed as

$$f(x) = x^q\tilde{f}(x), \quad \varphi(x) = x^p\tilde{\varphi}(x),$$

where $q, p \in \mathbb{Z}_+$, $q \geq 2$ and $\tilde{f}, \tilde{\varphi}$ are $C^\infty$ functions defined on a neighborhood of the origin with $\tilde{f}(0)\tilde{\varphi}(0) \neq 0$. Suppose that the support of $\chi$ is so small that $\tilde{f}, \tilde{\varphi}$ do not have any zero on the support.

It is easy to see that $f$ is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron, $\Gamma_+(f) = [q, \infty)$, $\Gamma_+(\varphi) = [p, \infty)$, $d(f, \varphi) = \frac{q}{p+1}$ and $m(f, \varphi) = 1$. Let $\alpha$ be the sign of $\tilde{f}(x)$ on the support of $\chi$. From a simple computation, for even $q$

$$\begin{aligned} Z_\alpha(s) &= \int_0^\infty x^{qs+p}\{|\tilde{f}(x)|^s\tilde{\varphi}(x)\chi(x) + (-1)^p|\tilde{f}(-x)|^s\tilde{\varphi}(-x)\chi(-x)\}dx, \\ Z_{-\alpha}(s) &= 0, \end{aligned} \tag{6.1}$$

and for odd $q$

$$\begin{aligned} Z_\alpha(s) &= \int_0^\infty x^{qs+p}|\tilde{f}(x)|^s\tilde{\varphi}(x)\chi(x)dx, \\ Z_{-\alpha}(s) &= (-1)^p \int_0^\infty x^{qs+p}|\tilde{f}(-x)|^s\tilde{\varphi}(-x)\chi(-x)dx. \end{aligned} \tag{6.2}$$

We can see that the poles of $Z_\pm(s)$ are simple and they are contained in the set $\{-\frac{p+1+\nu}{q}; \nu \in \mathbb{Z}_+\}$. Moreover, we can compute the explicit values of the coefficients of the term $(s + \frac{p+1}{q})^{-1}$ in the Laurent expansions of $Z_+(s)$ and $Z_-(s)$.

Next, applying the argument in Section 5, we have

$$I(\tau) \sim \tau^{-\frac{p+1}{q}} \sum_{j=0}^\infty C_j \tau^{-j/q} \quad \text{as } \tau \to \infty.$$

The relationship (5.5) gives the values of the coefficient $C_0$. As a result, we can see all the cases that $\beta(f, \varphi) = -1/d(f, \varphi)$ holds.

(i) ($q$:even; $p$:even) $C_0 = \frac{2}{q}\Gamma\left(\frac{p+1}{q}\right)|\tilde{f}(0)|^{-\frac{p+1}{q}}\tilde{\varphi}(0)e^{\alpha i\frac{p+1}{2q}\pi} \neq 0$, which implies $\beta(f, \varphi) = -1/d(f, \varphi)$;

(ii) ($q$:even; $p$:odd) $C_0 = 0$, which implies $\beta(f, \varphi) < -1/d(f, \varphi)$;

(iii) ($q$:odd; $p$:even) $C_0 = \frac{2}{q}\Gamma\left(\frac{p+1}{q}\right)|\tilde{f}(0)|^{-\frac{p+1}{q}}\tilde{\varphi}(0)\cos\left(\frac{p+1}{2q}\pi\right)$, which implies that $\beta(f, \varphi) = -1/d(f, \varphi)$ is equivalent to $\frac{p+1}{2q} \notin \mathbb{N} + \frac{1}{2}$;

(iv) ($q$:odd; $p$:odd) $C_0 = \alpha\frac{2i}{q}\Gamma\left(\frac{p+1}{q}\right)|\tilde{f}(0)|^{-\frac{p+1}{q}}\tilde{\varphi}(0)\sin\left(\frac{p+1}{2q}\pi\right)$, which implies that $\beta(f, \varphi) = -1/d(f, \varphi)$ is equivalent to $\frac{p+1}{2q} \notin \mathbb{N}$.

Let us compare the conditions (a),(b),(c),(d) in Theorem 2.7 with the condition of $p, q$. That $q$ (resp. $p$) is even is equivalent to the condition (b) (resp. (c), (d)). The condition (a) is equivalent to the inequality: $\frac{p+1}{2q}\pi < \frac{\pi}{2}$, which implies $C_0 \neq 0$ in (iii).

6.2. **Example 1.** Consider the following two-dimensional example:

$$\begin{aligned} f(x_1, x_2) &= x_1^4, \\ \varphi(x_1, x_2) &= x_1^2 x_2^2 + e^{-1/x_2^2}(=: \varphi_1(x_1, x_2) + \varphi_2(x_1, x_2)), \end{aligned}$$

and $\chi$ is radially symmetric about the origin. It is easy to see that $f$ is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron, $\Gamma_+(f) = \{(4,0)\} + \mathbb{R}^2_+$, $\Gamma_+(\varphi) = \Gamma_+(\varphi_1) = \{(2,2)\} + \mathbb{R}^2_+$, $\Gamma_+(\varphi_2) = \emptyset$, $d(f,\varphi) = 4/3$, $m(f,\varphi) = 1$. Define

$$Z_\pm^{(j)}(s) = \int_{\mathbb{R}^2} (f(x))_\pm^s \varphi_j(x)\chi(x)dx \quad j = 1, 2.$$

Note $Z_-(s) = 0$. A simple computation gives

$$Z_+^{(1)}(s) = 4\int_0^\infty \int_0^\infty x_1^{4s+2}x_2^2\chi(x_1,x_2)dx_1dx_2.$$

We see that the poles of $Z_+^{(1)}(s)$ are simple and they are contained in the set $\{-3/4, -4/4, -5/4, \ldots\}$. Similarly, the poles of

$$Z_+^{(2)}(s) = 4\int_0^\infty \int_0^\infty x_1^{4s}e^{-1/x_2^2}\chi(x_1,x_2)dx_1dx_2$$

are simple and contained in the set $\{-1/4, -2/4, -3/4, \ldots\}$. Moreover, the coefficient of $(s+1/4)^{-1}$ is computed as

$$\int_0^\infty e^{-1/x_2^2}\chi(0,x_2)dx_2 > 0.$$

Therefore, we have $\beta_+(f,\varphi) = \beta(f,\varphi) = -1/4$. As a result, $\beta(f,\varphi) > -1/d(f,\varphi)(= -3/4)$.

This example does not satisfy the condition (d) in Theorem 2.2. Noticing that $\Gamma_+(\varphi) = \{(2,2)\} + \mathbb{R}^2_+$, we see that the information of the Newton polyhedron is not sufficient to understand the behavior of oscillatory integrals in the case of $C^\infty$ amplitudes.

6.3. **Example 2.** Consider the following two-dimensional example with a real parameter $t$:

$$f(x_1,x_2) = x_1^5 + x_1^6 + x_2^5,$$
$$\varphi_t(x_1,x_2) = x_1^2 + tx_1x_2 + x_2^2.$$

It is easy to see that $f$ is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron, $(\varphi_t)_{\Gamma_0}(x) = \varphi_t(x)$, $d(f,\varphi_t) = 5/4$, and $m(f,\varphi_t) = 1$. $(\varphi_t)_{\Gamma_0}(x)$ is nonnegative on $\mathbb{R}^2$, if and only if $|t| \leq 2$. Thus, Theorem 4.8 implies that $\beta(f,\varphi_t) = -1/d(f,\varphi_t) = -4/5$ if $|t| \leq 2$. In this example, we understand the situation in more detail from the explicit computation below.

By applying the computation in Section 4, we see the properties of poles of the functions $Z_+(s)$ and $Z_-(s)$ in the following. The poles of the functions $Z_+(s)$ and $Z_-(s)$ are contained in the set $\{-4/5, -5/5, -6/5, \ldots\}$ and their order is at most one. Let $C_+(t)$, $C_-(t)$ be the coefficients of $(s-4/5)^{-1}$ in the Laurent expansions of $Z_+(s)$ and $Z_-(s)$. Then, we have $C_+(t) = C_-(t) = A + tB$ with

$$A := \frac{1}{5}\int_{-\infty}^{\infty} |u^5+1|^{-4/5}(u^2+1)du, \quad B := \frac{1}{5}\int_{-\infty}^{\infty} |u^5+1|^{-4/5}udu.$$

Note that $A$ is positive and $B$ is negative.

Next, by applying the argument in Section 5, $I(\tau)$ has the asymptotic expansion of the form:

$$I(\tau) \sim \tau^{-\frac{4}{5}} \sum_{j=0}^{\infty} C_j(t)\tau^{-j/5} \quad \text{as } \tau \to +\infty. \tag{6.3}$$

The relationship (5.5) gives $C_0(t) = 2\Gamma(\frac{4}{5})\cos(\frac{2}{5}\pi)(A+tB)$.

Set $t_0 = -A/B(>0)$. From the above value of $C_0(t)$, if $t \neq t_0$, then the equation $\beta(f,\varphi_t) = -1/d(f,\varphi_t)$ holds. This means that the condition (d) in Theorem 2.7 is not necessary to satisfy the above equation. Furthermore, this example shows that the oscillation index is determined by not only the geometry of the Newton polyhedra but also the values of the coefficients of $x^\alpha$ for $\alpha \in \Gamma_0$ in the Taylor expansion of the amplitude.

*Note* 6.1. The existence of the term $x_1^6$ in $f$ produces infinitely many non-zero coefficients $C_j(t)$ in the asymptotic expansion (6.3) for any $t$.

6.4. **Comments on results in** [1]. As mentioned in the Introduction, there have been studies in [1] in a similar direction to our investigations. In our language, their results can be stated as follows.

**"Theorem" 6.1** (Theorem 8.4 in [1], p 254). *If $f$ is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron, then*

(i) $\beta(f,\varphi) \leq -1/d(f,\varphi)$;
(ii) *If* $d(f,\varphi) > 1$ *and* $\Gamma_+(\varphi) = \{p\} + \mathbb{R}_+^n$ *with* $p \in \mathbb{Z}_+^n$, *then* $\beta(f,\varphi) = -1/d(f,\varphi)$.

Unfortunately, more additional assumptions are necessary to obtain the above assertions (i), (ii). Indeed, it is easy to see that Example 1 violates (i), (ii). As for (ii), even if $\varphi$ is real analytic, the one-dimensional case in Section 6.1 indicates that at least some condition on the power $p$ is needed. (It is easy to find counterexamples in higher dimensional case.) The same case shows that the evenness of $p$ is not always necessary to satisfy $\beta(f,\varphi) = -1/d(f,\varphi)$.

## References

[1] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko: *Singularities of Differentiable Maps II*, Birkhauser, 1988.

[2] B.-Y. Chen, J. Kamimoto and T. Ohsawa: Behavior of the Bergman kernel at infinity, Math. Z. **248** (2004), 695–708.

[3] K. Cho, J. Kamimoto and T. Nose: Asymptotics of the Bergman functions for semipositive holomorphic line bundles, Kyushu J. Math. **65** (2011), 349–382.
[4] K. Cho, J. Kamimoto and T. Nose: Asymptotic analysis of oscillatory integrals via the Newton polyhedra of the phase and the amplitude, To appear in J. Math. Soc. Japan.
[5] J. Denef, A. Laeremans and P. Sargos: On the largest nontrivial pole of the distribution $|f|^s$, RIMS Kokyuroku **999** (1997), 1–7.
[6] J. Denef, J. Nicaise and P. Sargos: Oscillating integrals and Newton polyhedra, J. Anal. Math. **95** (2005), 147–172.
[7] J. Denef and P. Sargos: Polyèdra de Newton et distribution $f_+^s$. I, J. Analyse Math. **53** (1989), 201–218.
[8] ———: Polyèdra de Newton et distribution $f_+^s$. II, Math. Ann. **293** (1992), 193–211.
[9] M. V. Fedorjuk, Non-homogeneous generalized functions of two variables, (Russian) Mat. Sb. (N.S.) **49** (1959), 431–446.
[10] W. Fulton: *Introduction to toric varieties*, Princeton University Press, Princeton, NJ, 1993.
[11] I. M. Gel'fand and G. E. Shilov: *Generalized Functions I.* Academic Press, New York, 1964.
[12] M. Greenblatt: A direct resolution of singularities for functions of two variables with applications to analysis. J. Anal. Math. **92** (2004), 233–257.
[13] ———: An elementary coordinate-dependent local resolution of singularities and applications. J. Funct. Anal. **255** (2008), 1957–1994.
[14] ———: The asymptotic behavior of degenerate oscillatory integrals in two dimensions, J. Funct. Anal. **257** (2009), 1759–1798.
[15] ———: Oscillatory integral decay, sublevel set growth, and the Newton polyhedron, Math. Ann. 346 (2010), 857–895.
[16] ———: Resolution of singularities, asymptotic expansions of oscillatory integrals, and related phenomena, preprint, arXiv:0709.2496v2.
[17] H. Hironaka: Resolution of singularities of an algebraic variety over a field of characteristic zero I, II, Ann. of Math. **79** (1964), 109–326.
[18] J. Igusa: Forms of higher degree. Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 59. New Delhi, 1978.
[19] J. Kamimoto: Newton polyhedra and the Bergman kernel, Math. Z. **246** (2004), 405–440.
[20] A. Kaneko: *Newton diagrams, Singular points and Oscillatory integrals*, Lecture Note at Sophia University, **11** (in Japanese), 1981.
[21] B. Malgrange: Intégrales asymptotiques et monodromie. Ann. Sci. École Norm. Sup. (4) 7 (1974), 405–430.
[22] M. Pramanik and C. W. Yang: Decay estimates for weighted oscillatory integrals in $\mathbb{R}^2$, Indiana Univ. Math. J., **53** (2004), 613–645.
[23] H. Schulz: Convex hypersurfaces of finite type and the asymptotics of their Fourier transforms, Indiana Univ. Math. J., **40** (1991), 1267–1275.
[24] E. M. Stein: *Harmonic Analysis. Real-variable methods, orthogonality and oscillatory integrals*, Princeton University Press, Princeton, NJ, 1993.
[25] V. A. Vassiliev: Asymptotic behavior of exponential integrals in the complex domain, Funct. Anal. Appl., (1979), 239–247.
[26] A. N. Varchenko: Newton polyhedra and estimation of oscillating integrals, Functional Anal. Appl., **10-3** (1976) 175-196.

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