数理解析研究所講究録
第 1783 巻 2012 年 198-199

Metric positivity of direct image sheaves of differential forms [1]

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We consider a proper surjective holomorphic map $f : X \longrightarrow Y$ with connected fibers, from a complex manifold X to a normal complex space Y with $\dim Y = m$ and $\dim X = n+m$, and an f-ample line bundle E on X. We discuss good properties of $R^{n-p}f_*(\Omega^p_{X/Y} \otimes E)$ or $R^{n-p}f_*(\Omega^{p+m}_X \otimes E)$, as Kollár did for $R^q f_*(K_{X/Y} \otimes E')$ with $q \geq 0$, where E' is a line bundle on X with weaker positivities. Our aim is to try to interpolate Kodaira-Spencer's deformation theory and the Hodge theory by introducing a polarization and by considering adjoint-type vector bundles associated to it. The result is as follows [T].

Theorem 1. *Let $f : X \longrightarrow Y$ and E be as above. Then*
(1) $R^{n+q}f_*(\Omega^{m-q}_X \otimes E) = 0$ if $q > 0$.
Let p be an integer with $0 \leq p \leq n$. Then
(2) $R^{n-p}f_*(\Omega^{p+m}_X \otimes E)$ is torsion free.
(3) Assume Y and f are smooth. Then $R^{n-p}f_*(\Omega^{p+m}_X \otimes E)$ is locally free.
(4) Assume Y and f are smooth. Then there exists an exact sequence

$$T_Y \otimes R^{n-p-1}f_*(\Omega^{p+1}_{X/Y} \otimes E) \xrightarrow{\delta_{p+1}} R^{n-p}f_*(\Omega^p_{X/Y} \otimes E) \xrightarrow{\tau_p} R^{n-p}f_*(\Omega^{p+m}_X \otimes E) \otimes K_Y^{-1} \longrightarrow 0,$$

which is induced by a natural inclusion $\Omega^p_{X/Y} \otimes f^*K_Y \subset \Omega^{p+m}_X$, i.e., $\Omega^p_{X/Y} \subset \Omega^{p+m}_X \otimes f^*K_Y^{-1}$. Moreover the surjection τ_p splits.

Theorem 2. *In Theorem 1, assume Y and f are smooth, and assume E admits a Hermitian metric h whose curvature is semi-positive $\sqrt{-1}\Theta_h \geq 0$ on X and positive $\sqrt{-1}\Theta_h|_{X_y} > 0$ on every fiber X_y. Then for every $0 \leq p \leq n$, the vector bundle $R^{n-p}f_*(\Omega^{p+m}_X \otimes E) \otimes K_Y^{-1}$ is Griffiths semi-positive. Moreover if E is positive on X, then $R^{n-p}f_*(\Omega^{p+m}_X \otimes E) \otimes K_Y^{-1}$ is Griffiths positive, if it is non-zero.*

In case when $f : X \longrightarrow Y$ with Y smooth, parametrizes canonically polarized manifolds i.e., every K_{X_y} is ample, we can take $E = K_{X/Y}$ in Theorem 1, and then $(R^{n-p}f_*(\Omega^p_{X/Y} \otimes E))^* = R^p f_* T^p_{X/Y}$ by the relative duality. When $p = 1$, $R^1 f_* T^1_{X/Y}$ is the sheaf where the Kodaira-Spencer class lives. By a recent remarkable result of Schumacher, $K_{X/Y}$ is semi-positive on X with respect to the fiberwise Kähler-Einstein metrics ω_y; the Hermitian metric h on $K_{X/Y}$ so that $h|_{X_y} = (\det \omega_y)^{-1}$ on every fiber X_y. Moreover if the parametrization is effective, $K_{X/Y}$ is positive. Thus we can apply Theorem 2 for such kind of $(E, h) = (K_{X/Y}, h)$. Earlier works of Siu and Schumacher tried to determine the sign of the curvature of the Weil-Petersson metric on T_Y. More recently, Schumacher also gives a curvature formula for $R^{n-p}f_*(\Omega^p_{X/Y} \otimes K^{\otimes m}_{X/Y})$ on some open subset of Y.

Regarding the method of proof, our argument is basically analytic. We put an appropriate Hermitian metric h on E with positive curvature, and we develope a theory of harmonic integrals for the cohomology groups $H^q(X, \Omega^p_X \otimes E)$ with $p + q = \dim X$ with respect to h and a Kähler form $\omega_X = \sqrt{-1}\Theta_h$ on X. Positive line bundle valued middle degree harmonic forms are special by the following reason. With respect to a Kähler form $\omega_X = \sqrt{-1}\Theta_h$ and h, Akizuki-Nakano's formula on bidegree (p, q) with $p + q = \dim X$ is $\Delta_{\bar\partial} = \Delta_{\partial_h}$, i.e., there is no curvature term $[\sqrt{-1}e(\Theta_h), \Lambda_{\omega_X}]$. Thus, at least in case X is

[1]RIMS 共同研究「ポテンシャル論とファイバー空間」2011 年 9 月 5 日〜7 日

compact, if $u \in C^{p,q}(X,E)$ is $\Delta_{\overline{\partial}}$-harmonic, it is Δ_{∂_h}-harmonic too, and hence u is not only $\overline{\partial}$-closed, but also $D_h = \partial_h + \overline{\partial}$ closed. We will have several vanishings like this, and have a theory of harmonic integrals when Y is localized as follows. Letting Y to be Stein, $H^q(X, \Omega_X^p \otimes E)$ with $p + q = \dim X$ is represented by

$$\mathcal{H}^{p,q}(X, \omega_X, E, h) = \{u \in C^{p,q}(X,E);\ \overline{\partial} u = \vartheta_h u = 0, u \text{ is } \omega_X\text{-primitive}, \\ u \wedge ds = 0 \text{ for any } s \in H^0(X, \mathcal{O}_X)\},$$

where ϑ_h is the formal adjoint of $\overline{\partial}$ with respect to ω_X and h. Since X is non-compact, we need practically a boundary condition to obtain a reasonable harmonic representation. The final condition $u \wedge ds = 0$ plays a role of the so-called $\overline{\partial}$-Neumann condition. We can moreover show that $\mathcal{H}^{p,q}(X, \omega_X, E, h)$ becomes an $H^0(X, \mathcal{O}_X)$-module in a natural way, which will lead the torsion freeness of $R^q f_*(\Omega_X^p \otimes E)$ via the above harmonic representation.

A Hermitian metric $g = g_p$ on $R^{n-p} f_*(\Omega_X^{p+m} \otimes E) \otimes K_Y^{-1}$ in Theorem 2 can be described as follows. The curvature of h on E gives a Kähler form $\omega_y = \sqrt{-1}\Theta_h|_{X_y}$ on every fiber X_y. We then consider a Hermitian inner product on $C^{p,n-p}(X_y, E_y)$ with respect to ω_y and $h_y = h|_{X_y}$, and put a "Hermitian metric" g on $R^{n-p} f_*(\Omega_{X/Y}^p \otimes E)$, via the fiberwise harmonic representation with respect to ω_y and h_y. Regarding $R^{n-p} f_*(\Omega_X^{p+m} \otimes E) \otimes K_Y^{-1}$ as a locally free subsheaf of $R^{n-p} f_*(\Omega_{X/Y}^p \otimes E)$ by the splitting in Theorem 1 (4), we put a "Hermitian metric" on $R^{n-p} f_*(\Omega_X^{p+m} \otimes E) \otimes K_Y^{-1}$. We then can show that this defines a Hermitian metric on $R^{n-p} f_*(\Omega_X^{p+m} \otimes E) \otimes K_Y^{-1}$ in the usual sense.

The computation of the curvature of the metric g on $R^{n-p} f_*(\Omega_X^{p+m} \otimes E) \otimes K_Y^{-1}$ will be done by trying to generalize that in Berndtsson [B] i.e. in case $p = n$. For simplicity, we let Y be a unit disc in $\mathbb{C}$ with coordinate t, and $\omega_X = \sqrt{-1}\Theta_h > 0$. Let $u \in C^{p,n-p}(X,E)$ and suppose that $u|_{X_y}$ is primitive on every fiber and $\partial_h u = \xi \wedge dt, \overline{\partial} u = \eta \wedge dt$ with $\xi \in C^{p,n-p}(X,E), \eta \in C^{p-1,n-p+1}(X,E)$. (This is not really correct as it stands.) We put $\|f_* u\|^2 \in C^\infty(Y, \mathbb{R})$ by $\|f_* u\|(y) = \|u|_{X_y}\|$: the L^2-norm of $u|_{X_y}$ with respect to the metrics ω_y and h_y. Then the curvature of g on $R^{n-p} f_*(\Omega_X^{p+m} \otimes E) \otimes K_Y^{-1}$ is like $-\sqrt{-1}\partial\overline{\partial}\|f_* u\|^2$, and

$$-\sqrt{-1}\frac{\partial^2 \|f_* u\|^2}{\partial t \partial \overline{t}}\bigg|_{t=0} = f_*\big(ch\sqrt{-1}\Theta_h \wedge u \wedge \overline{u}\big)_{t=0} - \|\xi|_{X_0}\|^2 + \|\eta|_{X_0}\|^2,$$

where c is a constant so that the pointwise pairing $ch\sqrt{-1}\Theta_h \wedge u \wedge \overline{u}$ is positive definite if u is primitive with respect to ω_X, and where $f_*(ch\sqrt{-1}\Theta_h \wedge u \wedge \overline{u})_{t=0} = \varphi(0)$ when we write the push-forward $(1,1)$-current $f_*(ch\sqrt{-1}\Theta_h \wedge u \wedge \overline{u}) = \varphi(t)\sqrt{-1}dt \wedge d\overline{t}$. We would like to show that the right hand side is positive. In case $p = n$ ([B]), the first term is positive since u is primitive simply by a bidegree reason, and one can manage to delete $\xi|_{X_0} = 0$. Thus $-\sqrt{-1}\partial\overline{\partial}\|f_* u\|^2 > 0$. One might think for general p that the first term is positive as well, since $\sqrt{-1}\Theta_h > 0$. However in general, it is not automatic, since u is not primitive at all. Thanks to detailed analysis of harmonic representatives, we can reduce it to primitive forms for some parts, but (it seems) we can not do it completely. We have a difficulty to show the Nakano positivity of $R^{n-p} f_*(\Omega_X^{p+m} \otimes E) \otimes K_Y^{-1}$, caused by this "non-primitive issue". But finally we can manage to show that $f_*(ch\sqrt{-1}\Theta_h \wedge u \wedge \overline{u})_{t=0} > 0$ and $\|\xi|_{X_0}\| \leq \|\eta|_{X_0}\|$, and then the Griffiths positivity.

[B] Berndtsson, *Curvature of vector bundles associated to holomorphic fibrations*, Ann. of Math. **169** (2009) 531–560.

[T] Takayama, *Higher direct images of twisted sheaves of differential forms*, preprint.