数理解析研究所講究録
第 1783 巻 2012 年 200-215

Semipositivity of relative canonical bundles via Kähler-Ricci flows

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Abstract

In this paper, we shall discuss the fact that the fiberwise Kähler-Ricci flow preserves the semipositivity on a smooth projective family. The full accounts will be given in [B-T].

1 Introduction

In [Ka1], Y. Kawamata proved a semipositivity of the direct image of a relative pluricanonical systems. The second author extended the result to the case of logpluricanonical systems in terms of the generalized Kähler-Einstein metric by using the method in [T4] ([T7]).

In February in 2010, the second aurhor attended the talk given by R. Berman in Luminy about [B].

Inspired by this talk the authors began to work on the stability of the semipositivity of the fiberwise Kähler-Ricci flows on a smooth projective family. This enables us to provide the homotopy version of the semipositivity of relative canonical bundles (cf. Theorem 7). This provides us a new tool to explore the projective (or possibly) Kähler families. For example, as a consequence we may give an alternative proof of the quasiprojectivity of the moduli space of polarized varieties with semiample canonical sheaves.

This is a reserch annoucement and the full accounts will be given in [B-T].

1.1 Kähler-Einstein metrics

Let X be a compact Kähler manifold. It is important to construct a canonical Kähler metric on X.

Let (X,ω) be a compact Kähler manifold. (X,ω) is said to be Kähler-Einstein, if there exists a constant c such that

$$\mathrm{Ric}(\omega) = c\cdot\omega$$

holds, where the Ricci tensor: $\mathrm{Ric}(\omega)$ is defined by

$$\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\det\omega.$$

This means that X admits a Kähler-Einstein metrics, then $c_1(X)$ is positive or negative or 0.

Theorem 1 *([A, Y1]) Let X be a compact Kähler manifold.*

(1) If $c_1(X) < 0$, then there exists a Kähler-Einstein metric ω such that

$$-\mathrm{Ric}(\omega) = \omega.$$

(2) If $c_1(X)$ is 0, for every Kähler class c, there exists a Ricci flat Kähler metric ω such that $[\omega] = c$ and

$$\mathrm{Ric}(\omega) = 0.$$

□

1.2 Twisted Kähler-Einstein metrics

Let X be a smooth projective variety defined over $\mathbb{C}$ and let (L, h_L): a (singular) hermitian $\mathbb{Q}$-line bundle on X with $\sqrt{-1}\Theta_{h_L} \geqq 0$.

ω is said to be a twisted Kähler-Einstein metrics associated with (L, h_L), if

$$-\mathrm{Ric}(\omega) + \sqrt{-1}\Theta_{h_L} = \omega$$

holds in the sense of current.

Theorem 2 *([T7]) If h_L is C^∞ on a nonempty Zariski open subset and $\mathcal{I}(h_L) \simeq \mathcal{O}_X$. Then there exists a closed positive current ω on X such that*

(1) There exists a nonempty Zariski open subset U of X such that $\omega|U$ is C^∞,

(2) $-\mathrm{Ric}(\omega) + \sqrt{-1}\Theta_{h_L} = \omega$ holds on U,

(3) $(\omega^n)^{-1} \cdot h_L$ is an AZD of $K_X + L$. □

1.3 Bergman metrics

Let X be a smooth projective variety and let (L, h_L) be a singular hermitian line bundle on X. We set

$$K(X, K_X + L, h_L) := \sum_i |\sigma_i|^2,$$

where $\{\sigma_i\}$ is an orthonormal basis of $H^0(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}(h_L))$ with respect to the inner product:

$$(\sigma, \tau) := \int_X \sigma \cdot \bar{\tau} \cdot h_L.$$

We call $K(X, K_X + L, h_L)$ the Bergman kernel of X with respect to (L, h_L). If $|H^0(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}(h_L))|$ is very ample, then the pull back of the Fubini-Study metric

$$\omega := \sqrt{-1}\partial\bar{\partial} \log K(X, K_X + L, h_L)$$

is a Kähler form on X. We call it the Bergman metric on X with respect to (L, h_L).

1.4 Dynamical construction of K-E-metrics

Let X be a smooth projectve n-fold with ample K_X and (A, h_A) be a sufficiently ample line bundle with C^∞-metric h_A. We set $K_1 = K(X, K_X + A, h_A), h_1 = K_1^{-1}$. And inductively we define

$$K_m = K(X, mK_X + A, h_{m-1}), h_m = K_m^{-1}$$

for $m \geqq 2$. Then we have the following rather unexpected result.

Theorem 3 *([T]) $dV_E = \lim_{m \to \infty} \sqrt[m]{(m!)^{-n} K_m}$ is the K-E volume form on X, i.e., $\omega_E = -\mathrm{Ric}\, dV_E$ is K-E-form.* □

1.5 Kähler-Ricci flow

Let X be a compact Kähler manifold and let ω_0: C^∞-Kähler form on X.

We consider the initial value problem:

$$\frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t)) - \omega(t) \tag{1}$$

on $X \times [0, T)$,

$$\omega(0) = \omega_0,$$

where $\mathrm{Ric}(\omega(t)) = -\sqrt{-1}\partial\bar{\partial} \log \det \omega(t)$ and T is the maximal existence time for the C^∞-solution. This type of Kähler-Ricci flow was first considered by the second author in [T1]. Then by taking the exterior derivative of the both sides of (1),

$$[\omega(t)] = (1 - e^{-t})2\pi c_1(K_X) + e^{-t}[\omega_0] \in H^{1,1}(X, \mathbb{R})$$

Let $\mathcal{K}(X)$ denote the Kähler cone of X. Then the following holds:

Proposition 1 *([T1])*

$$T = \sup\{t | [\omega(t)] \in \mathcal{K}(X)\}$$

holds. □

The next question is what happens on $\omega(t)$ after exiting the Kähler cone. Let $PE(X)$ denote the pseudoeffective cone $\subseteq H^{1,1}(X, \mathbb{R})$.

Definition 1 *Let T be a closed positive $(1,1)$ current on X. T is said to be of minimal singularities, if for every closed positive $(1,1)$-crrent T' with $[T'] = [T]$, there exists a L^1-function φ such that*

$$T' = T + \sqrt{-1}\partial\bar{\partial}\varphi$$

and is bounded from above. □

The following proposition is an easy consequence of [Le, p.26, Theorem 5].

Proposition 2 *Let $\eta \in PE(X)$ be a pseudoeffective class. Then there exists a closed positive $(1,1)$-current T_{min} with minimal singularities which represents η.* □

A closed semipositive current T with $[T] \in PE(X)$ is said to be of almost minimal singularities if we write T as $T = T_{min} + \sqrt{-1}\partial\bar{\partial}\varphi$ for some $\varphi \in L^1(X)$, $e^{-\varphi} \in L^p(X)$ holds for every $p \geqq 1$.

For a pseudoeffective $\mathbb{R}$-line bundle F on a smooth projective manifold M, we say that the decomposition:

$$F = P + N(P, N \in \mathrm{Div}(M) \otimes \mathbb{R})$$

is said to be a Zariski composition, if there exists a closed semipositive $(1,1)$ current T on M such that

(1) T is a closed semipositive current of almost minimal singularities in $2\pi c_1(F)$,

(2) $T_{sing} = 2\pi N$ in the sense of currents, where $T = T_{abc} + T_{sing}$ is the Lebesgue decomposition.

Let X be a smooth projective variety with pseudoeffective K_X. Then we have the following lemma by [B-C-H-M].

Lemma 1 *There exists a sequence: $T = T_0 < T_1 < \cdots < T_j < \cdots$ such that for each j, there exists a modification $\pi_j : X_j \longrightarrow X$ such that $\pi_j^*(e^{-t}L + (1 - e^{-t})K_X)$ admits a Zariski decomposition:*

$$\pi_j^*(e^{-t}L + (1 - e^{-t})K_X) = P_t + N_t$$

such that N_t is independent of $t \in [T_j, T_{j+1})$. □

Then we have the following theorem.

Theorem 4 *Let X be a smooth projective variety with pseudoeffective canonical class. Let (L, h_L) be a C^∞-hermitian line bundle such that $\omega_0 := \sqrt{-1}\Theta_{h_L}$ is a Kähler form on X. Then the initial value problem:*

$$\frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t)) - \omega(t) \quad on\ X \times [0, \infty), \tag{2}$$

$\omega(0) = \omega_0$ has the unique long time soluriton $\omega(t)$ such that

(1) For $t \in [T_j, T_{j+1})$, $\omega(t)$ is C^∞ on a nonempty Zariski open subset $U(T_j)$ depending on $T_j \in [0, \infty)$ defined as in Lemma 1.

(2) For $t \in [T_j, T_{j+1})$, $\omega(t)$ satisfies the equation (2) on $U(T_j)$.

(3) $\omega(t)$ is a closed semipositive current with almost minimal singularity in $(1 - e^{-t})2\pi c_1(K_X) + e^{-t}c_1(L)$. □

2 Proof of Theorem 4

Let X be a smooth projective variety with pseudoeffective canonical class and let (L, h_L) be a C^∞-hermitian line bundle on X such that $\omega_0 = \sqrt{-1}\Theta_{h_L}$ is a Kähler form.

2.1 Discretization of Kähler-Ricci flows

Let a be a positive integer. We consider the following successive equations:

$$a(\omega_{m,a} - \omega_{m-1,a}) = -\mathrm{Ric}_{\omega_{m,a}} - \omega_{m,a} \tag{3}$$

for $m \geqq 1$ under the initial condition $\omega_{0,a} = \omega_0$. We see that the cohomology class $[\omega_{m,a}]$ satisfies the equations:

$$a([\omega_{m,a}] - [\omega_{m-1,a}]) = 2\pi c_1(K_X) - [\omega_{m,a}] \tag{4}$$

Hence we see that

$$[\omega_{m,a}] = \left(1 - \left(1 + \frac{1}{a}\right)^{-m}\right) 2\pi c_1(K_X) + \left(1 + \frac{1}{a}\right)^{-m} [\omega_0] \tag{5}$$

We define the singular hermitian metric

$$h_{m,a} := n!(\omega_{m,a}^n)^{-\frac{1}{a+1}} \cdot h_{m-1}^{\frac{a}{a+1}} \tag{6}$$

on

$$(1 - t_{m,a})L + t_{m,a}K_X, \tag{7}$$

where

$$t_{m,a} = 1 - \left(1 + \frac{1}{a}\right)^{-m}. \tag{8}$$

$$\omega(m,a) := t_{m,a}(-\mathrm{Ric}\,\Omega) + (1 - t_{m,a})\omega_0 \tag{9}$$

Then the $\{u_{m,a}\}_{m=0}^{\infty}$ satisfies the successive differential equations:

$$a(u_{m,a} - u_{m-1,a}) = \log \frac{(\omega(m,a) + \sqrt{-1}\partial\bar{\partial}u_{m,a})^n}{\Omega} - u_{m,a}. \tag{10}$$

Now we introduce the following notation:

$$\delta_a u_{m,a} := a(u_{m,a} - u_{m-1,a}), \tag{11}$$

i.e., $\delta_a u_{m,a}$ denotes the (backward) difference at $u_{m,a}$.
Then (10) is denoted as:

$$\delta_a u_{m,a} = \log \frac{(\omega(m,a) + \sqrt{-1}\partial\bar{\partial}u_{m,a})^n}{\Omega} - u_{m,a}. \tag{12}$$

Later we shall see that the this equation corresponds to the parabolic Monge-Ampère equation:

$$\frac{\partial u}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} - u, \tag{13}$$

where

$$\omega_t := (1 - e^{-t})(-\mathrm{Ric}\,\Omega) + e^{-t}\omega_0 \tag{14}$$

with the initial condition: $u = 0$ on $X \times \{0\}$.

And there are correspondences:

$$\frac{m}{a} \leftrightarrow t, u_{m,a} \leftrightarrow u(\ ,t), \omega(m,a) \leftrightarrow \omega_t$$

and

$$\delta_a u_{m,a} \leftrightarrow \frac{\partial u}{\partial t}.$$

We set

$$T := \sup\{t \in \mathbb{R} | 2\pi(1-e^{-t})c_1(K_X) + e^{-t}[\omega_0] \in \mathcal{K}\}. \tag{15}$$

Since the Kähler-Ricci flow corresponds to the minimal model with scalings in [B-C-H-M] in an obvious manner, we have the following lemma.

Lemma 2 *([B-C-H-M]) The followings holds:*

(1) $e^{-T} \in \mathbb{Q}$,

(2) $(1-e^{-T})K_X) + e^{-T}L$ *is semiample.* □

By Lemma 2, there exists a C^∞-function ϕ such that

$$\omega_{T,\phi} := (1-e^{-T})(\mathrm{Ric}\,\Omega + \sqrt{-1}\partial\bar{\partial}\phi) + e^{-T}\omega_0 \tag{16}$$

is a C^∞-semipositive form on X and is strictly positive on a nonempty Zariski open subset of X. We set

$$\omega(m,a)_\phi := \left(1 - \left(1+\frac{1}{a}\right)^{-m}\right)(\mathrm{Ric}\,\Omega + \sqrt{-1}\partial\bar{\partial}\phi) + \left(1+\frac{1}{a}\right)^{-m}\omega_0 \tag{17}$$

$$= \omega(m,a) + \left(1 - \left(1+\frac{1}{a}\right)^{-m}\right)\sqrt{-1}\partial\bar{\partial}\phi$$

We set

$$m(a) := \sup\left\{m \left| \left(1 - \left(1+\frac{1}{a}\right)^{-m}\right)c_1(K_X) + \left(1+\frac{1}{a}\right)^{-m}[\omega_0] \in \mathcal{K}\right.\right\}. \tag{18}$$

Then since

$$\omega(m,a)_\phi = \frac{1-\left(1+\frac{1}{a}\right)^{-m}}{1-e^{-T}}\omega_{T,\phi} + \frac{\left(1+\frac{1}{a}\right)^{-m} - e^{-T}}{1-e^{-T}}\omega_0. \tag{19}$$

for every $m < m(a)$, $\omega(m,a)_\phi$ is a C^∞-Kähler form on X and for $m = m(a)$, $\omega(m,a)_\phi = \omega_{T,\phi}$ holds.

Theorem 5 *(3) has a smooth solution $\omega_{m,a}$ as long as $[\omega(m,a)] \in \mathcal{K}$. And (10) has C^∞-solution as $[\omega(m,a)] \in \mathcal{K}$.* □

Lemma 3 *Suppose that T is finite, then we see that*

$$\omega(T) := \lim_{t \uparrow T}\omega(t)$$

exists in C^∞-topology on $X \backslash E$ and is a well defined as a limit of closed positive current on X. □

2.2 Beyond the Kähler cone

After exiting the Kähler cone, the singular solution of the Kähler-Ricci flow can be constructed as follows.

Theorem 6 *There exists a sequence of closed semipositive currents* $\{\omega_{m,a}\}_{m=0}^{\infty}$ *such that*

(1) For every $m \geqq 0$, $\omega_{m,a}$ *is a closed semipositive current on* X,

(2) There exists a nonempty Zariski open subset U_m *of* X *such that* $h_{m,a}|U_m$ *is* C^{∞},

(3) $h_{m,a}$ *is an AZD of the* $\mathbb{Q}$*-line bundle* $(1 - t_{m,a})L + t_{m,a}K_X$,

(4) $\omega_{m,a} = \sqrt{-1}\Theta_{h_{m,a}}$ *is a well defined closed semipositive current on* X,

(5) $\{\omega_{m,a}\}_{m=0}^{\infty}$ *satisfies the equations (3) on* U_m. □

The following lemma is a slight refinement of Lemma 1.

Lemma 4 *There exists a sequence of positive number* $T = T_0 < T_1 < \cdots < T_j < \cdots$ *such that for every* $t \in [T_j, T_{j+1})$

(1) There exists a modification $\pi_j : X_j \longrightarrow X$ *such that* $\pi_j^*(e^{-t}L + (1-e^{-t})K_X)$ *admits a Zariski decomposition:*

$$\pi_j^*(e^{-t}L + (1 - e^{-t})K_X) = P_t + N_t (P_t, N_t \in \mathrm{Div}(X_j) \otimes \mathbb{R}),$$

where P_t *is nef and* N_t *is effective and*

$$H^0(X_j, \mathcal{O}_{X_j}(\lfloor mP_j \rfloor)) \simeq H^0(X_j, \mathcal{O}_{X_j}(m\pi_j^*(e^{-t}L + (1 - e^{-t})K_X))$$

holds for every m *such that* $me^{-t} \in \mathbb{Z}$.

(2) N_t *is independent of* $t \in [T_j, T_{j+1})$,

(3) If $e^{-t} \in \mathbb{Q}$, *then* P_t *is semiample.* □

We set $N_j := N_t(t \in [T_j, T_{j+1}))$. Let τ_j be the multivalued holomorphic section of N_j with divisor N_j. Then there exists a C^{∞}-hermitian metric $\| \ \|$ such that $\omega_{T_j} + \sqrt{-1}\partial\bar{\partial} \log \| \tau_j \|^2$ is a closed semipositive current. We set

$$\phi_j := \log \| \tau_j \|^2 . \qquad (20)$$

Suppose that we have already defined $u_{0,a}(\phi_j)$ such that for every $\varepsilon > 0$, there exists a constant $C(\varepsilon)$

$$u_{0,a}(\phi_j) \geqq \varepsilon\phi_j + C(\varepsilon) \qquad (21)$$

holds. We set

$$\omega_j(m, a) := \left(1 - e^{-T_j}\left(1 + \frac{1}{a}\right)^{-m}\right)(-\mathrm{Ric}\,\Omega) + e^{-T_j}\left(1 + \frac{1}{a}\right)^{-m}\omega_0. \qquad (22)$$

We consider the Ricci iteration:

$$\delta_a u_{m,a}(\phi_j) = \log \frac{(\omega(m, a)_{\phi_j} + \sqrt{-1}\partial\bar{\partial}u_{m,a}(\phi_j))^n}{\Omega \cdot e^{-\phi_j}} - u_{m,a}(\phi_j). \qquad (23)$$

The rest of the proof is similar to the case $t \in [0, T)$.

3 Semipositivity of a Kähler-Ricci flow

In this section we shall sketch the proof of the fact that the relative Kähler-Ricci flows preserve the semipositivity in the horizontal direction on projective families.

3.1 Main results

Let $f : X \longrightarrow S$ be a smooth projective family and let ω be a relative Kähler form on X. We set $n := \dim X - \dim S$ and $k := \dim S$. We define the relative Ricci form $\mathrm{Ric}_{X/S,\omega}$ of ω by

$$\mathrm{Ric}_{X/S,\omega} = -\sqrt{-1}\partial\bar{\partial}\log\left(\omega^n \wedge f^*|ds_1 \wedge \cdots \wedge ds_k|^2\right), \tag{24}$$

where $(s_1, \cdots, s_k)$ is a local coordinate on S. Then it is easy to see that $\mathrm{Ric}_{X/S,\omega}$ is independent of the choice of the local coordinate $(s_1, \cdots, s_k)$. The Kähler-Ricci flow preserves the semipositivity in the following sense.

Theorem 7 *Let $f : X \longrightarrow S$ be a smooth projective family of varieties with pseudoeffective canonical bundles. Let L be an ample line bundle on X and let h_L be a C^∞-hermitian metric on L with strictly positive curvature. Suppose that there exists a C^∞-relative volume form Ω on $f : X \longrightarrow S$ such that $\mathrm{Ric}\,\Omega + \sqrt{-1}\,\Theta_{h_L}$ is also a Kähler form on X. We set $\omega_0 := \sqrt{-1}\Theta_{h_L}$. We consider the normalized Kähler-Ricci flow:*

$$\frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}_{X/S,\omega(t)} - \omega(t)$$

on X with the initial condition $\omega(0) = \omega_0$, where $\mathrm{Ric}_{X/S,\omega(t)}$ denotes the relative Ricci form of $\omega(t)$ on X

Then $\omega(t)$ is a closed semipositive current on X for every $t \in [0, \infty)$. □

In Theorem 7, the semipositivity of $\omega(t)$ corresponds to the pseudoeffectivity of $(1 - e^{-t})K_{X/S} + e^{-t}L$. And as t goes to infinity, we observe that the relative canonical bundle $K_{X/S}$ is pseudoeffective.

Similarly we have the following theorem.

Theorem 8 *Let $f : X \longrightarrow S$ be a smooth projective family of varieties with pseudoeffective canonical bundles. Let L be an ample line bundle on X and let h_L be a C^∞-hermitian metric on L with strictly positive curvature. Let K be a closed semipositive current on X such that K is C^∞ on a nonempty Zariski open subset of X and $[K] \in 2\pi c_1(K_{X/S})$. We set $\omega_0 := \sqrt{-1}\Theta_{h_L}$. We consider the Kähler-Ricci flow:*

$$\frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}_{X/S,\omega(t)} - K$$

on X with the initial condition $\omega(0) = \omega_0$, where $\mathrm{Ric}_{X/S,\omega(t)}$ denotes the relative Ricci form of $\omega(t)$ on X

Then $\omega(t)$ is a closed semipositive current on X for every $t \in [0, \infty)$. Moreover as t goes to infinity, $\omega(t)$ converges to a current solution of $-\mathrm{Ric}_{X/S,\omega(t)} = K$. □

3.2 Some conjecture for the Kähler case

We expect that the similar statement holds even in the case that $f : X \longrightarrow S$ is a smooth Kähler fibration.

Conjecture 1 *Let X be a compact Kähler manifold with pseudoeffective canonical bundle. And let ω_0 be a C^∞-Kähler form on X. Suppose that there exists a C^∞-volume form Ω such that*

$$\mathrm{Ric}\,\Omega + \omega_0$$

is also a Kähler form on X. Then there exists a family of closed semipositive current $\omega(t)$ on X such that

(1) $\omega(0) = \omega_0$,

(2) For every $T > 0$, there exists a nonempty Zariski open subset $U(T)$ depending on T such that $\omega(t)$ is Kähler form on $U(T) \times [0, T)$,

(3) $[\omega(t)] = 2\pi(e^{-t}[\omega_0] + (1 - e^{-t})c_1(K_X))$ *holds for every $t \in [0, \infty)$,*

(4) On $U(t) \times [0, T)$ $\omega(t)$ satisfies the differential equation:

$$\frac{\partial \omega(t)}{\partial t} = -\mathrm{Ric}_{\omega(t)} - \omega(t).$$

□

Conjecture 2 *Let $f : X \longrightarrow S$ be a smooth Kähler family with pseudoeffective canonical bundles. Let ω_0 be a C^∞-Kähler form on X. Suppose that there exists a C^∞-relative volume form Ω on $f : X \longrightarrow S$ such that $\mathrm{Ric}\,\Omega + \omega_0$ is also a Kähler form on X. We consider the normalized Kähler-Ricci flow:*

$$\frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}_{X/S,\omega(t)} - \omega(t)$$

on X with the initial condition:$\omega(0) = \omega_0$, where $\mathrm{Ric}_{X/S\omega(t)}$ denotes the relative Ricci form of $\omega(t)$ on X

Then $\omega(t)$ is a closed semipositive current on X for every $t \in [0, \infty)$. □

This conjecture will lead us to the invariance of plurigenera in the Kähler case.

4 Proof of Theorem 7

The essential technical difficulty here is the fact that we cannot apply the direct calculation of the variation, since the Kähler-Ricci flow in Theorem 4 has singularities. We overcome this difficulty by using the dynamical construction of the solution of the Ricci iterations as in [LC]

4.1 The relative Ricci iterations to the relative Kähler-Ricci flow

Let $f : X \longrightarrow S$ be a smooth projective family of varieties with pseudoeffective canonical bundles. Let L be an ample line bundle on X and let h_L be a C^∞-hermitian metric on L with strictly positive curvature. Suppose that there exists a C^∞-relative volume form Ω on $f : X \longrightarrow S$ such that $\mathrm{Ric}\,\Omega + \sqrt{-1}\,\Theta_{h_L}$ is also a Kähler form on X. We set $\omega_0 := \sqrt{-1}\Theta_{h_L}$. We consider the normalized Kähler-Ricci flow:

$$\frac{\partial\omega(t)}{\partial t} = -\mathrm{Ric}_{X/S,\omega(t)} - \omega(t) \tag{25}$$

on X with the initial condition $\omega(0) = \omega_0$, where $\mathrm{Ric}_{\omega(t)}$ denotes the relative Ricci form on X.

For every $s \in S$, we consider Lemma 1. Then by the invariance of the twisted plurigenra, we see that for every $C > 0$ the sequence

$$T = T_0 < T_1 < \cdots < T_j < \cdots < C \tag{26}$$

in Lemma 1 are constant on a nonempty Zariski open subset $S(C)$ of S.

Suppose that we have already proven the (logarithmic) plurisubharmonic varitation property of the solution $\omega(t)$ of (25) for every $t < C$ on $f^{-1}(S(C))$. Then the removable singularity theorem for plurisubharmonic function implies the logarithmic plurisubharmonic variation property of the solution $\omega(t)$ over the whole X.

Hence we may and do assume that the sequence $T_0 < \cdots < T_j < \cdots$ are constant over the whole S without loss of generality. Moreover since the assertion of Theorem 7 is local in S, we may and do assume that S is the unit open polydisk Δ^k in $\mathbb{C}^k$.

The plurisubharmonic variation propety of the Ricci iteration is proven by the parallel argument as follows.

We set

$$m(a) := \sup\left\{ m \,\middle|\, \left(1 + \frac{1}{a}\right)^{-m} > e^{-T_0} \right\}. \tag{27}$$

First we shall consider the relative Ricci iteration:

$$\delta_a\omega_{m,a} = -\mathrm{Ric}_{\omega_{m,a},/S} - \omega_{m,a}, \omega_{0,a} = \omega_0 \tag{28}$$

on X for $0 \leqq m < m(a)$. This is equivalent to the fiberwise Ricci iteration:

$$\delta_a\omega_{m,a,z} = -\mathrm{Ric}_{\omega_{m,a}/S,s} - \omega_{m,a,s}, \omega_{0,a} = \omega_0|X_s, \tag{29}$$

on X_s for $0 \leqq m < m(a)$. Then by the proof of Theorem 4, letting a tends to infinity, we may construct the solution of the relative Kähler-Ricci flow:

$$\frac{\partial\omega(t)}{\partial t} = -\mathrm{Ric}_{X/S,\omega(t)} - \omega(t) \tag{30}$$

on $X \times [0, T_0)$.

Then as in the previous section, we may continue this process beyond the critical time T_0 and we obtain the long time existence of the current solution of the relative Kähler-Ricci flow on X.

4.2 Auxiliary Ricci iterations

We prove Theorem 7 by decomposing the Ricci iterations by a dynamical system of Bergman kernels and apply the plurisubharmonic variation properties of Bergman kernels due to Berndtsson. The main difficulty is to deal with $\mathbb{Q}$-line bundles. We deal with $\mathbb{Q}$-line bundles in terms of the auxiliary Ricci iterations.

Lemma 5 *For every $0 \leqq m \leqq m(a)$, $\omega_{m,a}$ is semipositive on X.* $\square$

We prove Lemma 5 by induction on m.

For $m = 0$ $\omega_{0,a} = \omega_0$ is a Kähler form on X by the assumption. Hence Lemma 5 holds for $m = 0$. Suppose that $\omega_{m,a}$ is semipositive on X. We shall prove that $\omega_{m+1,a}$ is also semipositive on X.

To prove this assertion, we consider the auxiliary Ricci iteration which connects $\omega_{m,a}$ and $\omega_{m+1,a}$.

First we define the $\mathbb{Q}$-line bundle L_m by

$$L_m := \left(1 - \left(1 + \frac{1}{a}\right)^{-m}\right) K_{X/S} + \left(1 + \frac{1}{a}\right)^{-m} L. \tag{31}$$

Let $q = q(m+1)$ be a postive integer such that qL_{m+1} is a genuine line bundle on X. Since

$$L_{m+1} = \left(1 - \left(1 + \frac{1}{a}\right)^{-(m+1)}\right) K_{X/S} + \left(1 + \frac{1}{a}\right)^{-(m+1)} L$$

is of the form $\beta(K_{X/S} + \alpha L)$ for some positive rational numbers α and β. By B-C-H-M, we have that the relative logcanonical ring:

$$R(X, K_{X/S} + \alpha L) = \oplus_{\nu=0}^{\infty} f_* \mathcal{O}_X(\lfloor \nu(K_{X/S} + \alpha L) \rfloor)$$

is a finitely generated algebra over $\mathcal{O}_S$. By the invariance of twisted plurigeera, we see that each $f_*\mathcal{O}_X(\lfloor \nu(K_{X/S} + \alpha L)\rfloor)$ is a vector bundle over S which is biholomorphic to the unit open polydisk Δ^k. We take a sufficiently large positive integer ν_0 and take a set of generators $\{\sigma_i\}$ of $f_*\mathcal{O}_X(\nu_0!(K_{X/S} + \alpha L))$ (In this casse $K_{X/S} + \alpha L$ is relatively ample. But later we also consider the case $K_{X/S} + \alpha L$ is big, but not relatively ample). Then we set

$$h_{m,a,0} := \left(\sum_i |\sigma_i|^2\right)^{-\frac{\beta}{\nu_0!}} \tag{32}$$

and

$$\omega_{m,a,0} := \sqrt{-1}\Theta_{h_{m,a,0}}. \tag{33}$$

Then $h_{m,a,0}$ is a hermitian metric of $L_{m+1} = \beta(K_{X/S} + \alpha L)$ with semipositive curvature on X. Now we shall consider the following Ricci iteration:

$$-\mathrm{Ric}_{\omega_{m,a,\ell}} + (q - a - 1)\omega_{m,a,\ell-1} + a\omega_{m,a} = q\omega_{m,a,\ell} \tag{34}$$

for $\ell \geqq 1$. The following lemma follows entirely the same way as the dynamical construction of Kähler-Einstein metrics.

Lemma 6 $\lim_{\ell \longrightarrow \infty} \omega_{m,a,\ell}$ *exists in* C^{∞}*-topology on* X. *And*

$$\lim_{\ell \longrightarrow \infty} \omega_{m,a,\ell} = \omega_{m+1,a} \tag{35}$$

holds. □

We use this auxiliary Ricci iteration to connect $\omega_{m,a}$ and $\omega_{m+1,a}$ by a dynamical system of Bergman kernels. This method is exactly the same one in [T7].

4.3 Dynamical systems of Bergman kernels

To prove the semipositivity of $\omega(t)$ on X for $t \in [0, T_0]$, it is enough to prove the following lemma.

Lemma 7 $h_{m,a}$ *has semipositive curvature on* X. □

We now use the strategy as in [T7]. We shall prove Lemma 7 by induction on m. Since h_L has positive curvature, $h_{0,a} = h_L$ has semipositive curvature.

Suppose that we have already proven that $h_{m-1,a}$ has semipositive curvature.

Let A be a sufficiently ample line bundle on X and let h_A be a C^{∞}-hermitian metric on X with strictly positive curvature.

Now we shall define the metric on L_{m+1} by

$$h_{m,a,\ell}|X_s = h_{m,a,\ell,s} (s \in S). \tag{36}$$

By induction on ℓ, we shall prove the following lemma.

Lemma 8 $h_{m,a,\ell}$ *has semipositive curvarue on* X *for every* $\ell \geqq 0$. □

Proof of Lemma 8. By the construction (cf. (32)), $h_{m,a,0}$ has semipositive curvature.

Suppose that we have already proven that $h_{m,a,\ell-1}$ is a hermitian metric with semipositive curvature on X. For every $s \in S$, we shall consider the dynamical system of Bergman kernels as follows. We set

$$K_{1,s} := K\left(X_s, A + K_{X_s} + (q-a-1)L_{m+1} + aL_m|X_s), h_A \cdot h_{m,\ell-1}^{q-a-1} \cdot h_{m,a}^{a}|X_s\right) \tag{37}$$

and

$$h_{1,s} := K_{1,s}^{-1}. \tag{38}$$

Suppose that we have already constructed $K_{p-1,s}$ and $h_{p-1,s}$ for some $p \geqq 2$. Then we define $K_{p,s}$ and $h_{p,s}$ by

$$K_{p,s} := K\left(X_s, A + p(K_{X_s} + (q-a-1)L_{m+1} + aL_m|X_s), h_{m,\ell-1}^{q-a-1} \cdot h_{m,a}^{a} \cdot h_{p-1}|X_s\right) \tag{39}$$

and

$$h_{p,s} := \frac{1}{K_{p,s}}. \tag{40}$$

Similarly as in [T4, T7] we have the following lemma.

Lemma 9

$$K_{\infty,s} := \limsup_{p\longrightarrow\infty}\left((p!)^{-n}h_A\cdot K_{p,s}\right)^{\frac{1}{pq}} \tag{41}$$

exists in L^1-topology and

$$h_{m,a,\ell,s} := K_{\infty,s}^{-1} \tag{42}$$

is a C^∞-hermitian metric on $L_{m+1}|X_s$. And the curvature

$$\omega_{m,a,\ell,s} := \sqrt{-1}\Theta_{h_{m,a,\ell,s}} \tag{43}$$

satisfies the differential equation:

$$-\mathrm{Ric}_{\omega_{m,a,\ell,s}} + (q-a-1)\omega_{m,a,\ell-1,s} + a\omega_{m,a,s} = q\omega_{m,a,\ell,s} \tag{44}$$

on X. □

We define the relative Bergman kernel K_p on X by

$$K_p|X_s = K_{p,s}.$$

Then $h_p = K_p^{-1}$ is a hermitian metric with semipostive curvature on $A + p(K_{X/S} + (q-a-1)L_{m+1} + aL_m)$ by induction on p by the following theorem mainly due to B. Berndtsson.

Theorem 9 *([B1, B2, B3, B-P]) Let $f : X \longrightarrow S$ be a projective family of projective varieties over a complex manifold S. Let S° be the maximal nonempty Zariski open subset such that f is smooth over S°.*

Let (L, h_L) be a pseudo-effective singular hermitian line bundle on X. Let $K_s := K(X_s, K_X + L\,|_{X_s}, h\,|_{X_s})$ be the Bergman kernel of $K_{X_s} + (L|\,X_s)$ with respect to $h \mid X_s$ for $s \in S^\circ$. Then the singular hermitian metric h of $K_{X/S} + L\,|\,f^{-1}(S^\circ)$ defined by

$$h \mid X_s := K_s^{-1}(s \in S^\circ)$$

has semipositive curvature on $f^{-1}(S^\circ)$ and extends to X as a singular hermitian metric on $K_{X/S} + L$ with semipositive curvature in the sense current. □

Now we prove the semipositivity of $\sqrt{-1}\Theta_{h_p}$ by induction on p. First the semipositivity of $\sqrt{-1}\Theta_{h_1}$ follows from Theorem 9 by the assumption that $\sqrt{-1}\Theta_{h_{m,a,\ell-1}}$ and $\sqrt{-1}\Theta_{h_{m-1,a}}$ are semipositive. Suppose that we have already proven the semipositivity of h_{p-1} for some $p \geqq 2$. We note that $h_{p-1}, h_{m,a,\ell-1}$ and $h_{m,a}$ has semipositive curvature on X by the induction assumption. Then by the inductive definition of h_p (cf. (39) and (40)) and Theorem 9, we see that $\sqrt{-1}\Theta_{h_p}$ is also semipositive.

Hence by induction, we see that $\{h_p\}_{p=1}^{\infty}$ has semipositive curvature on X. Then by Lemma 9, we see that $h_{m,a,\ell}$ has semiposive curvature. This completes the proof of Lemma 8. □

By Lemmas 6 and 8, we see that h_{m+1} is a metric on L_{m+1} with semipositive curvature. Hence by induction on m, we complete the proof of Lemma 7. □

Now by Lemma 7 and the proof of Theorem 1, we see that $\omega(t)$ is semipositive on X for $t \in [0, T_0]$.

Now we complete the proof of Theorem 7 by repeating the similar argument inductively for $t \in [T_j, T_{j+1}] (j \geqq 0)$. This completes the proof of Theorem 7. □

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