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Hochschild cohomology of quiver algebras defined by two cycles and a quantum-like relation

(2サイクルを持つ擬量子多元環のホッホシルトコホモロジー)

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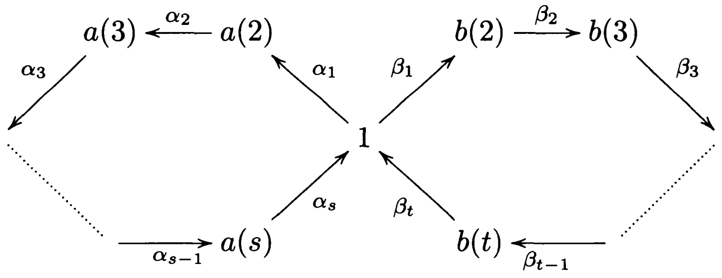
Abstract

This paper is based on my talk given at the Symposium on Cohomology Theory of Finite Groups and Related Topics held at Kyoto University, Japan, 29 August to 2 September 2011. In this paper, we consider quiver algebras A_q over a field k defined by two cycles and a quantum-like relation depending on a non-zero element q in k. We determine the ring structure of the Hochschild cohomology ring of A_q modulo nilpotence and give a necessary and sufficient condition for A_q to satisfy the finiteness conditions given in [4].

Introduction

Let A be an indecomposable finite dimensional algebra over a field k. We denote by A^e the enveloping algebra $A\otimes_k A^{op}$ of A, so that left A^e-modules correspond to A-bimodules. The Hochschild cohomology ring is given by $\mathrm{HH}^*(A) = \mathrm{Ext}^*_{A^e}(A, A) = \oplus_{n\geq 0}\mathrm{Ext}^n_{A^e}(A, A)$ with Yoneda product. It is well-known that $\mathrm{HH}^*(A)$ is a graded commutative ring, that is, for homogeneous elements $\eta \in \mathrm{HH}^m(A)$ and $\theta \in \mathrm{HH}^n(A)$, we have $\eta\theta = (-1)^{mn}\theta\eta$. Let $\mathcal{N}$ denote the ideal of $\mathrm{HH}^*(A)$ which is generated by all homogeneous nilpotent elements. Then $\mathcal{N}$ is contained in every maximal ideal of $\mathrm{HH}^*(A)$, so that the maximal ideals of $\mathrm{HH}^*(A)$ are in 1-1 correspondence with those in the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$.

Let q be a non-zero element in k and s, t integers with $s, t \geq 1$. We consider the quiver algebra $A_q = kQ/I_q$ defined by the two cycles Q with $s+t-1$ vertices and $s+t$ arrows as follows:



and the ideal I_q of kQ generated by

$$X^{sa}, X^sY^t - qY^tX^s, Y^{tb}$$

for $a, b \geq 2$ where we set $X{:=} \alpha_1 + \alpha_2 + \cdots + \alpha_s$ and $Y{:=} \beta_1 + \beta_2 + \cdots + \beta_t$. We denote the trivial path at the vertex $a(i)$ and at the vertex $b(j)$ by $e_{a(i)}$ and by $e_{b(j)}$ respectively. We regard the numbers i in the subscripts of $e_{a(i)}$ modulo s and j in the subscripts of $e_{b(j)}$ modulo t. In this paper, we describe the ring structure of $\mathrm{HH}^*(A_q)/\mathcal{N}$.

In [19], Snashall and Solberg used the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$ to define a support variety for any finitely generated module over A. This led us to consider the structure of $\mathrm{HH}^*(A)/\mathcal{N}$. In [19], Snashall and Solberg conjectured that $\mathrm{HH}^*(A)/\mathcal{N}$ is always finitely generated as a k-algebra. But a counterexample to this conjecture was given by Snashall [18] and Xu [23]. This example makes us consider whether we can give necessary and sufficient conditions on a finite dimensional algebra A for $\mathrm{HH}^*(A)/\mathcal{N}$ to be finitely generated as a k-algebra.

On the other hand, in the theory of support varieties, it is interesting to know when the variety of a module is trivial. In [4], Erdmann, Holloway, Snashall, Solberg and Taillefer gave the necessary and sufficient conditions on a module for it to have trivial variety under some finiteness conditions on A. In the paper, we show that A_q satisfies the finiteness conditions given in [4] if and only if q is a root of unity.

The content of the paper is organized as follows. In Section 1 we deal with the definition of the support variety given in [19] and precedent results about the Hochschild cohomology ring modulo nilpotence. In Section 2, we describe the finiteness conditions given in [4] and introduce precedent results about these conditions. In Section 3, we determine the Hochschild cohomology ring of A_q modulo nilpotence and show that A_q satisfies the finiteness conditions if and only if q is a root of unity.

1 Support variety

In [19], Snasall and Solberg defined the support variety of a finitely generated A-module M over a noetherian commutative graded subalgebra H of $\mathrm{HH}^*(A)$ with $H^0 = \mathrm{HH}^0(A)$. In this paper, we consider the case $H = \mathrm{HH}^*(A)$.

Definition 1.1 ([19]). *The support variety of M is given by*

$$V(M) = \{m \in \mathrm{MaxSpec}\,\mathrm{HH}^*(A)/\mathcal{N} |\, \mathrm{AnnExt}^*_A(M,M) \subseteq m'\}$$

where $\mathrm{AnnExt}^*_A(M,M)$ is the annihilator of $\mathrm{Ext}^*_A(M,M)$, m' is the pre-image of m for the natural epimorphism and the $\mathrm{HH}^*(A)$-action on $\mathrm{Ext}^*_A(A,A)$ is given by the graded algebra homomorphism $\mathrm{HH}^*(A) \xrightarrow{-\otimes M} \mathrm{Ext}^*_A(M,M)$.

Since A is indecomposable, we have that $\mathrm{HH}^0(A)$ is a local ring. Thus $\mathrm{HH}^*(A)/\mathcal{N}$ has a unique maximal graded ideal $m_{gr} = \langle \mathrm{rad}\,\mathrm{HH}^*(A), \mathrm{HH}^{\geq 1}(A)\rangle/\mathcal{N}$. We say that the variety of M is trivial if $V(M) = \{m_{gr}\}$.

In [18], Snashall gave the following question.

Question ([18]). *Whether we can give necessary and sufficient conditions on a finite dimensional algebra for the Hochschild cohomology ring modulo nilpotence to be finitely generated as a k-algebra.*

With respect to sufficient condition, it is shown that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated as a k-algebra for various classes of algebras by many authors as follows:

(1) In [6], [22], Evens and Venkov showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for any block of a group ring of a finite group.

(2) In [7], Friedlander and Suslin showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for any block of a finite dimensional cocommutative Hopf algebra.

(3) In [11], Green, Snashall and Solberg showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for finite dimensional self-injective algebras of finite representation type over an algebraically closed field.

(4) In [12], Green, Snashall and Solberg showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for finite dimensional monomial algebras.

(5) In [13], Happel showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for finite dimensional algebras of finite global dimension.

(6) In [17], Schroll and Snashall showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for the principal block of the Heche algebra $H_q(S_5)$ with $q = -1$ defined by the quiver

$$Q: \begin{tikzcd} 1 \arrow[loop left, "\varepsilon"] \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\overline{\alpha}"] \arrow[loop right, "\overline{\varepsilon}"] \end{tikzcd}$$

and the ideal I of kQ generated by

$$\alpha\overline{\varepsilon}, \overline{\alpha}\varepsilon, \overline{\varepsilon}\overline{\alpha}, \varepsilon^2 - \alpha\overline{\alpha}, \overline{\varepsilon}^2 - \overline{\alpha}\alpha.$$

(7) In [20], Snashall and Taillefer showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for a class of special biserial algebras.

(8) In [14], Koenig and Nagase produced many examples of finite dimensional algebras with a stratifying ideal for which $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated as a k-algebra.

(9) In [18] and [23], Snashall and Xu gave the example of a finite dimensional algebra for which $\mathrm{HH}^*(A)/\mathcal{N}$ is not a finitely generated k-algebra.

Example 1.2. ([18, Example 4.1]) Let $A = kQ/I$ where Q is the quiver

$$\begin{tikzcd} 1 \arrow[loop above, "a"] \arrow[loop below, "b"] \arrow[r, "c"] & 2 \end{tikzcd}$$

and $I = \langle a^2, b^2, ab - ba, ac\rangle$. Then Snashall showed the following in [18, Theorem 4.5].

(a) $\mathrm{HH}^*(A)/\mathcal{N} \cong \begin{cases} k \oplus k[a,b]b & \text{if } \operatorname{char} k = 2, \\ k \oplus k[a^2,b^2]b^2 & \text{if } \operatorname{char} k \neq 2. \end{cases}$

(b) $\mathrm{HH}^*(A)/\mathcal{N}$ is not finitely generated as a $k-$algebra.

Xu showed this in the case $\operatorname{char} k = 2$ in [23].

2 Finiteness conditions

In [4], Erdmann, Holloway, Snashall, Solberg and Taillefer gave the following two conditions **(Fg1)** and **(Fg2)** for an algebra A and a graded subalgebra H of $\mathrm{HH}^*(A)$.

(Fg1) H is a commutative Noetherian algebra with $H^0 = \mathrm{HH}^0(A)$.

(Fg2) $\mathrm{Ext}_A^*(A/\mathrm{rad}\,A, A/\mathrm{rad}\,A)$ is a finitely generated H-module.

In [4], under the finiteness conditions above, some geometric properties of the support variety and some representation theoretic properties are related. In particular, the following theorem hold.

Theorem 2.1 ([4, Theorem 2.5]). *Suppose that A satisfies the finiteness conditions.*

(a) *A is Gorenstein, that is, A has finite injective dimension both as a left A-module and as a right A-module.*

(b) *The following are equivalent for an A-module M.*

 (i) *The variety of M is trivial.*

 (ii) *The projective dimension of M is finite.*

 (iii) *The injective dimension of M is finite.*

There are some papers which deal with the finiteness conditions as follows.

(1) In [2], Bergh and Oppermann show that a codimension n quantum complete intersection satisfies the finiteness conditions if and only if all the commutators q_{ij} are roots of unity.

Definition 2.2. *A codimension n quantum complete intersection is defined by*

$$k\langle x_1, \ldots, x_n\rangle / I$$

where I generated by

$$x_i^{a_i}, x_j x_i - q_{ij} x_i x_j \quad \text{for } 1 \le i < j \le n, a_i \ge 2, q_{ij} \in k.$$

(2) In [5], Erdmann and Solberg gave the necessary and sufficient conditions on a Koszul algebra for it to satisfy the finiteness conditions.

Theorem 2.3 ([5, Theorem 1.3]). *Let A be a finite dimensional Koszul algebra over an algebraically closed field, and let $E(A) = \mathrm{Ext}_A^*(A/\mathrm{rad}\,A, A/\mathrm{rad}\,A)$. A satisfies the finiteness conditions if and only if $Z_{gr}(E(A))$ is Noetherian and $E(A)$ is a finitely generated $Z_{gr}(E(A))$-module.*

(3) In [9], Furuya and Snashall provided examples of (D, A)-stacked monomial algebras which are not self-injective but satisfy the finiteness conditions.

Example 2.4. ([9, Example 3.2]) Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[d, "\beta"] \\ 4 \arrow[u, "\delta"] & 3 \arrow[l, "\gamma"] \end{tikzcd}$$

and I the ideal of kQ generated by

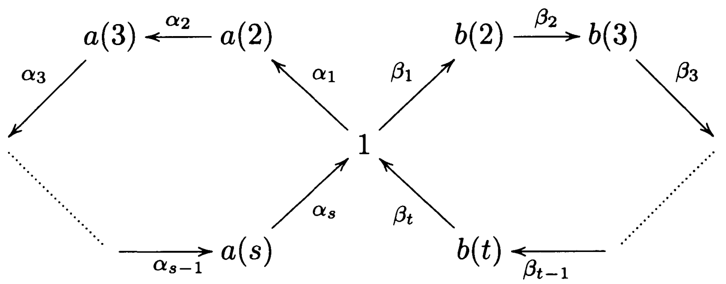
$$\alpha\beta\gamma\delta\alpha\beta, \gamma\delta\alpha\beta\gamma\delta.$$

Then, $A = kQ/\mathcal{I}$ is not self-injective but satisfies the finiteness conditions.

(4) In [17], Schroll and Snashall show that the finiteness conditions hold for the principal block of the Heche algebra $H_q(S_5)$ with $q = -1$.

3 Hochschild cohomology ring of quiver algebras defined by two cycles and a quantum-like relation

In this section, we consider the quiver algebras $A_q = kQ/I_q$ defined by the quiver Q as follows:



and the ideal I_q of kQ generated by

$$X^{sa}, X^sY^t - qY^tX^s, Y^{tb}$$

for $a, b \geq 2$ where we set $X := \alpha_1 + \alpha_2 + \cdots + \alpha_s$ and $Y := \beta_1 + \beta_2 + \cdots + \beta_t$, and q is non-zero element in k. Paths are written from right to left. We will determine the Hochschild cohomology ring of A_q modulo nilpotence $\mathrm{HH}^*(A_q)/\mathcal{N}$ and show that A_q satisfies the finiteness conditions if and only if q is a root of unity.

First, we note that the following elements in A_q form a k-basis of A_q.

$X^{sl+l'}e_{a(i)}$	for $2 \leq i \leq s, 0 \leq l \leq a-1, 0 \leq l' \leq s-1$,
$Y^{tl+l'}e_{b(j)}$	for $1 \leq j \leq t, 0 \leq l \leq b-1, 0 \leq l' \leq t-1$,
$X^{si+l}Y^{tj+l'}$	for $0 \leq i \leq a-1, 0 \leq j \leq b-1, 1 \leq l \leq s-1, 0 \leq l' \leq t-1$,
$X^{si}Y^{tj+l'}$	for $1 \leq i \leq a-1, 0 \leq j \leq b-1, 0 \leq l' \leq t-1$,
$X^{si}Y^{tj}X^l$	for $0 \leq i \leq a-1, 1 \leq j \leq b-1, 1 \leq l \leq s-1$,
$Y^{l'}X^{si}Y^{tj}$	for $1 \leq i \leq a-1, 0 \leq j \leq b-1, 1 \leq l' \leq t-1$,
$Y^{l'}X^{si}Y^{tj}X^l$	for $0 \leq i \leq a-1, 0 \leq j \leq b-1, 1 \leq l \leq s-1, 1 \leq l' \leq t-1$,
$X^{si+l}Y^{tj}X^{l'}$	for $0 \leq i \leq a-1, 1 \leq j \leq b-1, 1 \leq l, l' \leq s-1$,
$Y^lX^{si}Y^{tj+l'}$	for $1 \leq i \leq a-1, 0 \leq j \leq b-1, 1 \leq l, l' \leq t-1$.

So we have $\dim_k A_q = ab(s+t-1)^2$.

3.1 Projective resolution of A_q

For $n \geq 0$, we define left A_q^e-modules, equivalently A_q-bimodules

$$P_{2n} = \coprod_{l=0}^{2n} A_qe_1 \otimes e_1A_q \oplus \coprod_{i=2}^{s} A_qe_{a(i)} \otimes e_{a(i)}A_q \oplus \coprod_{j=2}^{t} A_qe_{b(j)} \otimes e_{b(j)}A_q,$$

$$P_{2n+1} = \coprod_{l=1}^{2n} A_qe_1 \otimes e_1A_q \oplus \coprod_{i=1}^{s} A_qe_{a(i+1)} \otimes e_{a(i)}A_q \oplus \coprod_{j=1}^{t} A_qe_{b(j+1)} \otimes e_{b(j)}A_q,$$

where $\coprod_{l=1}^{0} A_qe_1 \otimes e_1A_q = 0$. The generators $e_1 \otimes e_1$, $e_{a(i)} \otimes e_{a(i)}$ and $e_{b(j)} \otimes e_{b(j)}$ of P_{2n} are labeled ε_l^{2n} for $0 \leq l \leq 2n$, $\varepsilon_{a(i)}^{2n}$ for $2 \leq i \leq s$, and $\varepsilon_{b(j)}^{2n}$ for $2 \leq j \leq t$ respectively.

Similarly, we denote the generators $e_1 \otimes e_1$, $e_{a(i+1)} \otimes e_{a(i)}$ and $e_{b(j+1)} \otimes e_{b(j)}$ of P_{2n+1} by ε_l^{2n+1} for $1 \leq l \leq 2n$, $\varepsilon_{a(i)}^{2n+1}$ for $1 \leq i \leq s$, and $\varepsilon_{b(j)}^{2n+1}$ for $1 \leq j \leq t$ respectively. In [15], we give the minimal projective bimodule resolution of A_q as follows.

Theorem 3.1 ([15, Theorem 1.1]). *The following sequence $\mathbb{P}$ is a minimal projective resolution of the left A_q^e-module A_q:*

$$\mathbb{P}: \cdots \longrightarrow P_{2n+1} \xrightarrow{d_{2n+1}} P_{2n} \xrightarrow{d_{2n}} P_{2n-1} \longrightarrow \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} A_q \longrightarrow 0.$$

where π: $P_0 \longrightarrow A_q$ is the multiplication map, and we define left A^e-homomorphisms d_{2n+1} and d_{2n+2} by

$$d_{2n+1}:$$
$$\begin{cases}
\varepsilon_{b(j)}^{2n+1} \mapsto \varepsilon_{b(j+1)}^{2n} Y - Y\varepsilon_{b(j)}^{2n} \text{ for } 1 \leq j \leq t, \\
\varepsilon_{l''}^{2n+1} \mapsto \\
\begin{cases}
-\sum_{j=0}^{b-1} q^{j(al+1)} Y^{tj} \varepsilon_{2l+1}^{2n} Y^{t(b-1-j)} - X^s \varepsilon_{2l}^{2n} + q^{b(n-l)} \varepsilon_{2l}^{2n} X^s \\
\qquad\qquad\qquad\qquad \text{if } l'' = 2l+1 \text{ for } 0 \leq l \leq n-1, \\
-q^{al'} Y^t \varepsilon_{2l'}^{2n} + \varepsilon_{2l'}^{2n} Y^t + \sum_{i=0}^{a-1} q^{i(b(n-l')+1)} X^{s(a-1-i)} \varepsilon_{2l'-1}^{2n} X^{si} \\
\qquad\qquad\qquad\qquad \text{if } l'' = 2l' \text{ for } 1 \leq l' \leq n,
\end{cases} \\
\varepsilon_{a(i)}^{2n+1} \mapsto \varepsilon_{a(i+1)}^{2n} X - X\varepsilon_{a(i)}^{2n} \text{ for } 1 \leq i \leq s,
\end{cases}$$

$$d_{2n+2}:$$
$$\begin{cases}
\varepsilon_0^{2n+2} \mapsto \sum_{l=0}^{b-1} \sum_{l'=0}^{t-1} Y^{tl+l'} \varepsilon_{b(t-l')}^{2n+1} Y^{t(b-1-l)+t-l'-1}, \\
\varepsilon_{b(j)}^{2n+2} \mapsto \sum_{l=0}^{b-1} \sum_{l'=0}^{t-1} Y^{tl+l'} \varepsilon_{b(j-1-l')}^{2n+1} Y^{t(b-1-l)+t-l'-1} \text{ for } 2 \leq j \leq t, \\
\varepsilon_1^{2n+2} \mapsto \\
qY^t \varepsilon_1^{2n+1} - \varepsilon_1^{2n+1} Y^t - \sum_{j=1}^{t} X^s Y^{t-j} \varepsilon_{b(j)}^{2n+1} Y^{j-1} + q^{bn+1} \sum_{j=1}^{t} Y^{t-j} \varepsilon_{b(j)}^{2n+1} Y^{j-1} X^s, \\
\varepsilon_{l''}^{2n+2} \mapsto \\
\begin{cases}
\sum_{j=0}^{b-1} q^{alj} Y^{tj} \varepsilon_{2l}^{2n+1} Y^{t(b-1-j)} + \sum_{i=0}^{a-1} q^{ib(n-l+1)} X^{s(a-1-i)} \varepsilon_{2l-1}^{2n+1} X^{si} \\
\qquad\qquad\qquad\qquad \text{if } l'' = 2l \text{ for } 1 \leq l \leq n, \\
q^{al'+1} Y^t \varepsilon_{2l'+1}^{2n+1} - \varepsilon_{2l'+1}^{2n+1} Y^t - X^s \varepsilon_{2l'}^{2n+1} + q^{b(n-l')+1} \varepsilon_{2l'}^{2n+1} X^s \\
\qquad\qquad\qquad\qquad \text{if } l'' = 2l'+1 \text{ for } 1 \leq l' \leq n-1,
\end{cases} \\
\varepsilon_{2n+1}^{2n+2} \mapsto \\
q^{an+1} \sum_{i=1}^{s} Y^t X^{s-i} \varepsilon_{a(i)}^{2n+1} X^{i-1} - \sum_{i=1}^{s} X^{s-i} \varepsilon_{a(i)}^{2n+1} X^{i-1} Y^t - X^s \varepsilon_{2n}^{2n+1} + q\varepsilon_{2n}^{2n+1} X^s, \\
\varepsilon_{a(i)}^{2n+2} \mapsto \sum_{l=0}^{a-1} \sum_{l'=0}^{s-1} X^{sl+l'} \varepsilon_{a(i-1-l')}^{2n+1} X^{s(a-1-l)+s-l'-1} \text{ for } 2 \leq i \leq s, \\
\varepsilon_{2n+2}^{2n+2} \mapsto \sum_{l=0}^{a-1} \sum_{l'=0}^{s-1} X^{sl+l'} \varepsilon_{a(s-l')}^{2n+1} X^{s(a-1-l)+s-l'-1},
\end{cases}$$

for $n \geq 0$, where in the case $n = 0$, $\varepsilon_{l''}^1$ and $\varepsilon_{l''}^2$ vanish, and the image of ε_1^2 by d_2 is

$$-\sum_{j=1}^{t} X^s Y^{t-j} \varepsilon_{b(j)}^1 Y^{j-1} + q\sum_{j=1}^{t} Y^{t-j} \varepsilon_{b(j)}^1 Y^{j-1} X^s$$
$$+ q\sum_{i=1}^{s} Y^t X^{s-i} \varepsilon_{a(i)}^1 X^{i-1} - \sum_{i=1}^{s} X^{s-i} \varepsilon_{a(i)}^1 X^{i-1} Y^t.$$

3.2 Hochschild cohomology group of A_q

Next, we give a basis of the n-th Hochschild cohomology group $\mathrm{HH}^n(A_q) := \mathrm{Ext}^n_{A_q^e}(A_q, A_q)$ for $n \geq 0$, using the minimal projective A_q^e-resolution given in Theorem 3.1. Now we consider the case $s,t \geq 2$. In the case $s=1$ or $t=1$, we can give a basis of $\mathrm{HH}^n(A_q)$ by the same method.

Now, we consider the vector space structure of $\mathrm{HH}^n(A_q)$ for all $n \geq 0$. By the definition of P_n, we have isomorphisms

$$u_{2n} : \mathrm{Hom}_{A_q^e}(P_{2n}, A_q) \xrightarrow{\sim} \coprod_{l=0}^{2n} e_1 A_q e_1 \oplus \coprod_{i=2}^{s} e_{a(i)} A_q e_{a(i)} \oplus \coprod_{j=2}^{t} e_{b(j)} A_q e_{b(j)},$$

$$u_{2n+1} : \mathrm{Hom}_{A_q^e}(P_{2n+1}, A_q) \xrightarrow{\sim} \coprod_{l=1}^{2n} e_1 A_q e_1 \oplus \coprod_{i=1}^{s} e_{a(i+1)} A_q e_{a(i)} \oplus \coprod_{j=1}^{t} e_{b(j+1)} A_q e_{b(j)},$$

where $\coprod_{l=1}^{0} e_1 A_q e_1 = 0$. We denote the k-modules $\mathrm{Im}\, u_{2n}$ and $\mathrm{Im}\, u_{2n+1}$ by P_{2n}^* and P_{2n+1}^* respectively. We see that the dimension of P_n^* is given by

$$\dim_k P_n^* = nab + ab(s+t-1).$$

The elements e_1, $e_{a(i)}$ and $e_{b(j)}$ of P_{2n}^* are labeled e_l^{2n} for $0 \leq l \leq 2n$, $e_{a(i)}^{2n}$ for $2 \leq i \leq s$ and $e_{b(j)}^{2n}$ for $2 \leq j \leq t$ respectively. Similarly, we denote the elements e_1, $e_{a(i)}$ and $e_{b(j)}$ of P_{2n+1}^* by e_l^{2n+1} for $1 \leq l \leq 2n$, $e_{a(i)}^{2n+1}$ for $1 \leq i \leq s$ and $e_{b(j)}^{2n+1}$ for $1 \leq j \leq t$ respectively. These labels correspond to that of generators of P_n. Using the maps $u_{2n}, u_{2n+1}, d_{2n+1}, d_{2n+2}$ for $n \geq 0$, we obtain the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{A_q^e}(P_0, A_q) \arrow[r, "\overline{d}_1"] \arrow[d, "u_0", "\sim"'] & \mathrm{Hom}_{A_q^e}(P_1, A_q) \arrow[r, "\overline{d}_2"] \arrow[d, "u_1", "\sim"'] & \mathrm{Hom}_{A_q^e}(P_2, A_q) \arrow[r, "\overline{d}_3"] \arrow[d, "u_2", "\sim"'] & \cdots \\
0 \arrow[r] & P_0^* \arrow[r, "d_1^*"] & P_1^* \arrow[r, "d_2^*"] & P_2^* \arrow[r, "d_3^*"] & \cdots,
\end{tikzcd}$$

where we put $\overline{d}_n = \mathrm{Hom}_{A_q^e}(d_n, A_q)$, $d_n^* = u_n \overline{d}_n u_{n-1}^{-1}$ for $n \geq 1$. Hence we have the complex

$$\mathbb{P}^* : 0 \to P_0^* \xrightarrow{d_1^*} P_1^* \to \cdots \to P_{n-1}^* \xrightarrow{d_n^*} P_n^* \to \cdots.$$

See [16] for the homomorphism d_n^*. Now, we denote some elements of P_n^* as follows:

- For $n = 0$:

$$\begin{aligned} T_{l,l'} :=& X^{sl} Y^{tl'} e_0^0 + \textstyle\sum_{j=2}^{t} Y^{j-1} X^{sl} Y^{t(l'-1)+t-j+1} e_{b(j)}^0 \\ &+ \textstyle\sum_{i=2}^{s} X^{s(l-1)+i-1} Y^{tl'} X^{s-i+1} e_{a(i)}^0 \text{ for } 1 \leq l \leq a \text{ and } 1 \leq l' \leq b, \end{aligned}$$

- For n odd, $n \geq 1$:

$$\begin{aligned} U_{0,l,l'}^n &:= Y X^{sl} Y^{tl'} e_{b(1)}^n \text{ for } 0 \leq l \leq a-1 \text{ and } 0 \leq l' \leq b-1, \\ U_{m,l,l'}^n &:= X^{sl} Y^{tl'} e_m^n \text{ for } 1 \leq m \leq n-1, 0 \leq l \leq a-1 \text{ and } 0 \leq l' \leq b-1, \\ U_{n,l,l'}^n &:= X^{sl+1} Y^{tl'} e_{a(1)}^n \text{ for } 0 \leq l \leq a-1 \text{ and } 0 \leq l' \leq b-1, \end{aligned}$$

- For n even, $n \geq 2$:

$$W^n_{0,l,l'} := X^{sl}Y^{tl'}e^n_0 + \textstyle\sum_{j=2}^t Y^{j-1}X^{sl}Y^{t(l'-1)+t-j+1}e^n_{b(j)}$$
$$\text{for } 0 \leq l \leq a-1 \text{ and } 0 \leq l' \leq b,$$
$$W^{n*}_{0,l,b-1} := bX^{sl}Y^{t(b-1)}e^n_0 + b\textstyle\sum_{j=2}^t Y^{j-1}X^{sl}Y^{t(b-2)+t-j+1}e^n_{b(j)}$$
$$+ (q^{b(n/2-1)+1} - 1)X^{s(l+1)}e^n_1 \qquad \text{for } 0 \leq l \leq a-1,$$
$$W^n_{m,l,l'} := X^{sl}Y^{tl'}e^n_m \text{ for } 1 \leq m \leq n-1, 0 \leq l \leq a-1 \text{ and } 0 \leq l' \leq b-1,$$
$$W^n_{n,l,l'} := X^{sl}Y^{tl'}e^n_n + \textstyle\sum_{i=2}^s X^{s(l-1)+i-1}Y^{tl'}X^{s-i+1}e^n_{a(i)}$$
$$\text{for } 0 \leq l \leq a \text{ and } 0 \leq l' \leq b-1,$$
$$W^{n*}_{n,a-1,l'} := aX^{s(a-1)}Y^{tl'}e^n_n + a\textstyle\sum_{i=2}^s X^{s(a-2)+i-1}Y^{tl'}X^{s-i+1}e^n_{a(i)}$$
$$+ (q^{a(n/2-1)+1} - 1)Y^{t(l'+1)}e^n_{n-1} \qquad \text{for } 0 \leq l' \leq b-1.$$

In the following results we use the complex $\mathbb{P}^*$ to compute a basis of the Hochschild cohomology group $\mathrm{HH}^n(A_q) = \mathrm{Ker}\, d^*_{n+1}/\mathrm{Im}\, d^*_n$ of our algebra A_q for $n \geq 0$. First, we consider the case where q is an r-th root of unity for integer $r \geq 1$. Now, we set $\overline{z}$ is the remainder when we divide z by r for any integer z. Then we have $0 \leq \overline{z} \leq r-1$.

Theorem 3.2 ([15, Proposition 3.3] and [16, Theorem 2.1]). *If q is an r-th root of unity for $r \geq 1$ and $s, t \geq 2$, then the following elements form a basis of $\mathrm{HH}^n(A_q)$.*

(1) *Basis of $\mathrm{HH}^0(A_q)$:*

  (a) $1_A = e^0_0 + \sum_{j=2}^t e^0_{b(j)} + \sum_{i=2}^s e^0_{a(i)}$,

  (b) $T_{l,l'}$ *for* $1 \leq l \leq a-1$, $1 \leq l' \leq b-1$ *if* $\overline{l} = \overline{l'} = 0$,

  (c) $T_{l,b}$ *for* $1 \leq l \leq a-1$,

  (d) $T_{a,l'}$ *for* $1 \leq l' \leq b-1$,

(2) *Basis of $\mathrm{HH}^{2n}(A_q)$ for $n \geq 1$:*

  (a) $W^{2n}_{0,0,l'}$ *for* $0 \leq l' \leq b-1$ *if* $\overline{l'} = \overline{bn}$,

  (b) $W^{2n}_{0,l,l'}$ *for* $1 \leq l \leq a-1$, $1 \leq l' \leq b-1$ *if* $\overline{l} = 0$, $\overline{l'} = \overline{bn}$,

  (c) $W^{2n}_{0,l,b}$ *for* $1 \leq l \leq a-1$ *if* $\overline{l} = 0$, $\overline{b(n-1)} = 0$, $\mathrm{char}\, k | b$,

  (d) $W^{2n}_{0,a-1,l'}$ *for* $0 \leq l' \leq b-1$ *if* $\overline{a} = 1$, $\overline{bn - l'} \neq 0$,

  (e) $\begin{cases} W^{2n*}_{0,l,b-1} \text{ for } 0 \leq l \leq a-1 \text{ if } \overline{l} = 0, \overline{b(n-1)+1} \neq 0, \mathrm{char}\, k \nmid b, \\ W^{2n}_{1,l+1,0} \text{ for } 0 \leq l \leq a-2 \text{ if } \overline{l} = 0, \overline{b(n-1)+1} \neq 0, \mathrm{char}\, k | b, \end{cases}$

  (f) $W^{2n}_{1,l+1,1}$ *for* $1 \leq l \leq a-2$ *if* $\overline{l} = 0$, $\overline{b(n-1)} \neq 0$,

  (g) $\begin{cases} W^{2n}_{2l''-1,l,l'} \text{ for } 1 \leq l \leq a-1, 1 \leq l' \leq b-1 \text{ if } \mathrm{char}\, k \nmid a, \mathrm{char}\, k \nmid b, \\ W^{2n}_{2l''-1,l,l'} \text{ for } 1 \leq l \leq a-1, 0 \leq l' \leq b-1 \text{ if } \mathrm{char}\, k \nmid a, \mathrm{char}\, k | b, \\ W^{2n}_{2l''-1,l,l'} \text{ for } 0 \leq l \leq a-1, 1 \leq l' \leq b-1 \text{ if } \mathrm{char}\, k | a, \mathrm{char}\, k \nmid b \\ W^{2n}_{2l''-1,l,l'} \text{ for } 0 \leq l \leq a-1, 0 \leq l' \leq b-1 \text{ if } \mathrm{char}\, k | a, \mathrm{char}\, k | b, \end{cases}$
  for $1 \leq l'' \leq n$ *if* $\overline{l} = \overline{a(l''-1)+1}$, $\overline{l'} = \overline{b(n-l'')+1}$,

(h) $\begin{cases} W^{2n}_{2l'',l,l'} \textit{ for } 0 \le l \le a-2, 0 \le l' \le b-2 \textit{ if } \operatorname{char} k \nmid a, \operatorname{char} k \nmid b, \\ W^{2n}_{2l'',l,l'} \textit{ for } 0 \le l \le a-2, 0 \le l' \le b-1 \textit{ if } \operatorname{char} k \nmid a, \operatorname{char} k | b, \\ W^{2n}_{2l'',l,l'} \textit{ for } 0 \le l \le a-1, 0 \le l' \le b-2 \textit{ if } \operatorname{char} k | a, \operatorname{char} k \nmid b \\ W^{2n}_{2l'',l,l'} \textit{ for } 0 \le l \le a-1, 0 \le l' \le b-1 \textit{ if } \operatorname{char} k | a, \operatorname{char} k | b, \end{cases}$
for $1 \le l'' \le n-1$ *if* $\overline{l} = \overline{al''}$, $\overline{l'} = \overline{b(n-l'')}$,

(i) $W^{2n}_{2n-1,1,l'+1}$ *for* $1 \le l' \le b-2$ *if* $\overline{l'} = 0$, $\overline{a(n-1)} \ne 0$,

(j) $\begin{cases} W^{2n*}_{2n,a-1,l'} \textit{ for } 0 \le l' \le b-1 \textit{ if } \overline{l'} = 0, \overline{a(n-1)+1} \ne 0, \operatorname{char} k \nmid a, \\ W^{2n}_{2n-1,0,l'+1} \textit{ for } 0 \le l' \le b-2 \textit{ if } \overline{l'} = 0, \overline{a(n-1)+1} \ne 0, \operatorname{char} k | a, \end{cases}$

(k) $W^{2n}_{2n,l,0}$ *for* $0 \le l \le a-1$ *if* $\overline{l} = \overline{an}$,

(l) $W^{2n}_{2n,l,l'}$ *for* $1 \le l \le a-1$, $1 \le l' \le b-1$ *if* $\overline{l'} = 0$, $\overline{l} = \overline{an}$,

(m) $W^{2n}_{2n,a,l'}$ *for* $1 \le l' \le b-1$ *if* $\overline{l'} = 0$, $\overline{a(n-1)} = 0$, $\operatorname{char} k | a$,

(n) $W^{2n}_{2n,l,b-1}$ *for* $0 \le l \le a-1$ *if* $\overline{b} = 1$, $\overline{an-l} \ne 0$

(o) *Additionally in the case of* $q = -1$:

i. $W^{2n}_{0,l,b}$ *for* $0 \le l \le a-1$ *if* $\overline{an} = 0$, $\overline{b} = 0$, $\overline{l} = 1$,

ii. $W^{2n}_{1,l,0}$ *for* $1 \le l \le a-1$ *if* $\overline{b} = 0$, $\overline{l} = 0$,

iii. $W^{2n}_{2l''-1,0,0}$ *for* $1 \le l'' \le n$ *if* $\overline{a} = \overline{b} = 0$,

iv. $W^{2n}_{2l'',a-1,b-1}$ *for* $1 \le l'' \le n-1$ *if* $\overline{a} = \overline{b} = 0$,

v. $W^{2n}_{2n-1,0,l'}$ *for* $1 \le l' \le b-1$ *if* $\overline{a} = 0$, $\overline{l'} = 0$,

vi. $W^{2n}_{2n,a,l'}$ *for* $0 \le l' \le b-1$ *if* $\overline{a} = 0$, $\overline{bn} = 0$, $\overline{l'} = 1$,

(3) *Basis of* $\mathrm{HH}^{2n+1}(A_q)$ *for* $n \ge 0$:

(a) $U^{2n+1}_{0,l,l'}$ *for* $0 \le l \le a-1$, $1 \le l' \le b-1$ *if* $\overline{l'} = \overline{bn}$, $\overline{l} = 0$,

(b) $U^{2n+1}_{0,a-1,l'}$ *for* $1 \le l' \le b-1$ *if* $\overline{a} = 1$, $\overline{bn-l'} \ne 0$,

(c) $U^{2n+1}_{0,l,b-1}$ *for* $0 \le l \le a-1$ *if* $\overline{l} = 0$, $\overline{b(n-1)+1} \ne 0$,

(d) $\begin{cases} U^{2n+1}_{0,0,0} \textit{ if } \overline{bn} = 0, \operatorname{char} k \nmid b, \\ U^{2n+1}_{0,l,0} \textit{ for } 0 \le l \le a-1 \textit{ if } \overline{l} = 0, \overline{bn} = 0, \operatorname{char} k | b, \end{cases}$

(e) $U^{2n+1}_{1,l+1,0}$ *for* $0 \le l \le a-2$ *if* $\overline{l} = 0$, $\overline{bn} \ne 0$,

(f) $\begin{cases} U^{2n+1}_{2l'',l,l'} \textit{ for } 0 \le l \le a-2, 1 \le l' \le b-1 \textit{ if } \operatorname{char} k \nmid a, \operatorname{char} k \nmid b, \\ U^{2n+1}_{2l'',l,l'} \textit{ for } 0 \le l \le a-1, 1 \le l' \le b-1 \textit{ if } \operatorname{char} k | a, \operatorname{char} k \nmid b, \\ U^{2n+1}_{2l'',l,l'} \textit{ for } 0 \le l \le a-2, 0 \le l' \le b-1 \textit{ if } \operatorname{char} k \nmid a, \operatorname{char} k | b, \\ U^{2n+1}_{2l'',l,l'} \textit{ for } 0 \le l \le a-1, 0 \le l' \le b-1 \textit{ if } \operatorname{char} k | a, \operatorname{char} k | b, \end{cases}$
for $1 \le l'' \le n$ *if* $\overline{l} = \overline{al''}$, $\overline{l'} = \overline{b(n-l'')+1}$,

(g) $\begin{cases} U^{2n+1}_{2l''+1,l,l'} \textit{ for } 1 \le l \le a-1, 0 \le l' \le b-2 \textit{ if } \operatorname{char} k \nmid a, \operatorname{char} k \nmid b, \\ U^{2n+1}_{2l''+1,l,l'} \textit{ for } 1 \le l \le a-1, 0 \le l' \le b-1 \textit{ if } \operatorname{char} k \nmid a, \operatorname{char} k | b, \\ U^{2n+1}_{2l''+1,l,l'} \textit{ for } 0 \le l \le a-1, 0 \le l' \le b-2 \textit{ if } \operatorname{char} k | a, \operatorname{char} k \nmid b, \\ U^{2n+1}_{2l''+1,l,l'} \textit{ for } 0 \le l \le a-1, 0 \le l' \le b-1 \textit{ if } \operatorname{char} k | a, \operatorname{char} k | b, \end{cases}$
for $0 \le l'' \le n-1$ *if* $\overline{l} = \overline{al''+1}$, $\overline{l'} = \overline{b(n-l'')}$,

(h) $U^{2n+1}_{2n,0,l'+1}$ *for* $0 \le l' \le b-2$ *if* $\overline{l'} = 0,\ \overline{an} \ne 0$,

(i) $U^{2n+1}_{2n+1,l,l'}$ *for* $1 \le l \le a-1,\ 0 \le l' \le b-1$ *if* $\overline{l'} = 0,\ \overline{l} = \overline{an}$,

(j) $U^{2n+1}_{2n+1,a-1,l'}$ *for* $0 \le l' \le b-1$ *if* $\overline{l'} = 0,\ \overline{a(n-1)+1} \ne 0$,

(k) $U^{2n+1}_{2n+1,l,b-1}$ *for* $1 \le l \le a-1$ *if* $\overline{b} = 1,\ \overline{an-l} \ne 0$,

(l) $\begin{cases} U^{2n+1}_{2n+1,0,0} \ \textit{if}\ \overline{an} = 0, \mathrm{char}\, k \nmid a, \\ U^{2n+1}_{2n+1,0,l'} \ \textit{for}\ 0 \le l' \le b-1 \ \textit{if}\ \overline{l'} = 0, \overline{an} = 0, \mathrm{char}\, k | a, \end{cases}$

(m) *Additionally in the case of* $q = -1$:

i. $U^{2n+1}_{0,l,0}$ *for* $0 \le l \le a-1$ *if* $\overline{a(n-1)} = 0,\ \overline{b} = 0,\ \overline{l} = 1$,

ii. $U^{2n+1}_{1,l,b-1}$ *for* $1 \le l \le a-1$ *if* $\overline{b} = 0,\ \overline{l} = 0$,

iii. $U^{2n+1}_{2l''-1,0,b-1}$ *for* $1 \le l'' \le n$ *if* $\overline{a} = \overline{b} = 0$,

iv. $U^{2n+1}_{2l'',a-1,0}$ *for* $1 \le l'' \le n$ *if* $\overline{a} = \overline{b} = 0$,

v. $U^{2n+1}_{2n,a-1,l'}$ *for* $1 \le l' \le b-1$ *if* $\overline{a} = 0,\ \overline{l'} = 0$,

vi. $U^{2n+1}_{2n+1.0,l'}$ *for* $0 \le l' \le b-1$ *if* $\overline{a} = 0,\ \overline{b(n-1)} = 0,\ \overline{l'} = 1$.

In the case $q = 1$, *q is a first root of unity. Then* $\overline{z} = 0$ *for any integer z. Hence if* $q = 1$ *then a basis of* $\mathrm{HH}^n(A_q)$ *is formed by the elements of* (1), (2)(a),(b),(c),(g),(h),(k),(l),(m) *and* (3)(a), (d), (f), (g), (i), (l).

Next, in the case where q is not a root of unity, we give a basis of $\mathrm{HH}^n(A_q)$ for $n \ge 0$.

Theorem 3.3 ([16, Theorem 2.2]). *If q is not a root of unity and* $s, t \ge 2$, *then the following elements form a basis of* $\mathrm{HH}^n(A_q)$.

(1) *Basis of* $\mathrm{HH}^0(A_q)$:

(a) $1_{A_q} = e^0_0 + \sum_{j=2}^{t} e^0_{b(j)} + \sum_{i=2}^{s} e^0_{a(i)}$,

(b) $T_{l,b}$ *for* $1 \le l \le a-1$,

(c) $T_{a,l'}$ *for* $1 \le l \le b-1$,

(2) *Basis of* $\mathrm{HH}^{2n+1}(A_q)$ *for* $n \ge 0$:

(a) $U^1_{0,l+1,b-1}$ *for* $0 \le l \le a-2$ *if* $n = 0$,

(b) $U^{2n+1}_{0,0,b-1}$,

(c) $U^1_{0,0,0}$ *if* $n = 0$,

(d) $U^1_{1,a-1,l'+1}$ *for* $0 \le l \le b-3$ *if* $n = 0$,

(e) $U^{2n+1}_{2n+1,a-1,0}$,

(f) $U^1_{1,0,0}$ *if* $n = 0$,

(3) *Basis of* $\mathrm{HH}^{2n+2}(A_q)$ *for* $n \ge 0$:

(a) $W^2_{1,1,1}$ *if* $n = 0$,

(b) $W^{2n+2*}_{0,0,b-1}$,

(c) $W^{2n+2*}_{2n+2,a-1,0}$.

3.3 Hochschild cohomology ring of A_q

In this section, we determin the Hochschild cohomology ring of A_q modulo nilpotence. Now we recall the Yoneda product in $\mathrm{HH}^*(A)$ (see [8]). For homogeneous elements $\eta \in \mathrm{HH}^m(A)$ and $\theta \in \mathrm{HH}^n(A)$ represented by η: $P_m \to A$ and θ: $P_n \to A$ respectively, the Yoneda product $\eta\theta \in \mathrm{HH}^{m+n}(A)$ is given as follows: There exists a commutative diagram of A-bimodules

$$\begin{tikzcd}
\cdots \arrow[r] & P_{m+n} \arrow[r, "d_{m+n}"] \arrow[d, "\sigma_m"] & \cdots \arrow[r, "d_{n+2}"] & P_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "\sigma_1"] & P_n \arrow[d, "\sigma_0"] \arrow[dr, "\theta"] & & \\
\cdots \arrow[r] & P_m \arrow[r, "d_m"] & \cdots \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "\pi"] & A \arrow[r] & 0,
\end{tikzcd}$$

where σ_i are liftings of θ. Here we see that such liftings always exist but are not unique. Then we have $\eta\theta = \eta\sigma_m \in \mathrm{HH}^{m+n}(A)$. We note that $\eta\theta$ is independent of the choice of representation η, θ and liftings σ_i $(0 \le i \le m)$. See [16, Proposition 3.1] for the liftings of the basis of $\mathrm{HH}^n(A_q)$ $(n \ge 0)$. In the case where q is a root of unity, by the liftings given in [16, Proposition 3.1], we see that $\mathrm{HH}^{n+2r}(A_q)$ is generated by the elements in $\mathrm{HH}^n(A_q)$ and that in $\mathrm{HH}^{2r}(A_q)$ for $n \ge 2r$. By corresponding Yoneda product of the basis elements of $\mathrm{HH}^*(A_q) := \oplus_{n\ge 0}\mathrm{HH}^n(A_q)$ given in Theorem 3.2, we now have the generators of $\mathrm{HH}^*(A_q)$. In this paper. we consider the case where $s, t \ge 2$ and $\bar{a}, \bar{b} \neq 0$. In the other cases, we have similar results to the following theorem and corollary. See [16] for the other cases.

Theorem 3.4 ([16, Theorem 3.2]). *In the case where* $\bar{a}, \bar{b} \neq 0$, $\mathrm{HH}^*(A_q)$ *is generated as an algebra by the following generators*:

(1) *The generators of* $\mathrm{HH}^*(A_q)$ *in degree* 0:
$1_{A_q}, T_{r,r}, T_{l_1,r}, T_{r,l'_1}, T_{l_2,b}, T_{a,l'_2}$
for $1 \le l_1, l_2 \le a-1$, $1 \le l'_1, l'_2 \le b-1$ *if* $\overline{l_1} = \overline{l'_1} = 0$.

(2) *The generators of* $\mathrm{HH}^*(A_q)$ *in degree* 1:

- $U^1_{0,0,l'}$, $U^1_{1,l,0}$ *for* $0 \le l \le a-1$, $0 \le l' \le b-1$ *if* $\bar{l} = \bar{l'} = 0$,
- $\begin{cases} U^1_{0,a-1,l'} \text{ for } 1 \le l' \le b-1 \text{ if } \bar{a} = 1, \bar{l'} \neq 0, \\ U^1_{1,a-1,l'} \text{ for } 0 \le l' \le b-1 \text{ if } \bar{a} \neq 1, \bar{l'} = 0, \end{cases}$
- $\begin{cases} U^1_{1,l,b-1} \text{ for } 1 \le l' \le a-1 \text{ if } \bar{l} \neq 0, \bar{b} = 1, \\ U^1_{0,l,b-1} \text{ for } 1 \le l \le a-1 \text{ if } \bar{l} = 0, \bar{b} \neq 1, \end{cases}$
- $U^1_{1,0,l'}$ *for* $1 \le l' \le b-1$ *if* $\bar{l'} = 0$, $\operatorname{char} k | a$,
- $U^1_{0,l,0}$ *for* $1 \le l \le a-1$ *if* $\bar{l} = 0$, $\operatorname{char} k | b$.

(3) *The generators of* $\mathrm{HH}^*(A_q)$ *in degree* 2:

- $W^2_{0,0,l'_1}$, $W^2_{2,l_1,0}$, $W^2_{0,l_2,\bar{b}}$, $W^2_{2,\bar{a},l'_2}$
for $1 \le l_1, l_2 \le a-1$, $1 \le l'_1, l'_2 \le b-1$ *if* $\overline{l_1} = \bar{a}$, $\overline{l'_1} = \bar{b}$, $\overline{l_2} = \overline{l'_2} = 0$,
- $\begin{cases} W^{2*}_{0,l,b-1} \text{ for } 0 \le l \le a-1 \text{ if } \bar{l} = 0, \operatorname{char} k \nmid b, \\ W^2_{1,l+1,0} \text{ for } 0 \le l \le a-2 \text{ if } \bar{l} = 0, \operatorname{char} k | b, \end{cases}$

- $\begin{cases} W^{2*}_{2,a-1,l'} \text{ for } 0 \le l' \le b-1 \text{ if } \overline{l'} = 0, \operatorname{char} k \nmid a, \\ W^{2}_{1,0,l'+1} \text{ for } 0 \le l' \le b-2 \text{ if } \overline{l'} = 0, \operatorname{char} k | a, \end{cases}$
- $W^2_{0,a-1,l'}$ *for* $0 \le l' \le b-1$ *if* $\overline{a} = 1,\ \overline{l'} \neq \overline{b}$,
- $W^2_{2,l,b-1}$ *for* $0 \le l \le a-1$ *if* $\overline{b} = 1,\ \overline{l} \neq \overline{a}$.

(4) *The generators of* $\mathrm{HH}^*(A_q)$ *in degree* $2n$ *for* $2 \le n \le r$:

- $\begin{cases} W^{2n}_{0,0,\overline{bn}} \text{ if } \min\{\overline{bn'} | 1 \le n' \le n-1\} > \overline{bn}, \\ W^{2n}_{0,l,\overline{bn}} \text{ for } 1 \le l \le a-1 \text{ if } \overline{l} = 0, \overline{bn} \neq 0, \min\{\overline{bn'} | 1 \le n' \le n-1\} > \overline{bn}, \\ W^{2n}_{0,a-1,\overline{l'_1}+\overline{l'_2}-r} \\ \text{if } \overline{a} = 1, \overline{l'_1} + \overline{l'_2} \ge r+1, \overline{l'_1} = \min\{\overline{bn'} | 1 \le n' \le n-1\}, \\ \quad \overline{l'_2} \neq \overline{b(n-n')} \text{ where } n' \text{ is integer such that } 1 \le n' \le n-1 \text{ and} \\ \quad \overline{bn'} \le \overline{bn''} \text{ for any } 1 \le n'' \le n-1, \end{cases}$
- $\begin{cases} W^{2n*}_{0,l,b-1} \text{ for } 0 \le l \le a-1 \text{ if } \overline{l} = 0, \overline{b(n-1)+1} \neq 0, \operatorname{char} k \nmid b, \\ W^{2n}_{1,l+1,0} \text{ for } 0 \le l \le a-2 \text{ if } \overline{l} = 0, \overline{b(n-1)+1} \neq 0, \operatorname{char} k | b, \end{cases}$
- $W^{2n}_{1,l+1,1}$ *for* $1 \le l \le a-2$ *if* $\overline{l} = 0,\ \overline{b(n-1)} \neq 0$,
- $\begin{cases} W^{2n}_{1,l,0} \text{ for } 1 \le l \le a-1 \text{ if } \overline{l} = 1, \overline{b(n-1)+1} = 0, \operatorname{char} k | b, \\ W^{2n}_{2n-1,0,l'} \text{ for } 1 \le l' \le b-1 \text{ if } \overline{a(n-1)+1} = 0, \overline{l'} = 1, \operatorname{char} k | a, \\ W^{2n}_{2l''-1,0,0} \text{ for } 1 \le l'' \le n \\ \text{ if } \overline{a(l''-1)+1} = \overline{b(n-l'')+1} = 0, \operatorname{char} k | a, \operatorname{char} k | b, \end{cases}$
- $W^{2n}_{2n-1,1,l'+1}$ *for* $1 \le l' \le b-2$ *if* $\overline{l'} = 0,\ \overline{a(n-1)} \neq 0$,
- $\begin{cases} W^{2n*}_{2n,a-1,l'} \text{ for } 0 \le l' \le b-1 \text{ if } \overline{l'} = 0, \overline{a(n-1)+1} \neq 0, \operatorname{char} k \nmid a, \\ W^{2n}_{2n-1,0,l'+1} \text{ for } 0 \le l' \le b-2 \text{ if } \overline{l'} = 0, \overline{a(n-1)+1} \neq 0, \operatorname{char} k | a, \end{cases}$
- $\begin{cases} W^{2n}_{2n,\overline{an},0} \text{ if } \min\{\overline{an'} | 1 \le n' \le n-1\} > \overline{an}, \\ W^{2n}_{2n,\overline{an},l'} \\ \text{for } 1 \le l' \le b-1 \text{ if } \overline{l'} = 0, \overline{an} \neq 0, \min\{\overline{an'} | 1 \le n' \le n-1\} > \overline{an}, \\ W^{2n}_{2n,\overline{l_1}+\overline{l_2}-r,b-1} \\ \text{if } \overline{b} = 1, \overline{l_1} + \overline{l_2} \ge r+1, \overline{l_1} = \min\{\overline{an'} | 1 \le n' \le n-1\}, \\ \quad \overline{l_2} \neq \overline{a(n-n')} \text{ where } n' \text{ is integer such that } 1 \le n' \le n-1 \text{ and} \\ \quad \overline{an'} \le \overline{an''} \text{ for any } 1 \le n'' \le n-1. \end{cases}$

(5) *The generators of* $\mathrm{HH}^*(A_q)$ *in degree* $2n+1$ *for* $1 \le n \le r-1$:

- $U^{2n+1}_{0,l,\overline{bn}}$ *for* $0 \le l \le a-1$ *if* $\overline{l} = 0, \min\{\overline{bn'} | 1 \le n' \le n-1\} \ge \overline{bn}$,
- $U^{2n+1}_{0,l,b-1}$ *for* $0 \le l \le a-1$ *if* $\overline{l} = 0,\ \overline{b(n-1)+1} \neq 0,\ \operatorname{char} k | b$,
- $U^{2n+1}_{1,l+1,0}$ *for* $0 \le l \le a-2$ *if* $\overline{l} = 0,\ \overline{bn} \neq 0$,
- $\begin{cases} U^{2n+1}_{2,l,0} \text{ for } 0 \le l \le a-2 \text{ if } \overline{l} = \overline{a}, \overline{b(n-1)+1} = 0, \\ U^{2n+1}_{2l'',\overline{al''},0} \text{ for } 2 \le l'' \le n, 0 \le \overline{al''} \le \begin{cases} a-2 \text{ if } \operatorname{char} k \nmid a, \\ a-1 \text{ if } \operatorname{char} k | a, \end{cases} \\ \text{if } \overline{b(n-l'')+1} = 0, \min\{\overline{al'''} | 1 \le l''' \le l''-1\} > \overline{al''}, \end{cases}$ *if* $\operatorname{char} k | b$,

- $\begin{cases} U^{2n+1}_{2l''+1,0,\overline{b(n-l'')}} \ \textit{for}\ 0 \le l'' \le n-1, 0 \le \overline{b(n-l'')} \le \begin{cases} b-2 & \textit{if}\ \mathrm{char}\, k \nmid b, \\ b-1 & \textit{if}\ \mathrm{char}\, k | b, \end{cases} \\ \textit{if}\ \overline{al''+1} = 0, \quad \min\{\overline{b(n'-l'')} | l''+1 \le n' \le n-1\} > \overline{b(n-l'')}, \\ U^{2n+1}_{2n-1,0,l'} \ \textit{for}\ 0 \le l' \le b-2 \ \textit{if}\ \overline{a(n-1)+1} = 0, \overline{l'} = \overline{b}, \end{cases}$
  if $\mathrm{char}\, k | a$,
- $U^{2n+1}_{2n,0,l'+1}$ *for* $0 \le l' \le b-2$ *if* $\overline{l'} = 0$, $\overline{an} \ne 0$,
- $U^{2n+1}_{2n+1,\overline{an},l'}$ *for* $0 \le l' \le b-1$ *if* $\overline{l'} = 0, \min\{\overline{an'} | 1 \le n' \le n-1\} \ge \overline{an}$,
- $U^{2n+1}_{2n+1,a-1,l'}$ *for* $0 \le l' \le b-1$ *if* $\overline{l'} = 0$, $\overline{a(n-1)+1} \ne 0$, $\mathrm{char}\, k | a$.

(6) *The generators of* $\mathrm{HH}^*(A_q)$ *in degree* $2r+2n+1$ *for* $0 \le n \le r-2$:

- $U^{2r+2n+1}_{2l'',\overline{al''},0}$ *for* $n+1 \le l'' \le r, 0 \le \overline{al''} \le \begin{cases} a-2 & \textit{if}\ \mathrm{char}\, k \nmid a, \\ a-1 & \textit{if}\ \mathrm{char}\, k | a, \end{cases}$
  if $\min\{\overline{al'''} | 1 \le l''' \le l''-1\} > \overline{al''}, \overline{b(n-l'')+1} = 0, \mathrm{char}\, k | b$,
- $U^{2r+2n+1}_{2l''+1,0,\overline{b(n-l'')}}$ *for* $1 \le l'' \le r-1, 0 \le \overline{b(n-l'')} \le \begin{cases} b-2 & \textit{if}\ \mathrm{char}\, k \nmid b, \\ b-1 & \textit{if}\ \mathrm{char}\, k | b, \end{cases}$
  if $\overline{al''+1} = 0, \min\{\overline{b(n'-l'')} | l''+1 \le n' \le r+n-1\} > \overline{b(n-l'')}, \mathrm{char}\, k | a$.

It follows from the Theorem 3.4 that 1_{A_q}, $W^{2r}_{0,0,0}$ and $W^{2r}_{2r,0,0}$ are not nilpotent and the other generators are nilpotent. Thus we have the following corollary.

Corollary 3.5. *If* $s, t \ge 2$ *and* $\overline{a}, \overline{b} \ne 0$, *then the quotient of the Hochschild cohomology ring of* A_q *modulo nilpotence is isomorphic to the polynomial ring of two variables in all characteristic*:

$$\mathrm{HH}^*(A_q)/\mathcal{N} \cong k[W^{2r}_{0,0,0}, W^{2r}_{2r,0,0}].$$

Finaly, we consider the ring structure of $\mathrm{HH}^*(A_q)$ in the case where q is not a root of unity. It follows from the liftings given in [16] that all basis elements except 1_{A_q} of $\mathrm{HH}^n(A_q)$ are nilpotent elements for $n \ge 0$. Thus we have the following results.

Theorem 3.6. *If q is not a root of unity then* $\mathrm{HH}^*(A_q)$ *is not a finitely generated k-algebra.*

Corollary 3.7. *If q is not a root of unity then* $\mathrm{HH}^*(A_q)/\mathcal{N} \cong k$.

In general, our algebra A_q is not self-injective, monomial or Koszul. Moreover A_q does not have a stratifying ideal. Therefore A_q is new example of a class of algebras for which the Hochschild cohomology ring modulo nilpotence is finitely generated as a k-algebra. For example, in the case where $s = 2$, $t = 1$ and $a = b = 2$, our algebra A_q is not self-injective, monomial or Koszul. Moreover A_q does not have a stratifying ideal.

3.4 Finiteness conditions for A_q

Finally, we show that A_q satisfies the finiteness conditions in the case where q is a root of unity.

Now we consider the case where q is an r-th root of unity, $s, t \ge 2$ and $\overline{a}, \overline{b} \ne 0$. In the other case, we see that A_q satisfies the finiteness conditions by the same method. The Yoneda algebra or Ext algebra of A_q is given by $E(A_q) = \oplus_{n \ge 0} \mathrm{Ext}^n_{A_q}(A_q/\mathfrak{r}, A_q/\mathfrak{r})$

with the Yoneda product. We use the notation $E(A_q)^n = \mathrm{Ext}^n_{A_q}(A_q/\mathfrak{r}, A_q/\mathfrak{r})$ for the n-th graded component of $E(A_q)$. Then it is easy to see that $E(A_q)^n \simeq \coprod_{l=0}^n ke_l^n \oplus \coprod_{j=2}^t ke^n_{b(j)} \oplus \coprod_{i=2}^s ke^n_{a(i)}$.

Let φ: $\mathrm{HH}^*(A_q) \to E(A_q)$ be a homomorphism of graded rings given by $\varphi(\eta) = \eta \otimes_{A_q} A_q/\mathfrak{r}$. Then it is easy to see that the image of φ is precisely the graded ring $k[x, y]$ where $x := e_0^{2r} + \sum_{j=2}^t e^{2r}_{b(j)}$ and $y := e^{2r}_{2r} + \sum_{i=2}^s e^{2r}_{a(i)}$ in degree $2r$.

Proposition 3.8. *$E(A_q)$ is a finitely generated left $k[x, y]$-module with generators:*

$e_l^{2n}, e^{2n}_{b(j)}, e^{2n}_{a(i)}$ *for* $0 \le l \le 2n, 2 \le j \le t$ *and* $2 \le i \le s$

in degree $2n$ *for* $0 \le n \le r-1$,

$e_l^{2n+1}, e^{2n+1}_{b(j)}, e^{2n+1}_{a(i)}$ *for* $1 \le l \le 2n, 1 \le j \le t$ *and* $1 \le i \le s$

in degree $2n+1$ *for* $0 \le n \le r-1$,

e_l^{2r} *for* $1 \le l \le 2r-1$ *in degree* $2r$,

$e_l^{2r+2n+1}$ *for* $2n+1 \le l \le 2r$ *in degree* $2r+2n+1$ *for* $0 \le n \le r-1$,

$e_l^{2r+2n+2}$ *for* $2n+1 \le l \le 2r-1$ *in degree* $2r+2n+2$ *for* $0 \le n \le r-2$.

Now we consider the conditions **(Fg1)** and **(Fg2)**. The element $W^{2r}_{0,0,0} \in \mathrm{HH}^{2r}(A_q)$ is a pre-image of x and the element $W^{2r}_{2r,0,0} \in \mathrm{HH}^{2r}(A_q)$ is a pre-image of y. Let H be the graded subalgebra of $\mathrm{HH}^*(A_q)$ generated by $\mathrm{HH}^0(A_q)$, $W^{2r}_{0,0,0}$ and $W^{2r}_{2r,0,0}$, so that H is a pre-image of $k[x, y]$ in $\mathrm{HH}^*(A_q)$. Then we have the following immediate consequence of Proposition 3.8.

Theorem 3.9. *The conditions* **(Fg1)** *and* **(Fg2)** *hold for the algebra A_q with respect to the subring H of* $\mathrm{HH}^*(A_q)$.

By [2], Theorem 3.6 and 3.9, we have the necessary and sufficient conditions for A_q to satisfy the finiteness conditions.

Theorem 3.10. *A_q satisfies the finiteness conditions if and only if q is a root of unity.*

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